ANALYSIS OF PRICES IN THE HOUSING MARKET USING MIXED MODELS

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Abstract

Hedonic models, commonly applied for analyzing prices in the property market, do not always fulfil their role, mainly due to the application of simplified assumptions concerning the distribution of variables, the nature of relations or spatial heterogeneity. Classical regression models assumed that the variation of the explained variable (price) is explained by the effect of market features (fixed effects) and the residual component. The hierarchical structure of market data, both as regards market segments and the spatial division, suggests that statistical models of prices should also include random effects for selected subgroups of properties and interactions between variables. The mixed model provides an alternative for constructing various regression models for individual groups or for using binary variables within one model. With its appropriate structure, it makes it possible to take into account both the spatial heterogeneity and to examine the effects of individual features on prices within various property groups. It can also identify synergy effects.

The article presents the issue of mixed modelling in the property market and an example of its application in a market of dwellings in Olsztyn. The research used transaction data from the price and value register, supplemented with spatial data. The obtained model was compared with classical regression models and geographically weighted regression. The study also covered the usefulness of mixed models in the mass evaluation of properties, and the possibility of using them in spatial analyses and for the development of property value maps.

Keywords: *hierarchical analysis, mixed models, housing market, prices.*

JEL Classification: *C21, R21, R31.*

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1. Introduction and aim of the research

The principal aim of the paper is to present the possibilities of applying mixed models to analyze the development of key relations between the price and the attributes that can affect the value of a given property. The specific grouping of individual properties has a significant effect on the evaluation of the relations between the features describing these properties, thus contributing to obligatory disproval of the assumptions of the independence and identity of the distribution of the said features. This results in the autocorrelation of random components and/or heteroskedasticity in the classical linear regression model (HOX 2002; SNIJDERS, BOSKER 1999). The scarcity of simple univariate structures, the multitude of examined and mutually related features, as well as the requirement for the observation of independency causes difficulty in, or even prevents, applying a simple regression model or other research methods using linear models (RADKIEWICZ, ZIELIŃSKI 2010). Additionally, in such situations, unweighted estimators of regression parameters lose their effectiveness and their standard errors become negatively biased (HOX, MAAS 2004).

In a typical OLS regression analysis, three approaches are applied to grouped observations and their derivatives (Table 1).

Table 1

Methods for analyzing data in the group structure

Approach	Analysis
I - Disaggregation	Ignoring and passing up the analysis of the group structure of data
II - Aggregation	Data aggregation at the group level (units of analysis). The regression equation presents dependencies between mean values of explanatory variables in groups and average group predictors. The application of this method may cause the risk of drawing erroneous conclusions from equations intended for group data of individual observation features – the result in the form of the estimation of equations with other regression coefficients
III – Fixed effect model	Regression of a dependent variable on predictors, including observation grouping into the model. Explanatory variables represent a unit level of data, and identification of group membership is presented in the form of the so-called zero-one variables (dummy codes) of set g – 1 (where g is the number of groups). In effect, we obtain a slope coefficient for the X predictor, reflecting the weighted average of regression coefficient values from all observation groups (the effect of total interclass regression)

*Source:*own study on the base of RADKIEWICZ, ZIELIŃSKI 2010.

