

## Colourability and word-representability of near-triangulations

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**Abstract.** A graph  $G = (V, E)$  is word-representable if there is a word  $w$  over the alphabet  $V$  such that  $x$  and  $y$  alternate in  $w$  if and only if the edge  $(x, y)$  is in  $G$ . It is known [6] that all 3-colourable graphs are word-representable, while among those with a higher chromatic number some are word-representable while others are not.

There has been some recent research on the word-representability of polyomino triangulations. Akrobotu et al. [1] showed that a triangulation of a convex polyomino is word-representable if and only if it is 3-colourable; and Glen and Kitaev [5] extended this result to the case of a rectangular polyomino triangulation when a single domino tile is allowed.

It was shown in [4] that a near-triangulation is 3-colourable if and only if it is internally even. This paper provides a much shorter and more elegant proof of this fact, and also shows that near-triangulations are in fact a generalization of the polyomino triangulations studied in [1] and [5], and so we generalize the results of these two papers, and solve all open problems stated in [5].

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### 1 Introduction

Suppose that  $w$  is a word and  $x$  and  $y$  are two distinct letters in  $w$ . We say that  $x$  and  $y$  *alternate* in  $w$  if the deletion of all other letters from the word  $w$  results in either the pattern  $xyxy \cdots$  or  $yxyx \cdots$ .

A graph  $G = (V, E)$  is called *word-representable* if there exists a word  $w$  over the alphabet  $V$  such that letters  $x$  and  $y$  alternate in  $w$  if and only if  $(x, y)$  is an edge in  $E$ . For example, the cycle graph  $C_4$  on 4 vertices labelled by 1, 2, 3 and 4 consecutively can be represented by the word 14213243.

There is a long line of research on word-representable graphs, which is summarized in the book [7]. The roots of the theory of word-representable graphs are in the study of the celebrated Perkins semigroup [9] which has played a central role in semigroup theory since 1960, particularly as a source of examples and counterexamples. Two more papers [2, 3], closely related to this paper's study of word-representability of graphs with triangulated faces, have appeared recently.

A graph is  $k$ -colourable if its vertices can be coloured in at most  $k$  colours so that no pair of vertices with the same colour is connected by an edge.

**THEOREM 1.1** ([6]) *All 3-colourable graphs are word-representable.*

A *wheel*  $W_n$  is obtained from the cycle  $C_n$  by adding one universal vertex. See Figure 1 for  $W_5$ .

Note that, for  $k \geq 4$ , there are examples of non-word-representable graphs that are  $k$ -colourable, but not 3-colourable. For example, following from [8] all wheels  $W_n$  for odd  $n > 3$  are such graphs; in contrast, all even wheels are 3-colourable (and so word-representable). Note that  $W_3 = K_4$  is the complete graph on four vertices, and the only word-representable odd wheel.  $W_5$  is known to be the smallest non-word-representable graph.

A *near-triangulation* is a planar graph in which each inner bounded face is a triangle (where the outer face may possibly not be a triangle). In other words, a near-triangulation is a generalization of a triangulation that may have one untriangulated face. Let  $\mathcal{NT}_n$  denote the class of near-triangulations on  $n$  vertices, and let  $\mathcal{NT}_n^{even}$  denote members of  $\mathcal{NT}_n$  which are *internally even*, that is, each inner vertex (vertex not incident to the outer face) has even degree. See Figure 1 for examples of graphs in  $\mathcal{NT}_7$  and  $\mathcal{NT}_7^{even}$ . We let  $\mathcal{NT} = \bigcup_{n \geq 0} \mathcal{NT}_n$  and  $\mathcal{NT}^{even} = \bigcup_{n \geq 0} \mathcal{NT}_n^{even}$ . It is known [4] that a near-triangulation is 3-colourable if and only if it belongs to  $\mathcal{NT}^{even}$ . This result is of special importance in our paper, and we decided to provide a new and much more elegant proof of it (see Theorem 2.3).



Figure 1: From left to right: the wheel graph  $W_5$ , a simple near-triangulation in  $\mathcal{NT}_7$  and one in  $\mathcal{NT}_7^{even}$ . The vertex labelled  $v$  is an example of an inner vertex, and is the centre of a wheel  $W_4$ , and the vertex labelled  $w$  is an example of a boundary vertex.

