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Forests and pattern-avoiding permutations modulo pure descents

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Abstract. We investigate an equivalence relation on permutations based on the pure descent statistic. Generating functions are given for the number of equivalence classes for the set of all permutations, and the sets of permutations avoiding exactly one pattern of length three. As a byproduct, we exhibit a permutation set in one-to-one correspondence with forests of ordered binary trees, which provides a new combinatorial class enumerated by the single-source directed animals on the square lattice. Furthermore, bivariate generating functions for these sets are given according to various statistics.

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1 Introduction, definitions and notations

Many statistics on permutations have been studied for many years, but two of them appear more frequently in the literature: the number of descents and the number of excedances. These two statistics were introduced by MacMahon [15] and are closely related since they have the same distribution. However, many more articles deal with the descents which have links with other fields such as Coxeter groups [6] or the theory of lattice paths [11].

Recently [3, 4], two equivalence relations on permutations based on the excedance and descent statistics were introduced. The main results of these works consist of giving generating functions for the number of equivalence classes for several restricted sets of permutations such as involutions, cycles,

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derangements, and permutations avoiding at most one pattern of length three. So, it becomes natural to conduct a similar study for equivalence relations based on other statistics. This paper investigates an equivalence relation based on the pure descent statistic, first introduced in [5] and formally defined below. Moreover, we show how these equivalence classes are in one-to-one correspondence with certain forests of ordered trees, providing some links between several statistics on these sets. As a consequence, we exhibit a new set of pattern-avoiding permutations with the same cardinality as the set of single-source directed animals on the square lattice (see Barcucci and al. [2], Bousquet-Mélou [9] for two studies concerning directed animals, forests and pattern-avoiding permutations).

Now, we present some basic definitions and notation. Let S_n be the set of permutations of length n, *i.e.*, all one-to-one correspondences from $[n] = \{1, 2, \ldots, n\}$ into itself. The *one-line notation* of a permutation $\pi \in S_n$ is $\pi_1\pi_2 \cdots \pi_n$ where $\pi_i = \pi(i)$ for $i \in [n]$. The *graphical representation* of $\pi \in S_n$ is the set of points in the plane at coordinates (i, π_i) for $i \in [n]$ (see Figure 1).

Let π be a permutation in S_n. A *descent* of π is an integer $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$. Whenever there does not exist $j < i$ such that $\pi_{i+1} < \pi_j < \pi_i$, we call it a *pure descent*. Let $D(\pi)$ be the set of pure descents in π, and $DD(\pi)$ be the set of pairs (π_i, π_{i+1}) for $i \in D(\pi)$. By abuse of language, such a pair will be also called a *pure descent*. For instance, if $\pi = 1\ 4\ 2\ 7\ 5\ 3\ 8\ 6$ then $D(\pi) = \{2, 4\}$ and $DD(\pi) = \{(4, 2), (7, 5)\}$. In [5, Theorem 1], the authors prove that the number of length n permutations with k pure descents is given by the signless Stirling number of the first kind $c(n, k+1)$ where $c(n, k)$ satisfies

$$c(n, k) = (n-1) \cdot c(n-1, k) + c(n-1, k-1)$$

with the initial conditions $c(n, k) = 0$ if $n \leq 0$ or $k \leq 0$, except $c(0, 0) = 1$ ([20, 22] and the sequence A132393 in the Sloane's On-line Encyclopedia of Integer Sequences [19]).

We define the following equivalence relation on permutations of length n:

$$\pi \sim \sigma \iff DD(\pi) = DD(\sigma).$$

The set of equivalence classes in S_n (resp. in a restricted set $R \subset S_n$) is denoted $S_n^\sim$ (resp. $R^\sim$). For instance, the permutations $\pi = 1\ 4\ 2\ 7\ 5\ 3\ 8\ 6$ and $\sigma = 1\ 7\ 5\ 6\ 4\ 2\ 3\ 8$ belong to the same equivalence class (see Figure 1) because $DD(\pi) = DD(\sigma) = \{(4, 2), (7, 5)\}$, and $S_3^\sim$ is constituted of the 5 classes $\{123, 231\}$, $\{132\}$, $\{213\}$, $\{321\}$ and $\{312\}$.

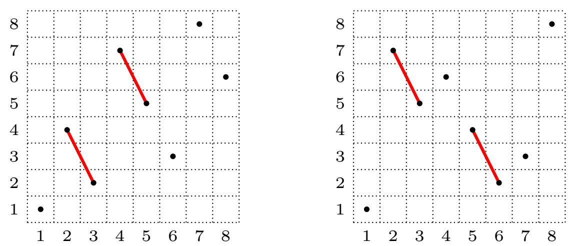

Figure 1: Two permutations $\pi = 1\ 4\ 2\ 7\ 5\ 3\ 8\ 6$ and $\sigma = 1\ 7\ 5\ 6\ 4\ 2\ 3\ 8$ in the same equivalence class of $S_8^\sim$ with $DD(\pi) = DD(\sigma) = \{(4, 2), (7, 5)\}$.

A permutation $\pi \in S_n$ *avoids* the pattern $\tau \in S_k$ if and only if there is no sequence of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ is order-isomorphic to τ (see [14, 18]). We denote by $S_n(\tau)$ the set of permutations of S_n avoiding the pattern τ. For example, if $\tau = 123$ then $52143 \in S_5(\tau)$ while $21534 \notin S_5(\tau)$. Many classical sequences in combinatorics appear as the cardinality of pattern-avoiding permutation sets. A large number of these results were firstly obtained in [7, 12, 14, 15, 18, 21]. Also, we refer to the books of Kitaev [13] and Bóna [8].

Later, Babson and Steingrímsson [1] defined *generalized patterns* (also called *vincular patterns*) where any pair of two adjacent values in the pattern may be underlined, which means that the corresponding values in the permutation must be adjacent. For instance, the pattern $2\underline{31}$ occurs in the permutation 316452 two times as the subsequences 352 and 452, while the pattern 231 occurs four times.

