On permutations avoiding 1243, 2134, and another 4-letter pattern

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Abstract. We enumerate permutations avoiding 1243, 2134, and a third 4-letter pattern \( \tau \), a step toward the goal of enumerating avoiders for all triples of 4-letter patterns. The enumeration is already known for all but three patterns \( \tau \), which are treated in this paper.

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1 Introduction

This paper is a companion to [2] which enumerates the permutations avoiding 1324, 2143, and a third 4-letter pattern \( \tau \), part of a project to enumerate avoiders for all triples of 4-letter patterns. As usual, \( S_n \) denotes the set of permutations of \( [n] = \{1, 2, \ldots, n\} \), considered as lists (words) of distinct letters. For a permutation \( \pi \) to avoid a pattern \( \tau \in S_k \) means that \( \pi \) contains no \( k \)-letter subsequence whose standardization (replace smallest letter by 1, second smallest by 2, and so on) is \( \tau \). For patterns \( \tau_1, \ldots, \tau_r \), \( S_n(\tau_1, \ldots, \tau_r) \) denotes the set of permutations of \( [n] \) that avoid each of \( \tau_1, \ldots, \tau_r \). Here, we count the set \( S_n(1243, 2134, \tau) \) for all 22 permutations \( \tau \in S_4 \setminus \{1243, 2134\} \) (for counting the set \( S_n(T) \) with \( T \subseteq S_4 \), see [1, 3, 4, 5, 6, 7, 8]). The three involutions reverse, complement, invert on permutations generate a dihedral group that divides pattern sets into so-called symmetry classes. All pattern sets in a symmetry class have the same counting sequence for their avoiders. The pattern sets with a given counting sequence form a Wilf class, by definition. We say a Wilf class is big if it contains more than one symmetry class. All 242 big Wilf classes of triples of 4-letter patterns are enumerated in [6]. Some small Wilf classes have been enumerated [2].

Table 1 below lists the generating function \( F_{1243,2134,\tau}(x) \) to count \( \{1243, 2134, \tau\} \)-avoiders for each of the 22 permutations \( \tau \). The 22 triples \( \{1243, 2134, \tau\} \) lie in precisely 11 Wilf classes, of which 3 are
big, hence covered by [6], 5 are small but can be counted by the INSENC algorithm (INSENC refers to regular insertion encodings, see [9]), and 3 are small and not treatable by the INSENC algorithm. These 3 are the triples with \( \tau = 3412, \tau = 2341 \) and \( \tau = 1423 \), which are treated in turn in Section 2. Our method is to consider the left-right maxima of an a\( \text{v} \)ider when \( \tau = 3412 \) and to focus on the initial letters in the other two cases. We use the usual left-right maxima decomposition of a nonempty permutation \( \pi \): \( \tau = i_{1}\pi^{(1)}i_{2}\pi^{(2)}\ldots i_{m}\pi^{(m)} \) where \( i_{1}, \ldots, i_{m} \) are letters, \( \pi^{(1)}, \ldots, \pi^{(m)} \) are words, \( i_{1} < i_{2} < \cdots < i_{m} \) and \( i_{j} > \max(\pi^{(j)}) \) for \( 1 \leq j \leq m \). Then \( i_{1}, i_{2}, \ldots, i_{m} \) are the left-right maxima of \( \pi \).

Throughout, \( C(x) = \frac{1 - \sqrt{1-4x}}{2x} \) denotes the generating function for the Catalan numbers \( C_{n} := \frac{1}{n+1}(\begin{pmatrix} 2n \\ n \end{pmatrix}) = (\begin{pmatrix} 2n \\ n \end{pmatrix}) - (\begin{pmatrix} 2n \\ n-1 \end{pmatrix}) \). The identity \( xc(x)^{2} = C(x) - 1 \) is used to simplify results.