The motivation for considering near-triangulations comes from a subclass of these graphs, called polyomino triangulations, which have been studied recently by Akrobotu et al. [1] and Glen and Kitaev [5], and which have had related work dealing with triangulations and face subdivisions in the recent papers [2, 3]. Polyomino triangulations will be defined in Section 3. Akrobotu et al. showed that a triangulation of a convex polyomino is word-representable if and only if it is 3-colourable, and Glen and Kitaev extended this result to the case of a rectangular polyomino triangulation with a single domino. This paper shows in Theorem 2.1 that a near-triangulation, that avoids the complete graph  $K_4$  as an induced subgraph, is word-representable if and only if it is 3-colourable, and so we obtain an elegant generalization of the previous results.

This paper is organized as follows. In Section 2 we look at near-triangulations, in particular proving Theorem 2.1 showing that a near-triangulation that avoids  $K_4$  is 3-colourable if and only if it is word-representable. Section 3 discusses polyomino triangulations as a specialization of near-triangulations, proving Theorem 3.3 as a corollary to Theorem 2.1. Finally, Section 4 concludes by stating a problem that is left open, namely that of the word-representability classification of our graphs that contain  $K_4$  as an induced subgraph.

## 2 Near-triangulations

The main result of this section is the following:

**THEOREM 2.1** *A  $K_4$ -free near-triangulation is 3-colourable if and only if it is word-representable.*

**LEMMA 2.2** *Let  $x$  be a vertex of odd degree in a graph  $G$ , and suppose the neighbours of  $x$  induce a path  $P$ . In any 3-colouring of  $G$ , the end-points of  $P$  must have the same colour.*

**Proof.** Let  $a$  and  $b$  denote the end-points of  $P$ . Without loss of generality and using colours from  $\{1, 2, 3\}$ , assume that  $x$  and  $a$  have colours 1 and 2, respectively. Starting from  $a$  and going along the path, the vertices must be coloured 2, 3, 2, 3, ... . Since the path is of even length,  $b$  must be coloured 2.  $\square$

The following theorem is proved in [4], but we provide an alternative (shorter) proof of it here.

**THEOREM 2.3** ([4]) *A graph  $G \in \mathcal{NT}$  is 3-colourable if and only if  $G \in \mathcal{NT}^{even}$ .*

**Proof.** First, if  $G \notin \mathcal{NT}^{even}$ , then it contains an odd wheel, so obviously it is not 3-colourable as odd wheels are not 3-colourable.

For the opposite direction, we proceed by induction on the number of vertices, with the trivial base case of the single vertex graph. We take a graph  $G \in \mathcal{NT}_n^{even}$ , remove a boundary vertex  $v$  from it, and colour the new graph, called  $G'$ , with three colours. We will then show that re-adding  $v$  preserves 3-colourability. Note that  $G' \in \mathcal{NT}_{n-1}^{even}$ . Using colours from  $\{1, 2, 3\}$ , it will be shown that in situations in which  $v$  has neighbours with all three colours, the graph can be recoloured in some way so that  $v$  does not require colour 4. If the neighbours of  $v$  are coloured with two colours then  $v$  can be coloured with the third colour and 3-colourability is preserved.

If the neighbours of  $v$  are coloured with three colours, then they can be recoloured with two colours as follows. There are two possible situations involving  $v$  and its neighbours: (i) there is a path  $P$  connecting all of the neighbors of  $v$ ; (ii)  $v$  is a cut-vertex, meaning that the neighbours of  $v$  in  $G'$  are in at least two different connected components.

