Mathematics and Arts

Allégorie de la Géométrie
A MATHEMATICAL INTERPRETATION

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Abstract: *In this work, we present a mathematical interpretation for the masterpiece* Allégorie de la Géométrie *(1649), painted by the French baroque artist Laurent de La Hyre (1606–1656).*

Keywords: Laurent de La Hyre, "Allégorie de la Géométrie", baroque art, mathematical interpretation, perspective.

Introduction

The main purpose of this text is to present a mathematical interpretation for the masterpiece *Allégorie de la Géométrie*, from a well-known series of paintings, *Les 7 arts libéraux*, by the French baroque artist Laurent de La Hyre (1606–1656).

Figure 1: *Allégorie de la Géométrie* (1649), oil on canvas.

Laurent de La Hyre painted the series *Les 7 arts libéraux* between 1649 and 1650 to decorate Gédéon Tallemant's residence. Tallemant was an adviser of Louis XIV (1638–1715). The king was 10 years old at the time of the commission. According to the artist's son, Philippe de La Hire (1640–1718), writing around 1690 [5],

> (...) une maison qui appartenoit autrefois a M. Tallemant, maistre des requestes, sept tableaux représéntant les sept arts liberaux qui font l'ornement d'une chambre.

Also, Guillet de Saint-Georges, a historiographer of the *Académie Royale de Peinture et de Sculpture*, mentioned that it was Laurent's work for the Capuchin church in the Marais which led to the commission for the "Seven Liberal Arts" in a house [2].

The seven liberal arts were seen as the crucial parts of learning, distinct from philosophy (knowledge) and manual arts (e.g., agriculture). They come from ancient Greece and have been depicted as women ever since. That is why geometry is personified by a woman in Laurent's painting.

As a general impression, *Allégorie de la Géométrie* shows rich color and nice forms (characteristic of Laurent's style). Geometry was personified as a young woman holding a compass and right angle in one hand and a "mysterious" sheet inscribed with mathematical diagrams in the other. She is surrounded by monuments, a globe, pyramids, a sphinx, all in an Egyptian-Greek athmosphere. The first obvious ideas suggest the opposites practice/theory and concrete/abstract (somehow related to ancient Greece/ancient Egypt) and for the impeccable technique (rigorous perspective). The dualities concrete-abstract and practice-theory strike us as *they go hand in hand* with the Egypt-Greece visual context. Accordingly, somehow, the theoretical mathematical results should be related to the concrete situations displayed in the painting.

Laurent's series was dispersed sometime between 1760 and 1793. At present we know nine paintings that belonged to the original group [9]. The *Allégorie de la Géométrie* was recently acquired by Fine Arts Museums of San Francisco. In early 2014, the work was valued between USD$800,000 and 1.2 million dollars [6].

Relevant facts about Laurent de La Hyre

Laurent de La Hyre, born in a bourgeois milieu, received a high-level education, having been a student of the painter Georges Lallemant (1575–1636). De La Hyre is well-known for his paintings incorporating scenes from classical antiquity. He regularly represented ruins backgrounds based on a very careful design: *Allégorie de la Géométrie* is one such example. Laurent's home was a stimulating place, receiving visits of prominent artists, scientists, and mathematicians: our analysis is consistent with this fact. Mathematics and geometry played an important role in Laurent's life. The great mathematician Girard Desargues (1591–1661), one of the founders of projective geometry, was among his regular visitors. One source for projective geometry was the theory of perspective, very important for the development presented in the next section. In that theory, a horizon line is a theoretical line that represents the eye level of the observer, vanishing points are points where parallel lines converge, vertical lines maintain the verticality in the paintings, etc [7]. Laurent de La Hyre was an expert using these geometric concepts in his paintings. Euclidean geometry and theory of perspective were also important for Laurent's descendants, perhaps influenced by the environment they lived in as children. Philippe de La Hire (1640–1718), Laurent's son, was a mathematician and astronomer; his works on conic sections and epicycloids were based on the teaching of Desargues, of whom he was the favorite pupil. Gabriel-Philippe de La Hire (1677–1719), Philippe's son and Laurent's grandson, was also a mathematician. These remarks suggest that the family was very mathematicaly minded.