The application of simple regression methods can involve certain problems related primarily to the issue of the observation of independence. The fact that interrelations between the residuals of the model can be the result of systematic changes in time poses a problem. A solution in this case may be more thorough sampling, a modified, hierarchical linear model or a correction of the research procedure. Other problems include the dependency of a series of examinations of the same features in the fixed set of objects showing minor differences in small time intervals, as well as collecting empirical material in specific groups, demonstrating minor differences in regression residuals from the sets and major differences between them. A solution in this situation may be hierarchical (multi-level) data analysis. Of course, if the data are grouped, it is possible to develop and estimate the number of models equal to the number of groups in the data set. However, the application of models with random coefficients and a random free term (i.e. multi-level, hierarchical models) allows a more common approach to modelling relations between features. This is a certain type of compromise between estimating all features without taking into account their grouping and doing it in each group separately. Additionally, multilevel models make it possible to determine the level and the source of differences for the explained variable caused by grouping the features on given hierarchy levels. This happens mainly as a result of taking into account the relations of explanatory variables of the first (or individual) level and higher/group levels (WĘZIAK 2007; WOLTMAN et al. 2012). Therefore, the hierarchical model is nothing more than a series of regression analyses for each group having their own coefficients (RADKIEWICZ, ZIELIŃSKI 2010). This statistical technique is quite popular in almost all fields of science, since it does not require significant assumptions, unlike other more stringent methods (OSBORNE 2000). Hierarchical models are highly useful for planning and managing the operations of an enterprise (COCHRANE, HEATLEY 1996), for analyzing data collected from visual sensors used with autonomous vehicles or in health care (SPEHR 2015), decision making and risk estimation based on qualitative or quantitative data (CHAN, WANG 2013), analysis of engineering structures (BUCALEM, BATHE 2011), analysis of gene structure (BENTSEN, KLEMETSDAL 1991) and in many other situations. However, its application is reasonable when the grouping affects the relations between features. For this reason, a null (or empty) model is used, taking into account the explained variables themselves (1), and the intergroup correlation coefficient is calculated (2), presenting the share of the intergroup variance in the global variance of the explained variable (SNIJDERS, BOSKER 1999):

$$Y_{ij} = \gamma_{00} + U_{0j} + R_{ij} \tag{1}$$

where:
Y_{ij} - value of dependent variable for the i-th respondent of the j-th group (i=1,…N_i, j=1,…,K),
γ_{00} - average value of the Y variable (the so-called global mean),
U_{0j} - random component at the group level,
R_{ij} - random component at the individual level.

$$\rho = \frac{Var\ (U_{0j})}{Var(U_{0j})+Var\ (R_{ij})} = \frac{r_0^2}{r_0^2+\sigma^2} \tag{2}$$

However, according to Muthén analysis (MUTHÉN 1997), the examination of the effect of the first level variables on the explained variable should not be based on classical linear modelling methods, since a relatively low factor of intergroup correlation with a high number of group members can already evoke significantly different results from simple and cluster sampling. The regression equation of the multilevel model is a set of equations explaining endogenous variables on individual levels of the hierarchy (HOX, MAAS 2005; HOX 2002):

$$\left.\begin{aligned} Y_{ij} &= \beta_{0j} + \beta_{1j}X_{ij} + R_{ij} \\ \beta_{0j} &= \gamma_{00} + \gamma_{01}Z_j + U_{0j} \\ \beta_{1j} &= \gamma_{10} + \gamma_{11}Z_j + U_{1j} \end{aligned}\right\} \qquad Y_{ij} = \gamma_{00} + \gamma_{10}X_{ij} + \gamma_{01}Z_j + \gamma_{11}Z_jX_{ij} + U_{1j}X_{ij} + U_{0j} \tag{3}$$

where:
γ_{00} - global mean,
$\gamma_{01}, \gamma_{10}, \gamma_{11}$ - regression coefficients,
X_{ij} - level of variable of the individual level X for the i-th unit of the j-th group,
Z_j- value of variables of the group level for the j-th group.

Linear mixed effects models are the combination of fixed parts/effects with random parts/effects (BIECEK 2013; BRYK, RAUDENBUSH 1992; PINHEIRO, BATES 2000). In view of the frequent application of data demonstrating a specific horizontal hierarchy structure for analyses, those models are considered a part of hierarchy models. Data used for estimating the mixed model contain only selected groups for a specific random factor. However, the aim of the analysis is to draw conclusions about the entire set, using selected clusters of the random factor, defined through explanatory variables with random effects (BRYK, RAUDENBUSH 1992).

Fixed factors, referred to as the explanatory/independent fixed effect variable, are described as qualitative, categorical or classification variables, covering all grouped data subject to the examination. In turn, random factors identify a given cluster selected at random for the examined population (BIECEK 2013).

In the context of real estate properties, i.e. objects demonstrating the complexity of location, physical and legal features, problems with regard to constructing functional relations between the price and the property attributes are also observed (EKELAND et al. 2004). Additionally, data from the property market are generally characterized by a hierarchical structure of spatial and functional information (BRAUNAUER et al. 2010). Therefore, mixed models are particularly useful in this case, since it is possible to take into account the random effects for individual subgroups of properties, as well as interactions between variables (EKELAND et al. 2004). Constructing one regression model for individual groups instead of constructing several models and using binary variables in each of them proves the highly promising effectiveness of this solution (BRAUNAUER et al. 2010; BROWN, UYAR 2004).