For situation (i), going along  $P$  starting from one of its end-points, take the first instance in which there are 3 consecutive vertices with different colours (there will be such an instance because all three colours are involved in  $P$ ). Assume without loss of generality that they are coloured 1, 2, 3, and call the vertices  $v_1$ ,  $v_2$  and  $v_3$ , respectively. Vertex  $v_2$  must be at the boundary, otherwise Lemma 2.2 can be used to show that  $v_1$  and  $v_3$  must have the same colour (in  $G$ ,  $v_2$  must be the centre of an even wheel, so in  $G'$  it must have odd degree with a path connecting all its neighbours, with  $v_1$  and  $v_3$  being its end-points). Because  $v_2$  is at the boundary, it is a cut-vertex in  $G'$  meaning that removing it increases the number of connected components, each of which is a graph in  $\mathcal{NT}^{even}$ . Take the component containing  $v_3$  and recolour its vertices, swapping 1 and 3, keeping 2 the same. Thus the vertices  $v_1$ ,  $v_2$  and  $v_3$  are now coloured 1, 2, 1 respectively. One can continue going through  $P$ 's as yet unvisited vertices, looking for more examples of three consecutive vertices with three different colours; when detecting such vertices, apply the re-colouring argument again. Once all of  $P$ 's vertices are visited, they will be coloured in two colours, so  $v$  can be coloured with the third colour.

For situation (ii), the recolouring argument from (i) can be applied to each connected component of  $G'$  to guarantee that the neighbours of  $v$  have at most two colours and  $v$  can be coloured with the third.  $\square$

For the following theorem, recall that it follows from [8] that all odd wheels  $W_{2n+1}$  for  $n \geq 2$  are non-word-representable, while  $W_3 = K_4$  is word-representable.

**THEOREM 2.4** *A  $K_4$ -free graph  $G \in \mathcal{NT}$  is word-representable if and only if  $G \in \mathcal{NT}^{even}$ .*

**Proof.** If  $G \in \mathcal{NT}^{even}$ , then it follows from Theorems 1.1 and 2.3 that it is word-representable.

For the other direction, if  $G$  is word-representable then it cannot contain any odd wheel as an induced subgraph as such graphs are non-word-representable, so  $G \in \mathcal{NT}^{even}$ .  $\square$

Theorem 2.1 now follows from Theorems 2.3 and 2.4.

Another observation to make as a corollary to Theorems 2.1 and 2.3 is the following:

**COROLLARY 2.5** *A  $K_4$ -free near-triangulation is word-representable if and only if it is perfect.*

A graph is *perfect* if the chromatic number of each of its induced subgraphs is equal to the size of the largest clique in that subgraph.

**Proof.** Any graph  $\in \mathcal{NT}^{even}$  has a maximum clique size of 3, as it avoids  $K_4$  as an induced subgraph, and so since Theorem 2.3 states that they are 3-colourable, these graphs are perfect. Conversely, a near-triangulation  $\notin \mathcal{NT}^{even}$  has chromatic number 4, and if it avoids  $K_4$  then it has no clique of size 4, so it is not perfect. So a  $K_4$ -free near-triangulation is 3-colourable if and only if it is perfect.

From this and Theorem 2.1 we have the fact that a  $K_4$ -free near-triangulation is word-representable if and only if it is perfect.  $\square$

### 3 Triangulations of polyominoes

A *polyomino* is a plane geometric figure formed by joining one or more equal squares edge to edge. Letting corners of squares in a polyomino be vertices, we can treat polyominoes as graphs. In particular, well-known *grid graphs* are obtained from polyominoes in this way. We are interested in *triangulations* of a polyomino, that is, subdividing each square into two triangles; Figure 2 shows an example of a polyomino triangulation. Note that no triangulation is 2-colourable—at least three colours are needed to properly colour a triangulation, while four colours are always enough to colour any triangulation, as it is a planar graph well-known to be 4-colourable by the 4 Colour Theorem.

The main result of Akrobotu et al. [1], Theorem 3.1, is related to *convex polyominoes*. A polyomino is said to be *column convex* if its intersection with any vertical line is convex (in other words, each column has no holes); and similarly, a polyomino is said to be *row convex* if its intersection with any horizontal line is convex. Finally, a polyomino is said to be *convex* if it is row and column convex.

**THEOREM 3.1** ([1]) *A triangulation of a convex polyomino is word-representable if and only if it is 3-colourable.*

The main result of Glen and Kitaev [5], Theorem 3.2, is related to a variation of the problem involving *polyominoes with domino tiles*. Polyominoes are objects formed by  $1 \times 1$  tiles, so that the induced graphs in question have only (chordless) cycles of length 4. A generalization of such graphs is allowing domino ( $1 \times 2$  or  $2 \times 1$ ) tiles to be present in polyominoes, so that in the respective induced graphs (chordless) cycles of length 6 would be allowed (in a triangulation of a domino tile, it would be subdivided into four triangles).