Moreover, we will consider a *barred pattern* $\bar{\tau}$, *i.e.*, a permutation in S_k having a bar over one value (see [17] and [21] for instance). Let τ be the permutation on $[k]$ identical to $\bar{\tau}$ but unbarred, and $\hat{\tau}$ be the permutation on $[k-1]$ made up of the $k-1$ unbarred values of $\bar{\tau}$ rewritten to be a permutation on $[k-1]$. Then $\pi \in S_n$ avoids the pattern $\bar{\tau}$ if and only if each pattern $\hat{\tau}$ in π can be expanded into a pattern τ in π where the expanded value corresponds to the barred value in $\bar{\tau}$. For instance, the permutation 3241 does not avoid $21\bar{3}$ since 41 cannot be expanded into a 213 pattern, while 3124 avoids it.

In the following, we will consider permutations avoiding the generalized and barred pattern $\underline{51}\bar{4}23$, consisting in permutations where any pattern $\underline{41}23$ can be expanded into a pattern $\underline{51}423$.

The main goal of this paper is to calculate the number of equivalence classes (modulo pure descents) for some subsets of permutations avoiding at most one pattern of length three. See Table 1 for an overview of our results.

The paper is organized as follows. In Section 2, we give a one-to-one correspondence between $S_n^\sim$ and the set of noncrossing partitions of $[n]$, proving that the cardinalities of $S_n^\sim$ for $n \geq 1$ are given by the Catalan numbers (see A000108 in the On-line Encyclopedia of Integer Sequences [19]). For the case of permutations avoiding the pattern 231, we prove that any equivalence class contains only one permutation on which we construct a forest of ordered trees. Also, we prove that $S_n(312)^\sim$ and $S_n(321)^\sim$ are enumerated by 2^{n-1} (A011782 in [19]).

In Section 3, we describe a bijection between forests of ordered binary trees with n nodes and the set $S_n(231, \underline{51}\bar{4}23)$, giving a new set of pattern-avoiding permutations in bijection with the single-source directed animals on the square lattice (see Barcucci et al. [2], and Bousquet-Mélou [9]). Bivariate generating functions are given for these sets according to various statistics.

In Section 4, we investigate the equivalence relation on the set $S_n(123)$ of permutations avoiding the pattern 123. We give a constructive bijection between forests of ordered binary trees and the classes in $S_n(123)^\sim$, proving that the cardinality of $S_n(123)^\sim$ is also given by the sequence A005773 that counts the single-source directed animals as above.

2 Enumeration of $S_n^\sim$, $S_n(231)^\sim$, $S_n(312)^\sim$ and $S_n(321)^\sim$

In this section, we provide the cardinality of $S_n^\sim$, $S_n(231)^\sim$, $S_n(312)^\sim$ and $S_n(321)^\sim$. Note that if a permutation π avoids the pattern 231 then any descent of π is a pure descent.

A *partition* Π of $[n]$ is a collection of non-empty pairwise disjoint subsets, called *blocks*, whose union

Pattern	Sequence	OEIS	$a_n, 1 \leq n \leq 9$
$\{\}, \{231\}$	Catalan	A000108	$1, 2, 5, 14, 42, 132, 429, 1430, 4862$
$\{312\}, \{321\}$	2^{n-1}	A011782	$1, 2, 4, 8, 16, 32, 64, 128, 256$
$\{231, \underline{51}\bar{4}23\}$	Directed animals	A005773	$1, 2, 5, 13, 35, 96, 267, 750, 2123$
$\{123\}$	Directed animals	A005773	$1, 2, 5, 13, 35, 96, 267, 750, 2123$

Table 1: Number of equivalence classes for the set of all permutations, and for some restricted sets of pattern-avoiding permutations.

is $[n]$ (see [16]). The *standard form* of Π is $\Pi_1/\Pi_2/\cdots$, where the blocks Π_i are arranged in increasing order of their smallest elements, and elements in a same block are in decreasing order. Let $\mathcal{P}_n$ be the set of partitions of $[n]$, and $\mathcal{NCP}_n \subset \mathcal{P}_n$ be the set of noncrossing partitions, *i.e.* all partitions Π where there do not exist four integers $p < q < r < s$ such that $p, r \in \Pi_i$ and $q, s \in \Pi_j$ with $i \neq j$.

We associate to a permutation $\pi \in S_n$ the unique partition Π defined as follows. Two elements x and y, $x > y$, belong to the same block in Π if and only if there exist i and j, $i < j$, such that the pairs $(x = \pi_i, \pi_{i+1}), (\pi_{i+1}, \pi_{i+2}), \ldots, (\pi_{j-1}, \pi_j = y)$ are pure descents in π. For instance, the two permutations in Figure 1 are associated to the same partition (in standard form) $\Pi = 1/42/3/75/6/8$. In fact, the associated partitions are always noncrossing partitions. Indeed, let us consider a block $\Pi_i = \pi_a\pi_{a+1}\cdots\pi_b$ with $a < b$, $\pi_a > \pi_{a+1} > \cdots > \pi_b$. Since $\pi_a\pi_{a+1}\cdots\pi_b$ is a subsequence of consecutive pure descents in π, there is no π_c, $c < a$, such that $\pi_c \in [\pi_b, \pi_a]$. So, let us assume that there is $c > b$ such that $\pi_c \in [\pi_b, \pi_a]$; then, for the same argument, all elements in the same block as π_c are greater than π_b and lower than π_a, which implies that there is no π_d in the block of π_c such that $\pi_d < \pi_b < \pi_c < \pi_a$ with $a < b < c < d$. *Mutatis mutandis*, there is no π_d in the block of π_c such that $\pi_b < \pi_c < \pi_a < \pi_d$ with $a < b < c < d$. Thus, the partition Π is noncrossing.