2. Research data and methodology

The analysis of price variables on the basis of property attributes with the use of the mixed model is based on 2014-2017 data, obtained from the Register of Property Prices and Values for the capital city of a Polish province - Olsztyn. It used information contained in the register concerning the transaction date and form, usable floor area of premises, the floor on which it was situated, the height of the building and the year in which it was constructed.

In order to identify the regularities and rules applicable in a selected data set, a reliable data set was selected, rejecting those values that contained errors or provided too high of a deviation from mean prices per m² of usable floor area of the premises (e.g. resulting from the failure to meet the specified standard and from the type of premises), as well as transactions with a price discount. Additionally, transactions distorting the model to the highest extent were also excluded, using, for this purpose, the popular measure for evaluating the effect of observations to evaluate coefficients in the model, the so-called Cook's distance, showing how an indicator changes when a given observation is deleted from the analysis. This measure is described as "the normalized sum of squared differences between the evaluation of the explained variable determined in the entire sample and the one determined in this sample with the *i*-th observation deleted" (BIECEK 2013):

$$D_i = \frac{\sum_{j=1}^{n}(x_j\hat{\beta} - x_j\widehat{\beta_{(-\imath)}})^2/p}{\sum_{j=1}^{n}(y - x_j\hat{\beta})^2/(n-p)} = \frac{\sum_{j=1}^{n}(\hat{y_j} - \widehat{y_{j,(-\imath)}})^2}{p\widehat{\sigma^2}} \tag{4}$$

where:
$\widehat{y_{j,(-\imath)}}$- evaluation of the explained variable for the values of explanatory variables x_i in the model without using observation x_i

In the next step, those transactions were geocoded using the Google Earth Pro software, which, due to a precise address, including the street name and the number of the building, provided spatial reference to 99% of data. This procedure made it possible to assign, to each record, all additional information required for further analyses. The set of transaction data was supplemented with information concerning the location within a given housing estate, distances to significant objects (i.e. bus stops, schools, the city center, inland water reservoirs or forests), traffic and industrial noise level, and evaluations provided by residents concerning the quality of life, cleanliness and safety. The information on spatial conditions was obtained from the Open Street Map portal and MSIPMO (Municipal Spatial Information System of Olsztyn), while the survey results were taken from the publicly available Report of a survey conducted among Olsztyn residents, reflecting their problems and vision of the city, prepared in 2016 and presenting detailed statistical data assigned to individual districts. The distribution of transactions in individual parts of Olsztyn is presented below, using the Open Street Map as a base. Final conditions taken into account in the analysis included the logarithmic distance of transactions from the city center, surface waters (major water reservoirs), safety evaluation in the opinion of residents (the remaining part was rejected due to a high correlation coefficient), and traffic noise emission (industrial noise was omitted due to the lack of significant industrial facilities).