<table>
<thead>
<tr>
<th>( \tau )</th>
<th>( F_{1243,2134,\tau}(x) )</th>
<th>Reference</th>
<th>Wilf class</th>
</tr>
</thead>
<tbody>
<tr>
<td>4321</td>
<td>(-9x^{2}+24x^{6}+23x^{7}+8x^{9}+2x^{7}+2x^{5}+2x^{4}-2x+1)</td>
<td>INSENC</td>
<td>7</td>
</tr>
<tr>
<td>3421, 4312</td>
<td>(-3x^{7}-5x^{6}+3x^{5}+10x^{4}+11x^{3}-5x^{2}+1)</td>
<td>INSENC</td>
<td>9</td>
</tr>
<tr>
<td>4231</td>
<td>(\frac{x^{5}+x^{3}-x^{2}}{(1-x)})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3412</td>
<td>(\frac{x^{10}-4x^{9}-6x^{8}-186x^{7}+291x^{6}+283x^{5}+170x^{4}+61x^{3}+12x-1}{(1-x)^{3}(1-2x)})</td>
<td>INSENC</td>
<td>53</td>
</tr>
<tr>
<td>1432, 3214</td>
<td>(\frac{1}{(1-x)^{2}+2x^{2}})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2341, 4123</td>
<td>(\frac{1-x^{2}+x^{3}}{(1-x)(1-2x+x^{4})})</td>
<td>Theorem 2.11</td>
<td>134</td>
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<tr>
<td>2413, 3142</td>
<td>(\frac{1-x^{3}}{(1-x)(1-2x^{2}+3x+1)(1-x^{3}+2x^{2}+2x+x^{4})})</td>
<td>INSENC</td>
<td>138</td>
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<td>1342, 1423, 2314, 3124</td>
<td>(\frac{1-x}{(1-x)^{2}(1-x^{3})})</td>
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<tr>
<td>2143</td>
<td>(\frac{1-x^{3}}{(1-x)^{2}(1-x^{2})})</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1234, 1324</td>
<td>(\frac{1-x^{2}}{2-9x+4x^{2}-x\sqrt{1-4x}})</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 1: Triples of 4-letter patterns containing 1243, 2134, divided into Wilf classes.

2 Proofs

2.1 Case 15: \( T = \{1243, 2134, 3412\} \).

We count \( T \)-avoiders by number of left-right maxima. Let \( G_{m}(x) \) denote the generating function for \( T \)-avoiders with exactly \( m \) left-right maxima. Clearly, \( G_{0}(x) = 1 \) and \( G_{1}(x) = xF_{T}(x) \).
Lemma 2.1 For $m \geq 4$, $G_m(x) = \frac{x^m(1+x)}{(1-x)^m}$.

Proof. Suppose $\pi = i_1\pi(1) \cdots i_m\pi(m) \in S_n(T)$ with $m \geq 4$ left-right maxima. Since $\pi$ avoids $T$, we have that $\pi^{(s)} = \emptyset$ for all $s = 1, 2, \ldots, m - 2$ and $\pi^{(m-1)}, \pi^{(m)} > i_2$. Moreover, $\pi^{(m-1)}\pi^{(m)}$ is decreasing. Thus, by considering whether $\pi^{(m)}$ has a letter between $i_1$ and $i_2$ or not, we obtain that $G_m(x) = \frac{x^m}{(1-x)^m} \left(1 + \frac{2x}{1-x}\right)$.

Lemma 2.2 We have $G_3(x) = \frac{x^3}{1-x+x^2+x^3}$.

Proof. Suppose $\pi = i_1\pi(1)i_2\pi(2)i_3\pi(3) \in S_n(T)$ with exactly 3 left-right maxima. Since $\pi$ avoids $T$, we have that $\pi^{(1)} = \emptyset$ and $\pi^{(2)} > i_2$. We consider four cases:

- $\pi^{(2)} = \emptyset$ and $\pi^{(3)}$ has no letter between $i_1$ and $i_2$: Since $\pi$ avoids $3412$, we can express $\pi$ as $\pi = i_1i_2\pi(2)i_3(i_1-1) \cdots 1$ where $\pi^{(2)}$ avoids $\{132, 213, 3412\}$. By a simple decomposition, we see that $K(x) = F_{\{132,213,3412\}}(x) = \frac{1}{1-x} + \frac{x}{(1-x)^2}$. Thus, we have a contribution of $\frac{x^3}{1-x}K(x)$.