Conversely, any noncrossing partition Π of standard form $\Pi = \Pi_1/\Pi_2/\ldots/\Pi_k$, $k \geq 1$, is associated to the permutation $\pi = \Pi_1\Pi_2\cdots\Pi_k$ that avoids the pattern 231. Indeed, the noncrossing property forces all descents of π to be pure, implying that π does not contain any pattern 231. As the set $\mathcal{NCP}_n$ (and also $S_n(231)$) is enumerated by the nth Catalan number (see A000108, [19]), we obtain Theorem 2.1. As an immediate consequence, equivalence classes in $S_n(231)^\sim$ are singletons, and the set $S_n(231)$ is a set of representatives of $S_n^\sim$.

Theorem 2.1 *The sets $S_n^\sim$ (resp. $S_n(231)^\sim$), $n \geq 1$, are enumerated by the Catalan numbers.*

As a byproduct of Theorem 2.1, we obtain the cardinalities of $S_n(312)^\sim$ and $S_n(321)^\sim$. Since $S_n(231)$ is a set of representatives of $S_n^\sim$, there is a unique $\pi' \in S_n(231)$ equivalent to $\pi \in S_n(312)$, and π' is obtained from the noncrossing partition associated to π (in standard form) $\Pi = \Pi_1/\Pi_2/\ldots/\Pi_k$ by deleting all '/', *i.e.*, $\pi' = \Pi_1\Pi_2\cdots\Pi_k$. Notice that for any permutation $\pi \in S_n(312)$, a pure descent in π is necessarily an adjacency, *i.e.*, a descent (π_i, π_{i+1}) with $\pi_{i+1} = \pi_i - 1$. Then, any block Π_j, $1 \leq j \leq k$, is an interval, which implies that π' avoids also the pattern 312. So, the set $S_n(312)^\sim$ is in one-to-one correspondence with the set of $S_n(231, 312)$ which induces Theorem 2.2 (see Simion and Schmidt [18]). Theorem 2.3 is obtained *mutatis mutandis*.

Theorem 2.2 *The sets $S_n(312)^\sim$, $n \geq 1$, are enumerated by 2^{n-1}.*

Theorem 2.3 *The sets $S_n(321)^{\sim}$, $n \geq 1$, are enumerated by 2^{n-1}.*

3 Forests and 231-avoiding permutations

In this section, we establish a constructive bijection between $S_n(231)$ and the set $\mathcal{F}_n$ of forests of ordered trees, *i.e.* collections of rooted trees in which children of each node are ordered and the total number of nodes is n. Taking advantage of the recursive definition of the forests, we exhibit a new set of permutations $S_n(231, \underline{51}\bar{4}23)$ having the same cardinality as the set of single-source directed animals on the square lattice (see [2, 9] and A005773, [19]). Moreover, we show how the bijection transports various statistics (see Table 2). As a byproduct, we provide several bivariate generating functions with respect to the length and these statistics for the two sets $S_n(231)$ and $S_n(231, \underline{51}\bar{4}23)$ (see Theorems 3.1 and 3.2).

Let π be a permutation in $S_n(231)$. We construct a forest $f_\pi \in \mathcal{F}_n$ as follows: we cross the graphical representation of π from left to right; if the point (i, π_i) is a left-to-right maximum (that is $\pi_i > \pi_j$ for all $j < i$), then it corresponds to the root of a new tree in f_π; otherwise we add an edge between (i, π_i) and (j, π_j) where j is the rightmost $j < i$ such that $\pi_j > \pi_i$. See Figure 2 for an example of this construction. Notice that in [10], the authors have a different way of converting a permutation to a graph (not necessary a forest) based on ascents.

By construction, the map $\phi: S_n(231) \to \mathcal{F}_n$ defined by $\pi \mapsto f_\pi$ is injective. Since $\mathcal{F}_n$ is enumerated by the nth Catalan number as $S_n(231)$, we deduce the bijectivity of ϕ.

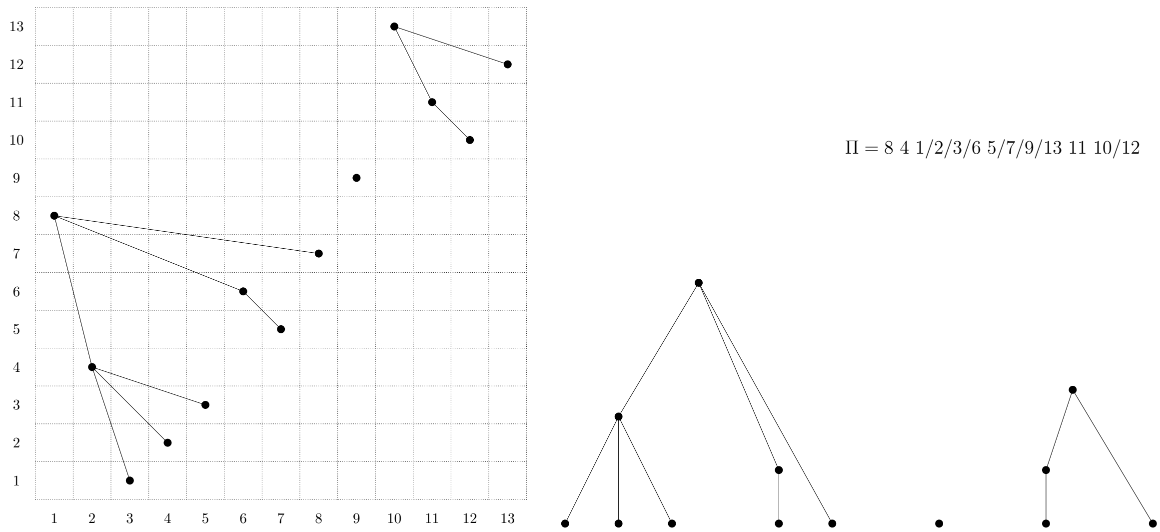

Figure 2: The permutation $\pi = 8\,4\,1\,2\,3\,6\,5\,7\,9\,13\,11\,10\,12$ with its corresponding forest f_π and the noncrossing partition $\Pi = 8\ 4\ 1/2/3/6\ 5/7/9/13\ 11\ 10/12$.