**THEOREM 3.2** ([5]) *A triangulation of a rectangular polyomino with a single domino tile is word-representable if and only if it is 3-colourable.*

We consider a generalization of both results. As a generalization of the results of Akrobotu et al. we consider polyominoes without *internal holes*. An internal hole is defined as a gap in a polyomino which is bounded on all sides. Notice that this restriction is much more general than the restriction in Theorem 3.1 to convex polyominoes, but it still prohibits any chordless cycles of length 4 or greater. However, the counter-examples by Akrobotu et al. are unavoidable: that is, a graph  $G$  containing internal holes can result in word-representable graphs even when  $G$  is non-3-colourable. It turns out that the exact same arguments can be used as in [1] while replacing the convex restriction with the internal holes restriction to instantly obtain a more general result. Note in particular that the boundary shape of the polyomino is not important. See Figure 2 for an example of a non-convex polyomino triangulation not covered by [1], but that our general result does cover.

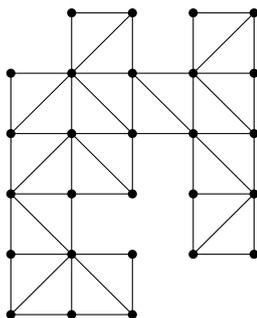


Figure 2: A non-convex polyomino triangulation.

As a generalization of Glen and Kitaev's result, we consider polyominoes in which any number of  $n$ -omino tiles are allowed. We call these shapes *polyominoes with  $n$ -omino tiles*. Additionally, we consider polyominoes that are not necessarily rectangular; as stated above, the boundary shape of the polyomino is unimportant. See Figure 3 for an example of a polyomino with  $n$ -omino tiles, without internal holes, and one of its triangulations.

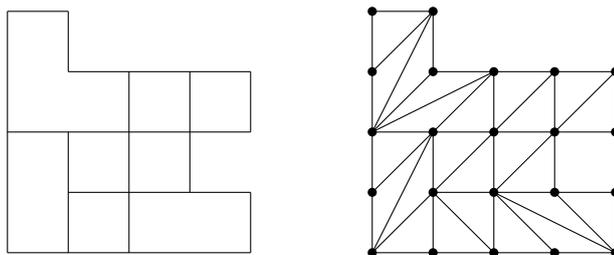


Figure 3: A polyomino with  $n$ -omino tiles (to the left) and one of its triangulations (to the right); notice that it is non-word-representable as it contains  $W_7$  as an induced subgraph.

The problem then is in finding a nice characterization of triangulations of such graphs that are word-representable, similar to Theorems 3.1 and 3.2. However, allowing arbitrary  $n$ -ominoes opens up the possibility of having  $W_3 = K_4$  as an induced subgraph, which as stated above is word-representable,

but it is not 3-colourable; see Figure 4 for the smallest such graph, containing a tromino and a square tile.

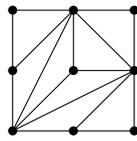


Figure 4: The smallest polyomino triangulation with  $n$ -omino tiles containing  $K_4$ .

Notice that the appearance of  $K_4$  here is because of the inner “bend” in the tromino, as such a bend is the only situation in a polyomino triangulation in which an inner vertex may have degree 3. From this it is clear that the result for polyominoes with  $n$ -omino tiles cannot be as elegant as the other results, as there exist non-3-colourable graphs that may or may not be word-representable. Therefore the main result of this section is the following:

**THEOREM 3.3** *A  $K_4$ -free triangulation of a polyomino with  $n$ -omino tiles and without internal holes is word-representable if and only if it is 3-colourable.*

It is clear to see that these types of polyomino triangulations are near-triangulations, with the requirement of no internal holes being equivalent to the requirement that every bounded face is a triangle. Theorem 3.3 therefore follows from Theorem 2.1.

## 4 Concluding remarks

It is still left as an open problem the word-representability classification of near-triangulations, and graphs in general, that contain  $K_4$  as an induced subgraph. As  $K_4$  is non-3-colourable, a very different approach from the one above involving colours is required to find an elegant classification; since planar graphs are always 4-colourable, a classification based on colourability is not possible.

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