- $\pi^{(2)} = \emptyset$ and $\pi^{(3)}$ has a letter between $i_1$ and $i_2$: Again, in this subcase, $\pi$ can be expressed as $\pi = i_1i_2\pi(2)i_3(i'-1) \cdots (i_1+1)(i_1-1) \cdots 21$ where $i_2 > \pi^{(2)} > i'$ and $\pi^{(2)}$ avoids $\{132, 213, 3412\}$. Thus, we have a contribution of $\frac{x^3}{1-x}K(x)$.

- $\pi^{(2)} \neq \emptyset$ and $\pi^{(3)}$ has no letter between $i_1$ and $i_2$: Similarly, in this subcase, $\pi$ can be written as $\pi = i_1i_2\pi(2)(i_1-1) \cdots (i_1+1)i_3(i'-1) \cdots 21$ where $\pi^{(2)}$ avoids $\{132, 213, 3412\}$. Thus, we have a contribution of $\frac{x^3}{1-x}K(x)$.

- $\pi^{(2)} \neq \emptyset$ and $\pi^{(3)}$ has a letter between $i_1$ and $i_2$: Since $\pi$ avoids $3412$, we can write $\pi$ as $i_1i_2(i_2-1) \cdots i_2(i_2-1) \cdots i_1(i_1-1)(i_1-2) \cdots i_3(i_3'-1) \cdots (i_1+1)(i_1+1) \cdots 21$. Thus, we have a contribution of $\frac{x^3}{1-x}K(x)$.

Hence, $G_3(x) = \frac{x^3}{1-x}K(x) + \frac{2x^4}{(1-x)^2}K(x) + \frac{x^5}{(1-x)^2}$, which simplifies to the stated expression.

Lemma 2.3 We have

$$G_2(x) = \frac{x^2(1 - 5x + 13x^2 - 16x^3 + 8x^4 - 7x^5 + 2x^6)}{(1 - x)^6(1 - 2x)}.$$

Proof. Let us write $G_2(x) = H(x) + J(x) + P(x)$, where $H(x)$ (respectively, $J(x)$ and $P(x)$) is the generating function for the number of $T$-avoiders $\pi$ with exactly 2 left-right maxima of form $\pi = (n-1)\pi'n\pi''$ (respectively, $\pi = \pi'\pi''$ with $i \leq n - 2$, and $\pi = i\pi'n\pi''$ with $i \leq n - 2$ and $\pi'$ is not empty).

First, we find $H(x)$. Let $\pi = (n-1)\pi'n\pi'' \in S_n(T)$ with exactly 2 left-right maxima and suppose $\pi'n$ has exactly $d \geq 1$ left-right maxima. Clearly, for $d = 1$, we have a contribution of $\frac{x^2}{1-x}$. For $d = 2$, we see that $\pi$ can be written as $\pi = (n-1)j_1\beta'n(n-2)(n-3) \cdots (j_1+1)\beta''$, where $\beta'$ is decreasing. Thus, by considering the two cases either $j_1 = n-2$ or $j_1 < n-2$, we have a contribution
of $xH(x) + \frac{x^4}{(1-x)(1-2x)}$. For $d \geq 3$, by Lemma 2.1 we obtain a contribution of $xG_d(x) = \frac{x^{d+1}(1+x)}{(1-x)^3}$. Hence,

$$H(x) = xH(x) + \frac{x^2}{1-x} + \frac{x^4}{(1-x)(1-2x)} + \sum_{d \geq 3} \frac{x^{d+1}(1+x)}{(1-x)^3},$$

which implies

$$H(x) = \frac{x^2(1-6x + 16x^2 - 22x^3 + 16x^4 - 8x^5 + x^6)}{(1-x)^6(1-2x)}.$$

For permutations in $S_n(T)$ with $n$ in position 2, we see by considering left-right maxima that their generating function is given by $H(x)$, while $|\{\pi \in S_n(T) : \pi_1 = n-1, \pi_2 = n\}| = 1$ for $n \geq 2$. Thus, $J(x) = H(x) - \frac{x^2}{1-x}$.

Next, write $P(x) = \sum_{d \geq 1} P_d(x)$, where $P_d(x)$ is the generating function for the number of $T$-avoiders $\pi$ with exactly $2$ left-right maxima and first letter $n-d-1$. Then $\pi = (n-d-1)j_1j_2 \cdots j_e \pi''$ with $j_1 > j_2 > \cdots > j_e$ and $e \geq 1$ (decreasing because $\pi$ avoids 1243 and $d \geq 1$). Write $\pi''$ as $\alpha^{(1)}(n-1) \cdots \alpha^{(d)}(n-d)\alpha^{(d+1)}$. Since $\pi$ avoids 3412, we see that $\alpha^{(1)}\alpha^{(2)} \cdots \alpha^{(d)}$ is decreasing.