Now we define some statistics on $S_n(231)$ and $\mathcal{F}_n$, and we show how the map ϕ establishes a correspondence between them.

For a permutation $\pi \in S_n(231)$, we define:

• **des**(π) = *number of descents* (which is also the number of pure descents);
• **ides**(π) = *number of descents in* π^{-1} (for $\pi \in S_n(231)$, we have **ides**$(\pi) =$ **des**(π));
• **adj**(π) = *number of adjacencies*, *i.e.* descent (π_i, π_{i+1}) such that $\pi_{i+1} = \pi_i - 1$;
• **lrM**(π) = *number of left-to-right maxima*, *i.e.* $i \geq 1$ such that $\pi_i > \pi_j$ for all $j < i$;
• **rlm**(π) = *number of right-to-left minima*, *i.e.* $i \geq 1$ such that $\pi_i < \pi_j$ for all $j > i$;
• **inv**(π) = *number of inversions*, *i.e.* pairs (π_i, π_j) with $\pi_i > \pi_j$ and $i < j$,
• **lmax**(π) = *maximum value of the Lehmer code* $\ell_1\ell_2\cdots\ell_n$ of π, *i.e.* $\max_{1\leq i\leq n} \ell_i$ where $\ell_i = |\{\pi_j > \pi_i,\ j < i\}|$;
• **lsum**(π) = *sum of all values of the Lehmer code* of π.

For instance, if $\pi = 8\,4\,1\,2\,3\,6\,5\,7\,9\,13\,11\,10\,12$ is the permutation in Figure 2, then we have **des**$(\pi) = 5$, **ides**$(\pi) = 5$, **adj**$(\pi) = 2$, **lrM**$(\pi) = 3$, **rlm**$(\pi) = 8$, **inv**$(\pi) = 15$, **lmax**$(\pi) = 2$, and **lsum**$(\pi) = 15$. For a forest $f \in \mathcal{F}_n$, we define

• **ledg**(f) = *number of left edges*, *i.e.*, leftmost edges among its siblings;
• **redg**(f) = *number of right edges*, *i.e.*, rightmost edges among its siblings (**ledg**(f)=**redg**(f));
• **nod1**(f) = *number of nodes with only one child*;
• **ordt**(f) = *number of ordered trees*;
• **leav**(f) = *number of leaves*, *i.e.*, nodes without child;
• **vpat**(f) = *number of vertical paths* (a vertical path is a path between a node and one of its ancestors);
• **dept**(f) = *depth*, *i.e.*, the maximal length of a vertical path;
• **inpl**(f) = *internal path length*, *i.e.*, the sum of the lengths of all paths from a node to the root.

For instance, if f is the associated forest of the permutation in Figure 2, then we have **ledg**$(f) = 5$, **redg**$(f) = 5$, **nod1**$(f) = 2$, **ordt**$(f) = 3$, **leav**$(f) = 8$, **vpat**$(f) = 15$, **dept**$(f) = 2$, and **inpl**$(f) = 15$.

In the following, the notation **st** will be refer to one of these statistics on the sets $S_n(231)$ or $\mathcal{F}_n$. According to these definitions, it is straightforward to check that ϕ transports these statistics as related in Table 2.

$S_n(231)$	**des** = **ides**	**adj**	**lrM**	**rlm**	**inv**	**lmax**	**lsum**
$\mathcal{F}_n$	**ledg** = **redg**	**nod1**	**ordt**	**leav**	**vpat**	**dept**	**inpl**

Table 2: Correspondences of statistics by the bijection ϕ from $S_n(231)$ to $\mathcal{F}_n$.

Using the correspondence between these statistics and taking advantage of the recursive structure of a forest, we derive several bivariate generating functions for two sets of pattern-avoiding permutations with respect to the length and the statistics above.

Theorem 3.1 *Let $F(z, y)$ be the bivariate generating function where the coefficient of $z^n y^k$ is the number of permutations $\pi \in S_n(231)$ with* **st**$(\pi) = k$. *Then, we have:*

- *if* **st** *is* **des, ides, or lmax**, *then* $F(z,y) = \frac{1-z+zy-\sqrt{z^2y^2-2\,z^2y+z^2-2\,zy-2\,z+1}}{2zy}$,
- *if* **st** *is* **adj**, *then* $F(z,y) = \frac{1-zy+z-\sqrt{z^2y^2+2\,z^2y-3\,z^2-2\,zy-2\,z+1}}{2z}$,
- *if* **st** *is* **lrM**, *then* $F(z,y) = \frac{2}{2-y+y\sqrt{1-4z}}$,

- *if* **st** *is* **rlm**, *then* $F(z,y) = \frac{1+z-zy-\sqrt{z^2y^2-2z^2y+z^2-2zy-2z+1}}{2z}$.

Whenever **st** *is* **inv** *or* **lsum**, *the generating function satisfies the functional equation*

$$F(z,y) = \frac{1}{1 - z(F(zy,y)-1) - z}.$$

Proof. Since a forest $f \in \mathcal{F}_n$ is a collection of ordered trees, we have $F(z,y) = \frac{1}{1-T(z,y)}$ where $T(z,y)$ is the generating function for the number of ordered trees with respect to the length and the statistic **st**. Now, using the fact that a nonempty ordered tree is a node connected to the roots of the trees of a forest, we easily derive functional equations for each statistic **st**:

- for **ledg**, **redg** and **dept**, $T(z,y) = zy(F(z,y)-1) + z$;
- for **nod1**, $T(z,y) = zyT(z,y) + z(F(z,y) - T(z,y))$;
- for **ordt**, $T(z,y) = zyF(z,1)$;
- for **leav**, $T(z,y) = z(F(z,y)-1) + zy$;
- for **vpat** and **inpl**, $T(z,y) = z(F(zy,y)-1) + z$.