A Mathematical Interpretation for the *Allégorie de la Géométrie*

First, we list some key ideas that can be found in other published interpretations (all inspired by [8]):

> Of all the works in the series, *Allégorie de la Géométrie* perhaps held the greatest personal significance for the artist, since knowledge of the discipline, in the form of perspective, was crucial for the practice of painting at the time. In the late 1630s, De la Hyre studied with the mathematician Girard Desargues, who is considered one of the founders of projective geometry [6].
>
> In the center of the composition sits the embodiment of geometry, leaning on a block of marble. In her right hand is a sheet of paper with a diagram of the golden section and three Euclidean proofs, which she holds up for the viewer to see. In her left hand is a compass and a right angle, tools of a mathematician and geometer. Surrounding her are references to the practical applications of geometry, beginning at the left with a painting set on an easel, the painter's palette and brushes affixed below. The globe next to it stands for the earth, but also is a product of geometry, as geometry underpins the process of map making. The snake above is an attribute for the goddess Ceres, an older representation of the earth. (...) At the right is a sphinx, representing Egypt, where an early form of geometry was invented, but the large crack in its back represents the flaws in the Egyptian approach; it was not until the advent of Euclid that modern geometry was developed [6].
>
> A relief is carved on the side of a tomb; the sacrificial scene may carry a hidden meaning. At the right sits a sphinx that symbolically represents Egypt, where an early form of geometry was developed. The painting relays the idea that the Egyptians developed an imperfect and inferior geometry based on practical necessity compared Greek geometry, which was valued as abstract thought and pure learning [1].

In the following interpretation, the idea that “the Egyptians developed an *imperfect and inferior geometry* based on practical necessity compared Greek geometry” is replaced by the idea of *complementarity*; the idea of “a diagram of the golden section” is simply not true; the idea of “three Euclidean proofs, for the viewer to see” is much more sophisticated because the three results are really crucial *to understand the painting*; the idea of a “globe standing for the earth and also a product of geometry” is quite important and *it is related to one of the Euclidean proofs*; the idea of “greatest personal significance for the artist” because of “perspective, crucial for the practice of painting at the time” may be greatly expanded.

THE “MYSTERIOUS” SHEET OF PAPER

The young woman holds a “mysterious” sheet containing mathematical illustrations (Figure 2).

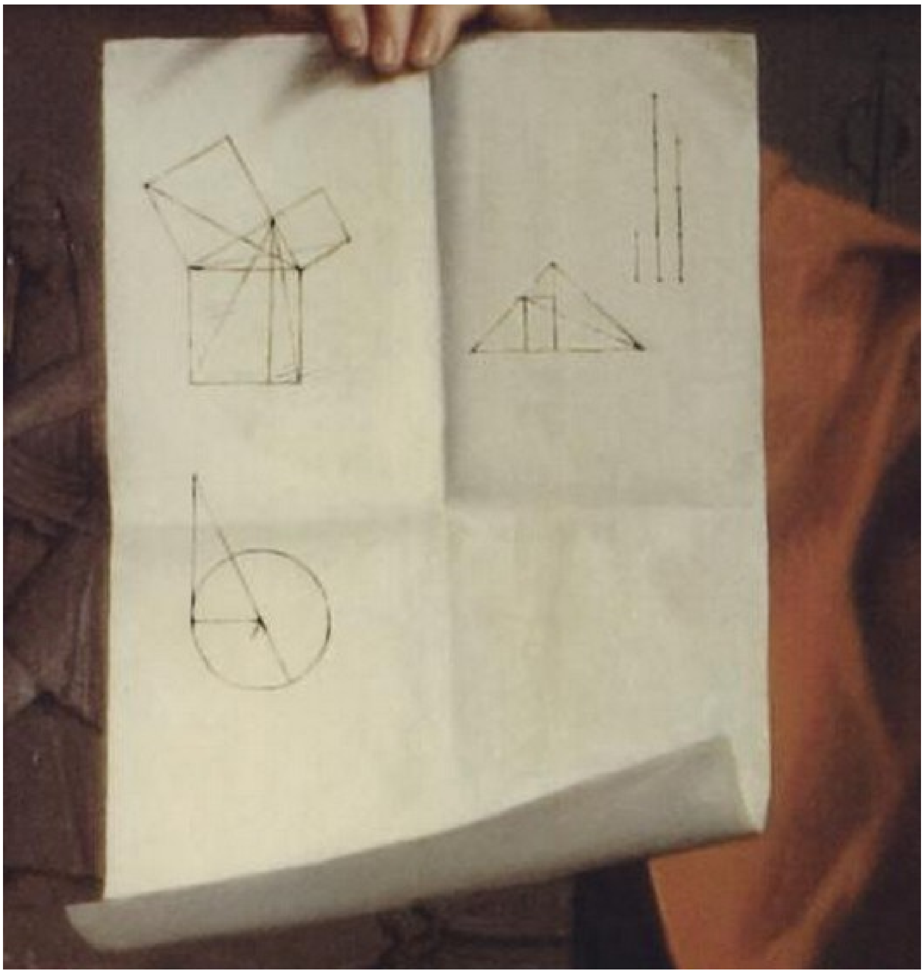

Figure 2: Mathematical diagrams.