- Case $d \geq 2$. Since $\pi$ avoids 2134, we see that $\alpha^{(1)} < j_e$. By considering whether $\alpha^{(1)}$ is empty or not, we have $P_d(x) = xP_{d-1}(x) + \frac{x^{d+4}}{(1-x)^{d+2}}$.

- Case $d = 1$. First, suppose that $\alpha^{(1)}$ is empty. In this case $\alpha^{(2)}$ is decreasing, so from the structure of $\pi$ we see that the contribution is given by $x^{e+3}/(1-x)^{e+1}$. Otherwise, $\alpha^{(1)}$ is not empty. So from the fact that $\alpha^{(1)}\alpha^{(2)}$ is decreasing we see that there two options: either $\alpha^{(1)} = \gamma'\gamma'$ with $\gamma > j_e > \gamma' > \alpha^{(2)}$ and $\gamma'\alpha^{(2)}$ is decreasing, or $\alpha^{(2)} = \gamma'\gamma'$ with $\alpha^{(1)} > \gamma > j_e > \gamma'$ and $\alpha^{(1)}\gamma'$ is decreasing. Each option gives a contribution of $x^{e+4}/(1-x)^3$. Thus,

$$P_1(x) = \frac{x^{e+3}}{(1-x)^{e+1}} + \frac{2x^{e+4}}{(1-x)^3},$$

and, summing over $e \geq 1$, we find that $P_1(x) = \frac{x^4(1-x-x^2-x^3)}{(1-x)^3(1-2x)}$.

Therefore, $P(x) - P_1(x) = xP(x) + \frac{x^6}{(1-x)^3(1-2x)}$, which gives $P(x) = \frac{x^4(1-x-2x^3)}{(1-x)^3(1-2x)}$. Hence, by adding $H(x)$, $J(x)$ and $P(x)$, we complete the proof.

Since $G_0(x) = 1$ and $G_1(x) = xF_T(x)$ and $F_T(x) = \sum_{d \geq 0} G_d(x)$, the preceding three lemmas imply

**Theorem 2.4** Let $T = \{1243, 2134, 3412\}$. Then

$$F_T(x) = \frac{1 - 8x + 28x^2 - 54x^3 + 65x^4 - 49x^5 + 18x^6 - 7x^7 + 2x^8}{(1-x)^7(1-2x)}.$$ 

**2.2 Case 134: $T = \{3421, 3214, 4312\}$.**

Here, $T$ is in the symmetry class of $\{1243, 2134, 2341\}$. Let $a(n; i_1, i_2, \ldots, i_m)$ be the number of permutations in $\pi = i_1i_2 \cdots i_m \pi' \in S_n(T)$ and $a_n = |S_n(T)|$. Thus $|S_n(T)| = \sum_{i=1}^n a(n; i)$. 

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Lemma 2.5 We have
\[ L(x) := \sum_{n \geq 3} a(n; n, 2)x^n = x \left( \frac{1 - 2x + 2x^3 + x^4}{(1 - 2x)(1 - x - x^2)} - 1 \right). \]

Proof. First we find the generating function \( A(x) = F_{\{312,2314,3421\}}(x) \). By symmetry, \( A(x) = F_{\{132,2341,2134\}}(x) \). For \( \pi \in S_n(132,2341,2134) \), by considering the position of \( n \), we obtain
\[ A(x) = 1 + xF_{\{132,213,2341\}}(x) + \frac{x}{1 - x}(A(x) - 1) \]
and
\[ F_{\{132,213,2341\}}(x) = 1 + \frac{x}{1 - x} + (x + x^2)(F_{\{\{132,213,2341\}}(x) - 1). \]
Thus,
\[ F_{\{312,3214,3421\}}(x) = \frac{1 - 2x + 2x^3 + x^4}{(1 - 2x)(1 - x - x^2)}, \quad F_{\{132,213,2341\}}(x) = \frac{1 - x + x^3}{(1 - x)(1 - x - x^2)}. \]

Note that \( \pi = n2\pi' \in S_n \) avoids \( T \) if and only if \( \pi' \) avoids \( \{312,3214,3421\} \). Thus, \( L(x) = x(A(x) - 1) \), which ends the proof.