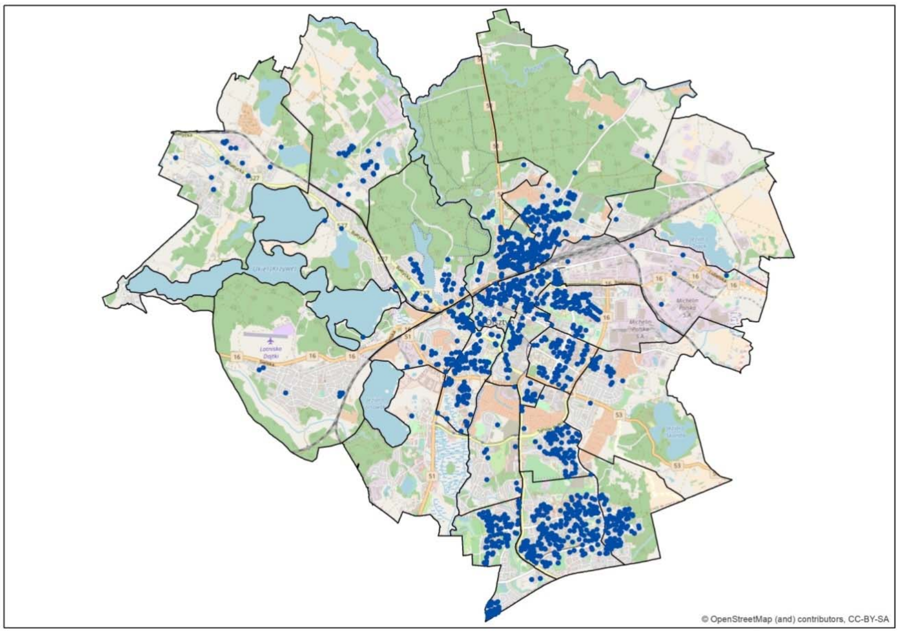

Fig. 1. Location of transactions obtained from the Register of Property Prices and Values (2014-2017).
Source: own study.

Based on the above distribution of identified transactions in relation to the residential estates of Olsztyn, it can be concluded that their highest number was located in the city center and in its direct vicinity, as well as in the so-called "dormitory suburbs" of Olsztyn, i.e. the estates of Nagórki, Pieczewo, Generałów and Jaroty. In each region of the city, the intensity of traffic noise is concentrated along the major lines of transportation. Most water reservoirs are located in the western part of the city, where, due to the declining rate of development, only a few transactions were concluded in the last three years.

The analysis of the location of a flat in terms of the floor also took into account the number of storeys in a given building, since the top floors have a different significance in a high building with a lift than in a low building without a lift. For this reason, the following criterion for grouping the buildings based on their height was assumed: low buildings of up to 2 floors, buildings of medium height - up to 5 floors, and high buildings - 6 floors and higher. The following scores were assumed for individual floors: ground floor – 0 points, first floor – 5 points, second floor – 4 points, third floor – 3 points., fourth floor – 2 points, fifth floor and above – 1 point. Additionally, 1 point was deducted for the highest floors.

Finally, one explained variable (Y_1), 8 explanatory variables (X_1- X_8) and two grouping variables (Z_1-Z_2) were assumed for the analysis:

Table 2

Variables assumed for the analysis

Indicator	Notation	Symbol
Price of the dwelling	PLN/m^2	Y_1
Useful floor area of the dwelling	m^2	X_1
Transaction date	consecutive number of the month, starting from January 2014	X_2
Floor location of the dwelling	explained above	X_3
Construction year of the building	year	X_4
Distance from the city center	natural logarithm of distance in km	X_5
Distance from lakes	natural logarithm of distance in km	X_6
Traffic noise level	L_{DWN} in db	X_7
Safety evaluation	on the 1-5 scale – explained above	X_8
Building height	low, medium, high – explained above	Z_1
Housing estate name	administrative name	Z_2

Source: own study.

The next step was to carry out the final analysis of collected and organized data in environment R under the R CRAN GNU license, available as an Open Source, using the lme4 package that permits defining fixed and random effects in one formula (BIECEK 2013; BATES 2015). First, linear modelling was carried out in order to compare it later with the mixed model obtained from the estimation of coefficients using the REML method.

3. Results and discussion

The study was carried out in four stages, with the first stage involving the linear model estimation, and the next three stages referring to mixed modelling, respectively: with random effects for location (Z_2), taking into account built-up density (Z_1) and taking into account interactions between variables.

Since the applied models are of a diagnostic and not predictive nature, all variables were used, even those without statistical significance.

As a result of the linear simple regression model estimation, five significant variables were obtained along with two variables of low significance (transaction date and traffic noise level); the obtained results are presented in Table 3. The unit prices of dwellings increase along with reduced distance from the city center and from major water reservoirs. A slight decrease is also observed for the useful floor area and traffic noise emission. The analysis of floor location also demonstrated a negative relationship with the price.