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For sure, the top-left diagram illustrates the famous Pythagorean Proposition (*Elements* I, 47, [3]), stating that, in a right triangle, the sum of the squares on the two legs equals the square on the hypotenuse (Figure 3).

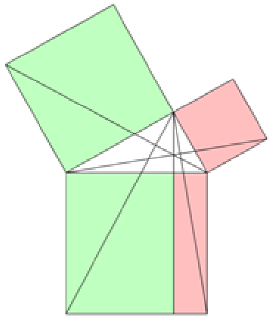

Figure 3: *Elements* I, 47.

In the painting, there are objects with right angles everywhere. Furthermore, the young woman holds a compass and a right angle on the other hand. But we think there are more hidden reasons for the Theorem's occurrence. Later, a possible use of the Pythagorean Theorem for the painting process will be presented.

The top-right diagram is very interesting. First of all, *it is not related to the golden ratio.* It is more likely to be the *Elements* II, 9 [3]:

If a straight line is cut into equal and unequal segments, then the sum of the squares on the unequal segments of the whole is double the sum of the square on the half and the square on the straight line between the points of section.

Let a segment $[AB]$ be cut into equal segments at C, and into unequal segments at D (Figure 4). Then, $\overline{AD}^2 + \overline{DB}^2 = 2(\overline{AC}^2 + \overline{CD}^2)$.

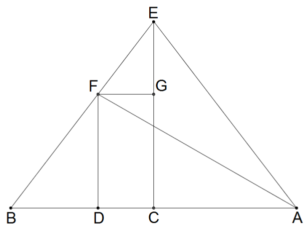

Figure 4: *Elements* II, 9.

The diagram for the proof is the same as the drawing exposed in the sheet of paper. That is the likely reason for the idea that *all diagrams* are related to Euclidean results. However, there is a big but. Proposition 9 is not related, in any way, to the atmosphere of the painting. It is much more logical to prove the following result:

Theorem 1. *Consider ABC, a isosceles triangle, D, the midpoint of $[AB]$ and E, the midpoint of $[DC]$. Let G be the intersection of AE with BC and H its perpendicular foot in $[AB]$. Then,*

$$\frac{\overline{GH}}{\overline{DE}} = \frac{4}{3}\,.$$

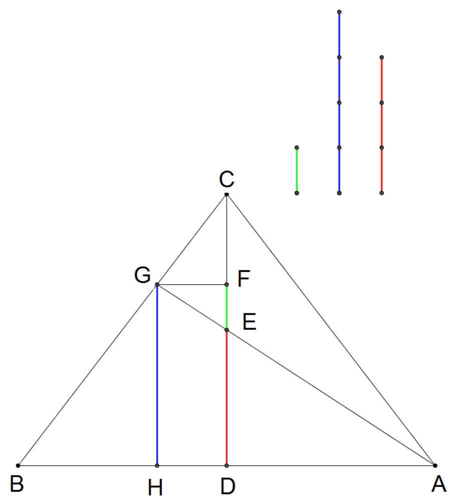

Figure 5: Four thirds rule.

Proof.

First,

$$\frac{\overline{CD}}{\overline{AD}} = \frac{\overline{CF}}{\overline{FG}} \Leftrightarrow \frac{2 \times \overline{DE}}{\overline{AD}} = \frac{\overline{CE} - \overline{EF}}{\overline{FG}} \Leftrightarrow \frac{2 \times \overline{DE}}{\overline{AD}} = \frac{\overline{DE}}{\overline{DH}} - \frac{\overline{EF}}{\overline{DH}}\,.$$

Second,

$$\frac{\overline{EF}}{\overline{FG}} = \frac{\overline{DE}}{\overline{AD}} \Leftrightarrow \frac{\overline{EF}}{\overline{DH}} = \frac{\overline{DE}}{\overline{AD}}\,.$$