A simple calculation (using Maple for instance) completes the proof. □

THEOREM 3.2 *The sets* $S_n(231, \underline{51}\bar{4}23)$, $n \geq 1$, *are enumerated by the number of single-source directed animals on the square lattice (A005773, [19]). Let* $G(z,y)$ *be the bivariate generating function where the coefficient of* $z^n y^k$ *is the number of permutations* $\pi \in S_n(231, \underline{51}\bar{4}23)$ *with* $\mathbf{st}(\pi) = k$. *Then, we have:*

- *if* **st** *is* **des, ides, or lmax**, *then* $G(z,y) = \frac{2yz}{3yz-1+\sqrt{y^2z^2-4yz^2-2yz+1}}$,
- *if* **st** *is* **adj**, *then* $G(z,y) = \frac{2z}{2z-1+yz+\sqrt{y^2z^2-2yz-4z^2+1}}$,
- *if* **st** *is* **lrM**, *then* $G(z,y) = \frac{2z}{yz-y+2z+y\sqrt{-3z^2-2z+1}}$,
- *if* **st** *is* **rlm**, *then* $G(z,y) = \frac{2z}{3z-1+\sqrt{-4yz^2+z^2-2z+1}}$.

Whenever **st** *is* **inv** *or* **lsum**, *the generating function satisfies the functional equation*

$$\begin{cases} R(z,y) &= z + zR(yz,y) + zR(yz,y)^2 \\ G(z,y) &= \frac{1}{1-R(z,y)} \end{cases}.$$

Proof. Let $\mathcal{G}_n \subset \mathcal{F}_n$ be the set of forests of ordered binary trees, *i.e.*, ordered trees where each node has at most two children. Let us prove that we have $\phi^{-1}(\mathcal{G}_n) = S_n(231, \underline{51}\bar{4}23)$. Let π be a permutation in $S_n(231, \underline{51}\bar{4}23)$ and $f_\pi = \phi(\pi)$. Since π avoids $\underline{51}\bar{4}23$, any pattern $\underline{41}23$ can be expanded into a pattern $\underline{51}423$ which implies that the corresponding forest f_π does not contain a node with more than two children. Conversely, if the forest f_π belongs to $\mathcal{G}_n$, then the degree of any node of f is at most two. Let us suppose that π contains the pattern $\underline{41}23$ on $\pi_i\pi_{i+1}\pi_j\pi_k$, $i+1 < j < k$. If there does not exist ℓ, $i+1 < \ell < j$ such that $\pi_k < \pi_\ell < \pi_i$, by construction the forest f_π has the node π_i with at least three children π_{i+1}, π_j and π_k. So, any pattern $\underline{41}23$ in π can be expanded into a pattern $\underline{51}423$, which proves that π avoids $\underline{51}\bar{4}23$.

Let $G(z,y)$ be the generating function where the coefficient of $z^n y^k$ is the number of forests $f \in \mathcal{G}_n$ with $\mathbf{st}(f) = k$. Since a forest $f \in \mathcal{G}_n$ is a collection of the ordered binary trees, we have $G(z,y) = \frac{1}{1-R(z,y)}$ where $R(z,y)$ is the generating function for the number of ordered binary trees with respect to the length and the parameter **st**. Now, using the fact that a nonempty ordered binary tree is a node connected to the roots of at most two ordered binary trees, we can easily derive functional equations for each statistic **st**:

- for **ledg**, **redg** and **dept**, $R(z,y) = z + zyR(z,y) + zyR(z,y)^2$;
- for **nod1**, $R(z,y) = z + zyR(z,y) + zR(z,y)^2$;
- for **ordt**, $R(z,y) = zy + zyR(z,1) + zyR(z,1)^2$;
- for **leav**, $R(z,y) = zy + zR(z,y) + zR(z,y)^2$;
- for **vpat** and **inpl**, $R(z,y) = z + zR(z,y) + zR(z,y)^2$.

A simple calculation (using Maple for instance) completes the proof. □

Notice that for **inv** and **lsum**, functional equations provide generating functions as continued fractions instead of closed forms.

4 Enumeration of $S_n(123)^\sim$

In this section we prove that the set $S_n(123)^\sim$ is enumerated by the number of single-source directed animals on the square lattice (A005773, [19]). To achieve this, we construct a bijection between $S_n(123)^\sim$ and the set of forests of ordered binary trees, *i.e.*, trees where nodes have at most two ordered children (if a node has only one child then the corresponding link is called 0-*edge* the corresponding link, and if a node has two children then the two corresponding links are called 0-*edge* and 1-*edge*, which defines an order on siblings).

A *run* of pure descents (also called *run* for short) in $\pi = \pi_1 \cdots \pi_n \in S_n$ is a maximal subsequence $\pi_i\pi_{i+1}\cdots\pi_j$, $1 \le i \le j \le n$, of successive pure descents, *i.e.* (π_k, π_{k+1}) is a pure descent for $i \le k \le j-1$, and the two pairs $(\pi_{i-1}\pi_i)$, $(\pi_j\pi_{j+1})$ are not pure descents (a run contains at least one entry, that is π_i). To any run R of $\pi \in S_n$, we associate the interval $I(R) = [a,b] \subseteq [n]$ where a and b are the extremities of R, that is $a = \min R$ and $b = \max R$.

In a permutation $\pi \in S_n(123)$, there do not exist three runs R, S and T such that $I(S) \subset I(R) \subset I(T)$ (otherwise a pattern 123 would be created on the three entries $\min T, \min R$ and $\min S$). So, whenever there are two runs S, R such that $I(S) \subset I(R)$, we will say that S is a *secondary run*, and S appears necessarily in π at the right of R. A run R that is not secondary will be called *primary*. The family of intervals $I(R)$ associated to the primary runs of $\pi \in S_n(123)$ forms a partition of $[n]$. We denote by $p \ge 1$ the cardinality of this partition, and let I_i, $1 \le i \le p$, be the ith interval (considered in decreasing order), and let P_i be its associated primary run ($I(P_i) = I_i$). For $1 \le i \le p$, let L_i be the restriction of π to the interval I_i. It can be decomposed as $L_i = P_i S_i^1 S_i^2 \cdots S_i^{s_i}$ where P_i is the ith primary run of π, S_i^j is the jth secondary run of the interval I_i and s_i is the number of secondary runs in L_i.