Table 3

Classical linear model estimation results

Variable	Estimate	Std. error	t value	Pr (>|t|)	Signif. codes
(Intercept)	5378.296	178.104	30.197	<0.001	***
X_1	-11.377	0.449	-25.346	<0.001	***
X_2	-1.620	0.797	-2.033	0.0421	*
X_3	7.464	5.409	1.380	0.1678	
X_4	177.435	4.262	41.628	<0.001	***
X_5	-335.695	15.663	-21.432	<0.001	***
X_6	-76.078	16.157	-4.709	<0.001	***
X_7	-3.067	1.242	-2.469	0.0136	*
X_8	53.028	9.114	5.818	<0.001	***
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.'					

Source: own study.

Estimation results for the mixed model with random effects for location are presented in Table 4. At the same time, slight differences are demonstrated between the values of the free term in comparison to the linear model. The values of probability were subject to significant changes, among which only four variables demonstrate the highest level. A reverse effect in relation to the previous model is observed only in the context of traffic noise immission.

Table 4

Results of estimation for a mixed random effects model for location (fixed effects presented in the table)

Variable	Estimate	Std. Error	t value	Pr(>|t|)	Signif. codes
(Intercept)	6248.927	426.954	14.636	<0.001	***
X_1	-12.349	0.454	-27.220	<0.001	***
X_2	-0.574	0.804	-0.714	0.475	
X_3	4.272	5.332	0.801	0.423	
X_4	167.663	4.695	35.712	<0.001	***
X_5	-357.130	36.031	-9.912	<0.001	***
X_6	-182.698	34.009	-5.372	<0.001	***
X_7	0.859	1.318	0.652	0.515	
X_8	40.498	40.223	1.007	0.329	
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.'					

Source: own study.

Based on the grouping of data under analysis in the residential estate units, taking into account all conditions prevailing in the estates, it can be concluded that the effects of location in relation to the estate are mainly concentrated south of the railway tracks and in the Nad Jeziorem Długim and Gutkowo estates. The highest values are observed in the residential estates of Generałów, Brzeziny and Kościuszki, and the lowest are related to the estates of Zielona Górka and Likusy (Fig. 2).

The subsequent stage involved an analysis of not only the location of a given dwelling in regards to the specific floor, but also the height of the building. In this case, the estimation results are more similar to the linear model than the estimation results for the model with random effects for location (Table 5).

It can be observed that buildings of medium height have the highest effect on the development of unit prices, followed by low buildings, up to two storeys high; the highest buildings demonstrate the lowest impact (Table 6). However, differences in prices obtained for individual properties in the context of this attribute are not significant.

Name of estate	Random effect
Brzeziny	385.745
Dajtki	-101.402
Generałów	336.766
Grunwaldzkie	-65.933
Gutkowo	11.158
Jaroty	176.43
Kętrzyńskiego	25.422
Kormoran	-116.812
Kortowo	18.264
Kościuszki	204.214
Likusy	-241.073
Mazurskie	136.947
Nad Jez. Dł.	89.698
Nagórki	-106.85
Pieczewo	117.919
Podgrodzie	93.43
Podleśna	-59.857
Pojezierze	63.762
Redykajny	-192.505
Śródmieście	2.759
Wojska Polsk.	-145.504
Zacisze	123.218
Zatorze	-134.534
Zielona Górka	-621,271