Joining first and second,

$$\frac{2 \times \overline{DE}}{\overline{AD}} = \frac{\overline{DE}}{\overline{DH}} - \frac{\overline{DE}}{\overline{AD}} \Leftrightarrow \frac{\overline{DH}}{\overline{AD}} = \frac{1}{3}\,.$$

Finally,

$$\frac{\overline{GH}}{\overline{DH} + \overline{AD}} = \frac{\overline{DE}}{\overline{AD}} \Leftrightarrow \frac{\overline{GH}}{\overline{DE}} = \frac{\overline{DH} + \overline{AD}}{\overline{AD}} \Leftrightarrow \frac{\overline{GH}}{\overline{DE}} = 1 + \frac{\overline{DH}}{\overline{AD}} \Leftrightarrow \frac{\overline{GH}}{\overline{DE}} = \frac{4}{3}\,.$$

□

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What is interesting about this result is its generality: it works for *all* isosceles triangles. Moreover, it corresponds exactly to the diagram in the sheet of paper. We observe that the painting has several pyramids of *different shapes*. If someone wanted to make a tunnel from the ground, passing through a room in the center of a pyramid, this result would provide the location of the exit on the other side (Figure 6).

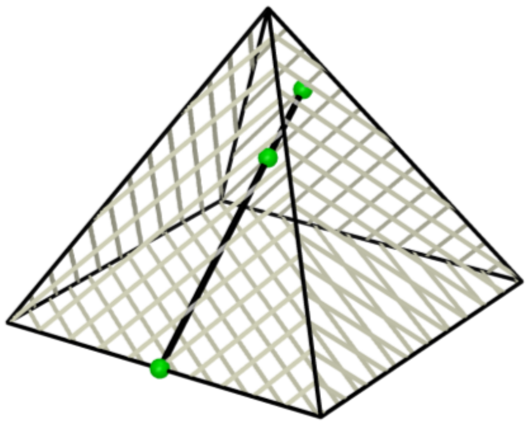

Figure 6: Pyramid.

Laurent de La Hyre did not have the modern knowledge about the interior of the Egyptian pyramids, their astronomical significance and construction methods. Nevertheless, apart from the fact that looting had been common practice for a long time, the existence of chambers, both interior and underground is old knowledge. For example, Herodotus (484–425 BC) mentioned

> (...) The aforesaid ten years went to the building of this road and of the underground chambers in the hill where the pyramids stand [4].

The use of mathematics in human buildings is a central issue in this Laurent's artwork. Egyptian pyramids in the background make it natural that Laurent de La Hyre may have thoughts about the architecture and the interior of the pyramids.

The bottom diagram is also interesting. *Elements* I, 32 [3] states that three interior angles of a triangle are equal to two right angles. Using that, the relation illustrated in the Figure 7 has a very easy proof.

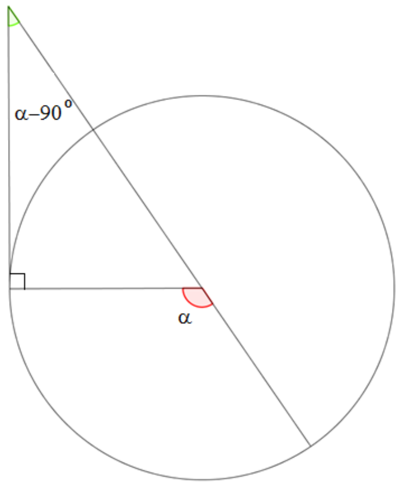

Figure 7: Bottom diagram.

Of course, the fundamental question is to know *why* that particular relation is there. If we contextualize we risk a guess (Figure 8).