We say there is a *break* between two consecutive secondary runs S_i^j and S_i^{j+1} in L_i if $\min S_i^j = 1 + \max S_i^{j+1}$. We refer to Figure 3 for an illustration of such a decomposition.

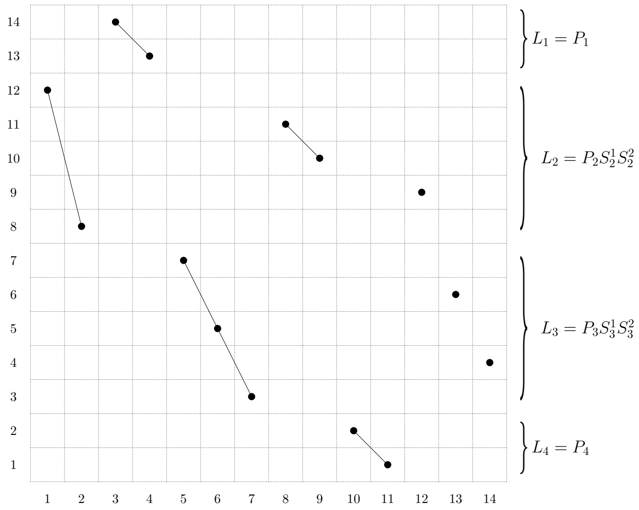

Figure 3: Illustration of the decomposition into runs of the permutation π = 12 8 14 13 7 5 3 11 10 2 1 9 6 4; P_1 = 14 13, P_2 = 12 8, S_2^1 = 11 10, S_2^2 = 9, P_3 = 7 5 3, $S_3^1 = 6$, $S_3^2 = 4$, $P_4 = 2$ 1. A break occurs between the two consecutive secondary runs S_2^1 and S_2^2.

With the above definitions we have Lemma 4.1.

LEMMA 4.1 *Let $\pi \in S_n(123)$ and $1 \leq i \leq p$. If there is a break between two consecutive secondary runs S_i^j and S_i^{j+1} in L_i, then there exists a unique primary run P_k between S_i^j and S_i^{j+1} in the one-line notation of π, and we necessarily have $k > i$.*

Proof. If there is a break between S_i^j and S_i^{j+1}, then we have $\min S_i^j = 1 + \max S_i^{j+1}$ and the pair $(\min S_i^j, \max S_i^{j+1})$ is not a pure descent. So, there exists an entry x of π between S_i^j and S_i^{j+1}, *i.e.* S_i^j and S_i^{j+1} are not contiguous. As π avoids 123, x is necessarily less than $\min P_i$ where P_i is the primary run of L_i, and it does not belong to a secondary run. For a contradiction, let us assume that there are two entries x and y, $x > y$, between S_i^j and S_i^{j+1}, that do not belong to the same primary run (we take y maximal such that $x > y$). Obviously, x is on the left of y, otherwise it would create a 123 pattern on the entries $yx \min S_i^{j+1}$. Let P_k (resp. P_ℓ, $\ell > k$) be the primary run that contains x (resp. y). Since $x > y$ and P_k is on the left of P_ℓ, the two primary runs are not contiguous, and there exists a value z between P_k and P_ℓ. The maximality of y implies that z is either below P_ℓ or above P_k, which creates a 123 pattern in both cases. Thus, we obtain the desired contradiction. □

Lemma 4.1 allows us to define an injective map α from the set $\mathcal{B}$ of breaks to the set $\mathcal{P}$ of primary runs, where the image of a break under α is the unique primary run defined in Lemma 1. Moreover, it is easy to check that the map α is increasing, *i.e.*, if $B_1, B_2, \ldots, B_r$ are the breaks of $\mathcal{B}$ ordered in decreasing order (from top to bottom in the graphical representation of π), then the two primary runs $P_k = \alpha(B_i)$ and $P_\ell = \alpha(B_j)$, $1 \leq i < j \leq r$, satisfy $k < \ell$ (*i.e.*, $P_k > P_\ell$, which means that P_k is above P_ℓ in the graphical representation of π). The existence of this increasing map α allows us to define another increasing map β (possibly equal to α) from $\mathcal{B}$ to $\mathcal{P}$:

- $\beta(B_1)$ is the highest primary run below B_1 (it always exists since $\alpha(B_1)$ is below B_1).
- Let us assume that β is defined on $U_i = \{B_1, \ldots, B_i\}$, $i \geq 1$, and β is increasing such that $\beta(B_j) \leq \alpha(B_j)$ for $1 \leq j \leq i$. Setting $V_i = \beta(U_i)$, we define $\beta(B_{i+1})$ by the highest primary run below B_{i+1} that does not lie in V_i (it always exists since $\alpha(B_{i+1})$ is below B_{i+1} and $\alpha(B_{i+1})$ cannot lie in V_i since $\alpha(B_{i+1})$ is below $\alpha(B_i)$ and thus, also below $\beta(B_i)$).

A crucial property of β is that it depends only on the set of primary and secondary runs, which means that two permutations in the same equivalence class provide the same map β.

Using the map β, we construct a forest $\chi(\pi)$ of ordered binary trees from the graphical representation of π by adding 0-edges and 1-edges between some entries of π using the following process.
(i) If (π_i, π_{i+1}) is a pure descent, then we add a 0-edge between π_i and π_{i+1}.
(ii) If there is break between S_i^j and S_i^{j+1}, then we add a 0-edge between $\min S_i^j$ and $\max S_i^{j+1}$.
(iii) If S_i^j is a secondary run and there is no break just before S_i^j, then we add a 1-edge between x and $\max S_i^{j+1}$ where x is the smallest entry greater than $\max S_i^{j+1}$ in the primary run P_i.
(iv) If there is a break B between S_i^j and S_i^{j+1}, then we add a 1-edge between $\max S_i^j$ and $\max \beta(B)$.
At the end of this process, we read the different connected components (rotated clockwise by $\frac{\pi}{4}$) from top to bottom, and we draw the corresponding trees so that any 0-edge points to the left child and 1-edge points to the right child. See Figure 4 for an illustration of this construction. Black line (resp. blue dash-dotted line, red dotted line, green dashed line) edges come from (i) (resp. (ii), (iii), (iv)). In what follows, an edge e in $\chi(\pi)$ will be denoted (a, b) where a and b are the extremities of e such that a is the parent of b.