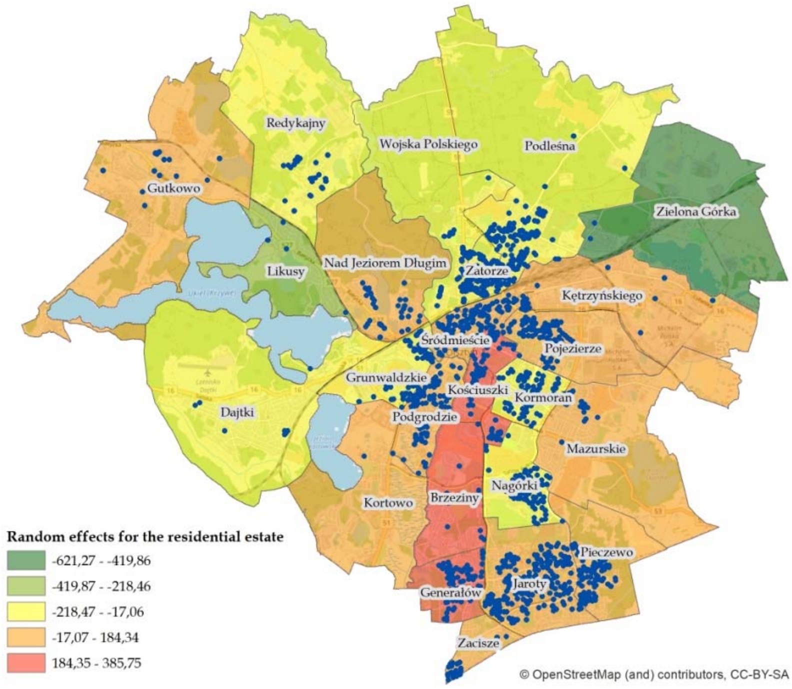

Fig. 2. Random effects for the residential estate as a grouping variable
Source: own study.

Table 5

Estimation results for mixed random effects model for building height (fixed effects presented in the table)

Variable	Estimate	Std. Error	t value	Pr(>|t|)	Signif. codes
(Intercept)	5360.531	178.818	29.978	<0.001	***
X_1	-11.264	0.687	-16.400	0.004	**
X_2	-1.699	0.798	-2.130	0.033	*
X_3	8.315	5.424	1.533	0.125	
X_4	175.095	4.443	39.409	<0.001	***
X_5	-328.180	16.199	-20.259	<0.001	***
X_6	-74.607	16.178	-4.612	<0.001	***
X_7	-3.285	1.246	-2.637	0.008	**
X_8	51.661	9.137	5.654	<0.001	***
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.'					

Source: own study.

Table 6

Random effects for build-up density

Building height	Random effect	X_1
low	-9.506	-10.936
average	21.946	-12.022
high	-12.438	-10.834

Source: own study.

The estimation of the next model brought about the following results (Table 7), which demonstrate certain similarity to the model with random effects for location. Undoubtedly, the highest impact is found for residential estates as grouping variables, but at the same time, they cause higher standard errors. Additionally, a classical OLS model and the model with random effects for the building height demonstrate a higher degree of significance of variables assumed for the analysis.

Table 7

The results obtained from the mixed model taking into account interaction between variables

Variable	Estimate	Std. Error	t value	Pr(>|t|)	Signif. codes
(Intercept)	6212.411	436.031	14.248	<0.001	***
X_1	-12.179	0.454	-26.826	<0.001	***
X_2	-0.579	4.507	-0.128	0.907	
X_3	-11.777	13.142	-0.896	0.370	
X_4	163.996	4.736	34.630	<0.001	***
X_5	-343.108	35.796	-9.585	<0.001	***
X_6	-177.754	33.769	-5.264	<0.001	***
X_7	0.308	1.318	0.234	0.815	
X_8	17.954	43.476	0.413	0.683	
X_2: X_8	0.366	0.694	0.527	0.598	
X_3: X_8	6.552	4.405	1.487	0.137	
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.'					

Source: own study.

The values of random effects for an average dwelling in a given residential estate are as follows (Fig. 3): the highest values are observed in the center of Olsztyn (Kościuszki state) and in the residential estates of Brzeziny and Generałów; slightly lower values concern areas located in the southeastern part of the city, i.e. in Brzeziny and near Długie Lake and Kortowo. The lowest values, as can be seen in the model presented in Table 5, are observed in case of Likusy and Zielona Górka estates. Comparing the results from two models, taking into account a group variable in the form of residential estates, the results are very similar, and differences are observed only in the case of Gutkowo.

Name of estate	Random effect
Brzeziny	371,097
Dajtki	-107,283
Generałów	342,909
Grunwaldzkie	-83,671
Gutkowo	-16,359
Jaroty	174,671
Kętrzyńskiego	35,36
Kormoran	-134,983
Kortowo	21,259
Kościuszki	190,13
Likusy	-235,538
Mazurskie	132,656
Nad Jez. Dł.	82,017
Nagórki	-97,305
Pieczewo	105,135
Podgrodzie	98,730
Podleśna	-33,514
Pojezierze	68,141
Redykajny	-184,162
Śródmieście	-14,943
Wojska Polsk.	-128,700
Zacisze	124,536
Zatorze	-125,099
Zielona Górka	-585,081