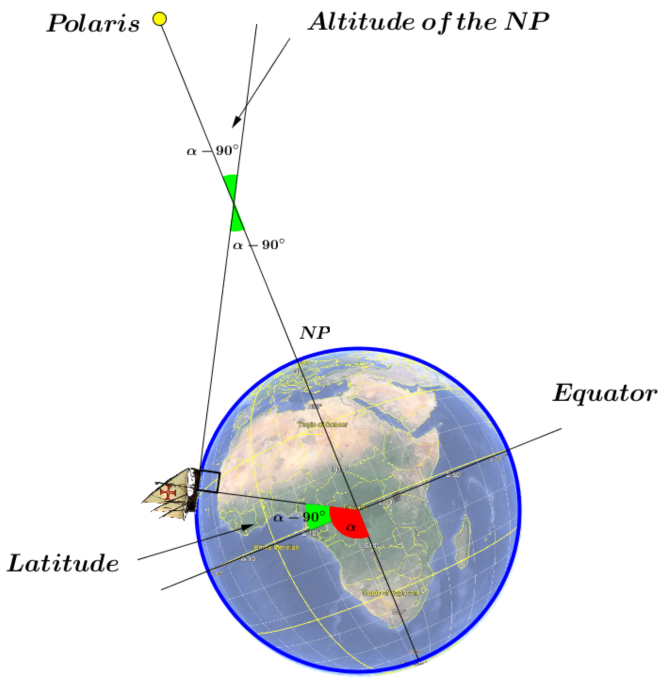

Figure 8: Latitude determination.

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The result in question gives a direct argument for the fact that the altitude of the northern celestial pole is exactly the latitude. The globe, as well as Ceres' snake, draws attention to the use of geometry in surveying, navigation, map making, and so on. The chosen geometric relation is a concrete example of that use.

We observe that the Pythagorean theorem, the relation latitude/altitude of the northern pole and the Four thirds rule are general results, showing the incredible power of geometry, the main theme of the painting. The Pythagorean theorem holds for *all right triangles*; the relation latitude/altitude of the NP holds for *any place on Earth*, the Four thirds rule holds for *all isosceles triangles* (and, thus, for all pyramids). That strong power and generality of geometry is exactly what Laurent de La Hyre wanted to transmit in his masterpiece.

PERSPECTIVE

Laurent de La Hyre learned techniques on perspective from Girard Desargues. Basically, perspective is a representation, on a flat surface (canvas), of an image as it is seen by the observer's eye. Italian Renaissance painters, including Filippo Brunelleschi, Masaccio, Paolo Uccello, Piero della Francesca, and Luca Pacioli, studied perspective, incorporating it into their artworks. An important characteristic feature of perspective is that vanishing points are points where parallel lines converge. In we consider some obvious parallel lines (not the horizontal lines of the painting inside the painting!), we find that the head of the sphinx is a vanishing point (Figure 9).

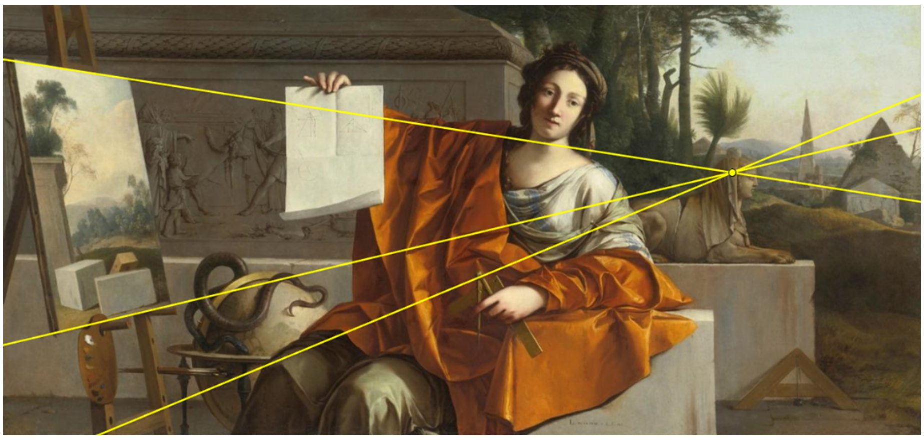

Figure 9: Head of the sphinx.

This fact is full of meaning. Everything takes place in the head of the sphinx, symbolizing the complementarity between ancient Greece and ancient Egypt and, more generally, between practical knowledge and mathematical theory. This makes even more sense considering that the sphinx is an important element of both Greek and Egyptian cultures.

Perspective arises as a self-referential piece. This hymn to geometry begs to be approached geometrically. The painting within the painting is a natural challenge. Its lack of human presence seems to tell us that the practical applications derived from mathematical theory enrich the world in an essential way.

We did some experiments to assess if the perspective within the perspective is accurate. Mathematically speaking, it is a composition of perspectives. Of course, it is almost impossible to know the exact perspective techniques used by the artist, but we can easily verify whether the perspective is rigorous. Notice that the cuboids are the easiest objects to represent. That reinforces the idea that one of the artist's purposes was to execute a composition of perspectives. After analyzing several possibilities (canvas and block dimensions, observer's position, block positions), we got the figure of the appendix 1 as a *possible* "inner painting". In Appendix 1, the observer lives in the painting world. Then, we studied the perspective made *with that possibility* (canvas dimensions, observer's position, height and inclination of the canvas) and we got the figure of the Appendix 2. In that case, the observer lives in our world and, as expected, his eye's level is the sphinx's head level. The matching is really impressive.