Since any node in $\chi(\pi)$ has at most two children, $\chi(\pi)$ is a forest of ordered binary trees. Let T be a binary tree of $\chi(\pi)$. For any node $v \in T$ we denote by $r(v)$ (resp. $l(v)$) the number of 1-edges (resp. 0-edges) in the path connecting the root of T with v. We say that a node v is isolated when it has no siblings.

Remark 4.2 Let $e = (a, b)$ be a 0-edge in $\chi(\pi)$.

- (a, b) is a pure descent in a primary run of π if and only if $r(a)$ is even.
- (a, b) is a pure descent in a secondary run of π if and only if a is isolated and $r(a)$ is odd.
- There is a break between a and b if and only if a is not isolated and $r(a)$ is odd.

Let $\mathrm{rath}(v)$ be the binary word consisting of edge labels in the path from the root to v. Using a lexicographical order over such binary words (e.g. $101 > 011$, $1 > 01$), we define a total order on the set of nodes V in $\chi(\pi)$. For $a, b \in V$, we set

$$a < b \Longleftrightarrow \begin{cases} \text{either } a \text{ belongs to a tree before that of } b \text{ in the forest } \chi(\pi), \\ \text{or } \begin{cases} r(a) < r(b) \text{ or} \\ r(a) = r(b) \text{ and } \mathrm{rath}(a) > \mathrm{rath}(b) \end{cases} \end{cases} \qquad (\bigstar)$$

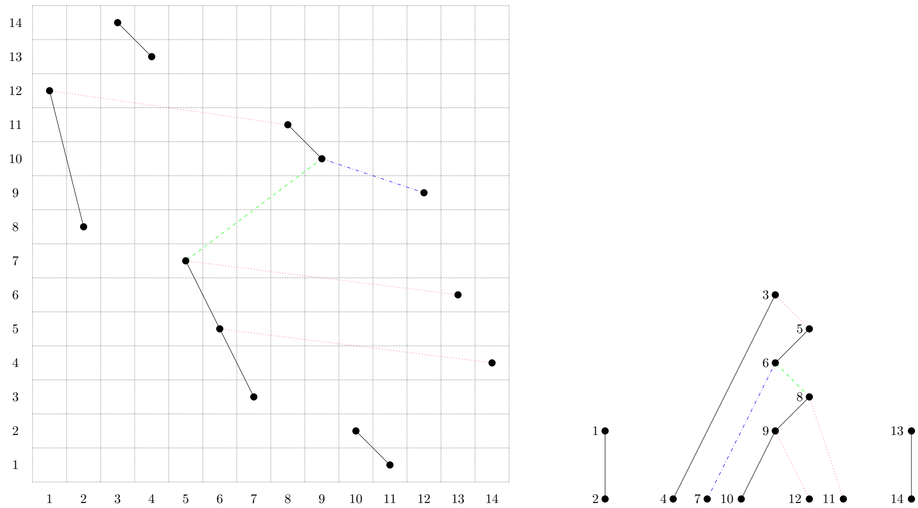

Figure 4: The permutation $\pi = 12\,8\,14\,13\,7\,5\,3\,11\,10\,2\,1\,9\,6\,4$ and the corresponding forest with nodes labeled using (★) order relation.

where $r(a)$ and is defined before Remark 4.2. We extend this order for paths $v_1v_2\cdots$ where v_i and v_{i+1} are nodes of $\chi(\pi)$ connected by a 0-edge: two disjoint paths $v_1v_2v_3\cdots$ and $u_1u_2u_3\cdots$ are compared by their heads, i.e.

$$(v_1v_2v_3\cdots) < (u_1u_2u_3\cdots) \iff v_1 < u_1.$$

Extracting from the forest $\chi(\pi)$ certain subsets of disjoint paths and taking into account the above order relation, we obtain the following.

Remark 4.3 The three statements hold:

- The ith primary run P_i in π (ordered from the top) corresponds to the ith maximal path $v_1v_2v_3\cdots$ of consecutive nodes joined by 0-edges in $\chi(\pi)$ where $r(v_1)$ is even.
- The ith secondary run in π (ordered from the top) corresponds to the ith maximal path $v_1v_2v_3\cdots$ of consecutive isolated nodes joined by 0-edges in $\chi(\pi)$ where $r(v_1)$ is odd.
- The ith break in π (ordered from the top) corresponds to the the ith 0-edge (a, b) in $\chi(\pi)$ such that $r(a)$ is odd and a is isolated.

Consequence Let (a, b) be a pure descent in a primary run of π, and $e = (a, b)$ its associated 0-edge in $\chi(\pi)$. Then, the number of entries of π in the interval (b, a), *i.e.* $a - b - 1$, is equal to the number of nodes in the maximal path of 0-edges starting on the right child of a.

Proposition 4.4 *Let π and π' be two permutations in $S_n(123)$.*
- If π and π' belong to the same equivalence class, then $\chi(\pi) = \chi(\sigma)$.
- If π and π' belong to different equivalence classes, then $\chi(\pi) \neq \chi(\sigma)$.

Proof. As we have seen above, the map β depends only on the set of primary and secondary runs. Thus, our construction applied on two permutations lying in the same class provides the same forest.