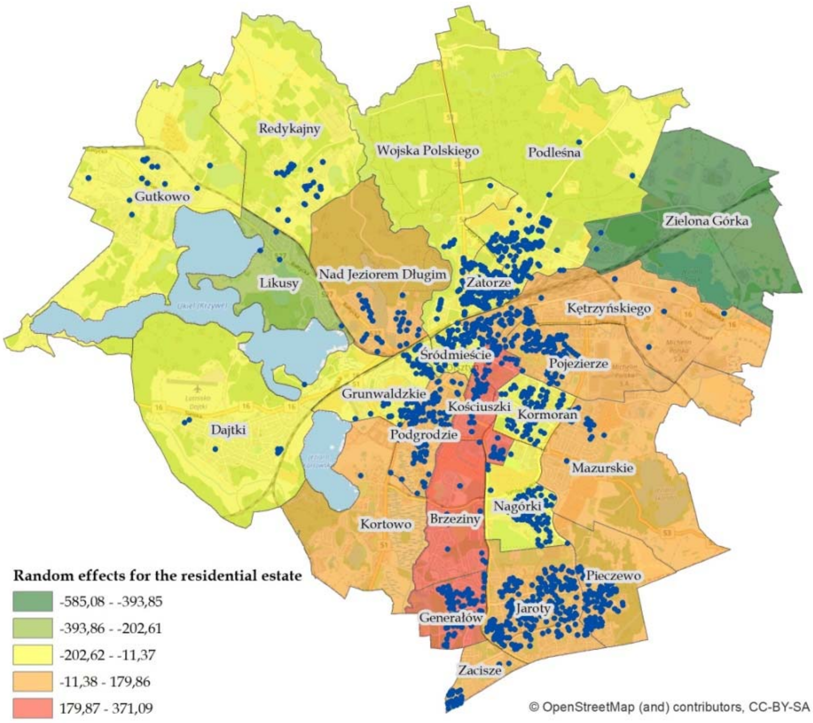

Fig. 3. Random effects for residential estates as grouping variables obtained from the mixed model taking into account interactions between variables. *Source*: own study.

In the case presented in Table 8, it can be observed that high and medium-height buildings have the highest effect on unit prices of residential premises, while the lowest effect is observed for low buildings.

Table 8

Random effects for the building height as a grouping variable obtained from the mixed model taking into account interactions between variables

Building height	Random effect	X_2
low	-114.021	4.026
average	-49.036	1.401
high	163.057	-7.165

Source: own study.

Additionally, the quality of the above-presented models was compared by applying the ANOVA function. The model taking into account grouping variables in the form of residential estates and the height of the building, and interactions between the safety assessment and floor location of the premises and the transaction date, demonstrated the highest degree of significance.

Table 9

Comparison of the quality of estimated models

Building height	Df	AIC	BIC	logLik	deviance	Chisq	Df	Pr(>Chisq)
Model2	11	87466	87539	-43722	87444			
Model3	13	87622	87708	-43798	87596	0.00	2	1
Model4	16	87451	87557	-43710	87419	176.75	3	<0.001

Source: own study.

4. Summary and conclusions

The paper, based on extensive literature, described a complex and relatively new linear model with random effects, along with its components, and put forward methodological arguments for applying this method, particularly in the context of the property market. For the analysis of the property market in Olsztyn, both the classical model and the models with random effects were used. On the basis of the above study, it can be concluded that the mixed model proves superior to the classical model in view of its quality as well as convenience in the aspects of time and technical issues.

The analyses conducted in this study also confirmed the possibility of taking into account the hierarchy of location in the given residential estate and the build-up density in regression modelling, and the estimated random effects, at the same time, provide information on their effect on price variability in individual groups. In view of the above, mixed models applied for the examination of transaction prices of properties provide a highly effective tool both for developing the value map of residential premises and for determining the relationships between variables. The advantages of statistically correct models, the possibilities of using them as a base to develop maps of dwelling or land values, and their constantly developing methodology lead to the increased use of the above models in the field of property markets. Additionally, in view of the promising results obtained, their possible application in other types of markets is suggested.

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