Observe that the inclination of the inner painting is an important parameter (our conclusion suggests $\alpha = 77°$). Because the heights are needed informations, maybe the artist used the diagram of the Figure 10. An easy application of the Pythagorean Theorem.

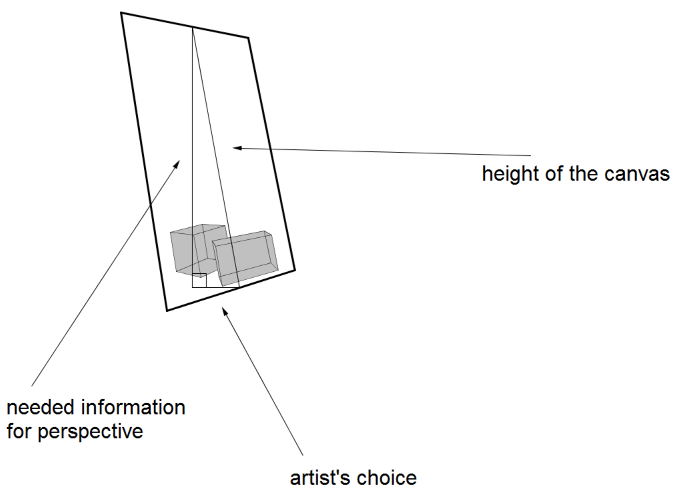

Figure 10: Canvas' inclination.

There are aspects of this interpretation that are pretty speculative. There are other aspects that are very likely: the matching with rigorous perspective and the Four thirds rule are mathematical facts. We do not know how truthful our interpretation is, but its internal logic consistency made it worth pursuing.

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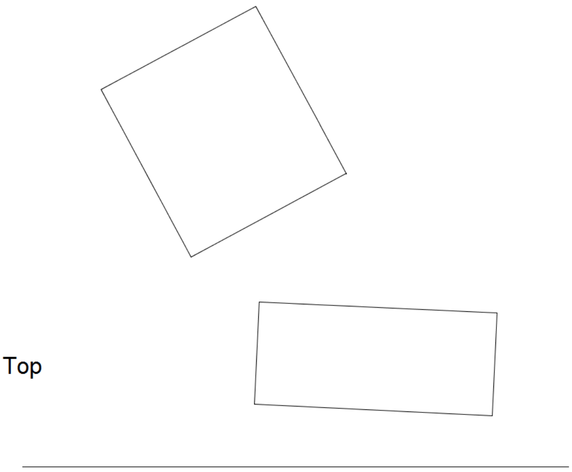
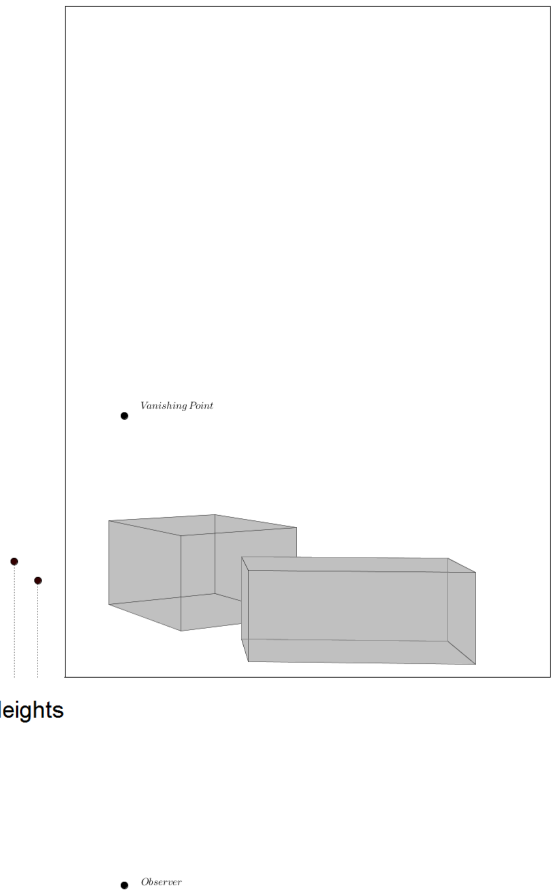
Appendix 1



Top



Heights

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Appendix 2