Moreover, if two permutations π and π' do not belong to the same class then their sets of primary and secondary runs necessarily differ. Due to the statements of Remark 2 and the above consequence, we deduce easily that $\chi(\pi)$ and $\chi(\pi')$ are different. □

THEOREM 4.5 *The sets $S_n(123)^{\sim}$, $n \geq 1$, are enumerated by the numbers of single-source directed animals on the square lattice (A005773, [19]).*

Proof. Proposition 1 proves that χ is injective. So, it suffices to show the surjectivity of χ, *i.e.*, any forest of ordered binary trees is the image by χ of a permutation avoiding 123.

First, we prove that any binary tree can be obtained from a permutation $\pi \in S_n(123)$ by the above construction. Let T be a binary tree with n nodes. By Remark 2, a maximal path P of nodes connected by 0-edges in T such that $r(P)$ is even corresponds to a primary run of π. Moreover, if a 0-edge e corresponds to a pure descent (a, b) in a primary run of π, $b(e) = a - b - 1$ is the number of nodes in the maximal path (possibly reduced to one node) of 0-edges starting on the right child of a.

Then the primary runs of π are entirely determined by the sequence $b_1 b_2 \cdots b_k$ with $b_1 = n$ and $b_i = b(e)$ where e is the ith 0-edge of T (using the ($\bigstar$) order relation) such that $r(e)$ is even. If p is the number of primary runs and P_i is the ith primary run of π then the sequence $P_1 P_2, \ldots P_p$ is decreasing.

Consequently, secondary runs of π take values from $[n] \setminus \cup_i^p P_i$, and the breaks correspond to the 0-edges $e = (a, b)$ where a is non-isolated and $r(a)$ is odd, which entirely determines secondary runs and breaks. If q is the number of secondary runs and S_i is the ith secondary run (using the ($\bigstar$) order relation) then the sequence $S_1 S_2 \cdots S_q$ is decreasing.

Now we construct a permutation π avoiding 123 by a shuffle of the two decreasing sequences $P_1 P_2 \cdots P_p$ and $S_1 S_2 \cdots S_q$. We read $P_1 S_1 S_2 \cdots S_q$ from left to right, and whenever we meet a break between S_i and S_{i+1} we insert between them the first primary run not yet inserted (this is exactly the correspondence given by the increasing map β defined above). Obviously, the sequence obtained at the end of the process is a permutation avoiding the pattern 123 since it is a shuffle of two decreasing sequences. Finally, the image of π by χ provides the tree T, which means that $\chi(S_n(123))$ contains the set of all ordered binary trees of size n.

So, let us assume that f is a forest of ordered binary trees $T_1, T_2, \ldots, T_k$. For $1 \leq i \leq k$, we construct the permutation π_i from the tree T_i by the previous process, *i.e.*, $\pi_i = \chi^{-1}(T_i)$. Let π be the permutation obtained by the skew sum $\pi_1 \ominus \pi_2 \ominus \ldots \ominus \pi_k$ where $\pi \ominus \pi'$ is the permutation σ such that

$$\sigma(i) = \begin{cases} \pi(i) + m' & \text{for } 1 \leq i \leq m, \\ \pi'(i - m) & \text{for } m + 1 \leq i \leq m + m' \end{cases}$$

where m (resp. m') is the length of π (resp. π').

Now we read $\pi_1 \ominus \pi_2 \ominus \ldots \ominus \pi_k$ from left to right. Whenever a pure descent is created between π_i and π_{i+1}, it is easy to see that π_i is necessarily a decreasing sequence. In this case we permute π_i and the first primary run of π_{i+1}. At the end of the process, the permutation π satisfies $\chi(\pi) = f$. □

For instance, the previous construction applied on the forest illustrated in Figure 4 provides the permutation $\pi = 12\,8\,14\,13\,11\,10\,7\,5\,3\,9\,6\,4\,2\,1$. Indeed, we have $\pi_1 = \chi^{-1}(T_1) = 2\,1$, $\pi_2 = \chi^{-1}(T_2) =$

10 6 9 8 5 3 1 7 4 2 and $\pi_3 = \chi^{-1}(T_3) = 2\,1$. Since a pure descent $(13, 12)$ is created in the permutation $\pi_1 \ominus \pi_2 \ominus \pi_3 = \mathbf{14\,13}\,12\,8\,11\,10\,7\,5\,3\,9\,6\,4\,\mathbf{2\,1}$, we permute the two primary runs $12\,8$ and $14\,13$, which gives $\pi = 12\,8\,\mathbf{14\,13}\,11\,10\,7\,5\,3\,9\,6\,4\,2\,1$.

5 Going further

We conclude this paper by giving several open questions and possible research directions.

We experimentally obtained the numbers of classes in $S_n(132)^\sim$ and $S_n(213)^\sim$ for small values of n, $1 \leq n \leq 9$. For $S_n(132)^\sim$, we obtain the sequence $1, 2, 4, 10, 26, 66, 169, 437, 1130$ and for $S_n(213)^\sim$, we obtain the sequence $1, 2, 4, 9, 22, 56, 146, 388, 1048$. The first sequence does not appear in [19], while the second sequence seems to be $A152225$ which corresponds to the number of Dyck paths of semilength n with no peaks at height $0 \mod 3$ and no valleys at height $2 \mod 3$. Is it possible to obtain the generating functions for these sets and to construct a bijection with Dyck paths?

In [2], the authors give a one-to-one correspondence between the set $\mathcal{F}_n$ of forests of ordered trees and the set $S_n(321, 4\bar{1}523)$ that transports various parameters. However, they do not give an interpretation for the number of inversions, the degree of the root less one and the internal-path-length. In Section 2, we exhibit a bijection between $\mathcal{F}_n$ and $S_n(231, \underline{51}\bar{4}23)$, which gives a new set of pattern-avoiding permutations enumerated as the single-source directed animals on the square lattice. This bijection has the advantage that it transports many parameters (see Section 3), and in particular the three previous parameters. Is it possible to give an interpretation of these parameters in term of the single-source directed animals?

In Section 4, we prove that $S_n(123)^\sim$ is enumerated by the number of directed animals (or equivalently directed polyominoes). Is it possible to give an interpretation of the equivalence relation in term of polyominoes?

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