Lemma 2.6 We have
\[ B(x, v) := \sum_{n \geq 3} \sum_{i=3}^{n-1} a(n; i, n)v^ix^n = \frac{x^3v^3}{1 - xv} \left( \frac{1 - 2x + 2x^3 + x^4}{(1 - 2x)(1 - x - x^2)} - 1 \right). \]

Proof. Let \( \pi = in\pi' \in S_n(T) \). Since \( \pi \) avoids 3421, we see that \( \pi \) contains the subsequence \( in12 \cdots (i - 1) \). Since \( \pi \) avoids 4312, there exists \( \pi'' \) such that \( \pi = in12 \cdots (i - 2)\pi'' \in S_n(T) \). Thus,
\[ a(n; i, n) = |S_{n-i}(312,3421,3214)| = |S_{n-i}(132,2341,2134)|, \]
which leads to \( B(x, v) = \frac{x^3v^3}{1 - xv} \sum_{n \geq 1} |S_{n}(132,2341,2134)|x^n \). Hence, by Lemma 2.5
\[ B(x, v) = \frac{x^3v^3}{1 - xv} \left( \frac{1 - 2x + 2x^3 + x^4}{(1 - 2x)(1 - x - x^2)} - 1 \right), \]
as required.

Lemma 2.7 We have
\[ K(x, v) := \sum_{n \geq 3} \sum_{i=3}^{n} a(n; i, 2)v^ix^n = \frac{x^2v^3}{1 - xv}L(x) + \frac{x^3v^3}{1 - xv}L(xv) \]
\[ + \frac{x^3v^3(3x^3v^3(1 - x)(1 - xv) + x^2v^2(1 - 3x) - xv(2 - 3x) + 1 - x)}{(1 - x)(1 - xv)^2(1 - 2xv)}, \]
where \( L(x) \) is given in Lemma 2.5.
Proof. Let $K'(x, v) = \sum_{n \geq 5} \sum_{i=4}^{n-3} a(n; i, 2)v^i x^n$. Let $\pi = i2\pi' \in S_n(T)$. Since $\pi$ avoids $3214$, we can write $\pi$ as $i2\alpha\beta$ such that $2 < \beta < i$. So $a(n; 3, 2) = |S_{n-3}(4312, 231, 3214)| = |S_{n-3}(132, 2134, 2341)|$; for $4 \leq i \leq n - 1$, we have that $a(n; i, 2) = a(n - 1, i - 1, 2) + 1$, and $a(n; n, 2)$ is given by Lemma 2.5.

So

$$\sum_{n \geq 5} \sum_{i=4}^{n-3} a(n; i, 2)v^i x^n = v \sum_{n \geq 5} \sum_{i=4}^{n-3} a(n - 1, i - 1, 2)v^{i-1} x^n + \sum_{n \geq 5} \sum_{i=4}^{n-3} v^i x^n,$$

which implies

$$K'(x, v) - \sum_{n \geq 5} a(n; n, 2)v^n x^n - v^3 \sum_{n \geq 5} a(n; 3, 2)x^n = vx \sum_{n \geq 4} \sum_{i=3}^{n-1} a(n; i, 2)v^i x^n + \frac{v^4 x^4}{(1-v)(1-x)} - \frac{v^4 x^4}{(1-v)(1-vx)}.$$ 

By Lemma 2.5, we have $\sum_{n \geq 5} a(n; 3, 2)v^3 x^n = x^2 v^3(L(x) - x^2)$ and

$$\sum_{n \geq 5} a(n; n, 2)v^n x^n = xv(L(xv) - x^2 v^2 - 2x^3 v^3),$$

so

$$K'(x, v) = xvL(xv) - x^2 v^2 - 2x^3 v^3 + x^2 v^3(L(x) - x^2) + vxK'(x, v) + v^3 x^4 - xv(L(xv) - x^2 v^2 - 2x^3 v^3) + \frac{v^4 x^5}{(1-x)(1-xv)}.$$ 

We have $K(x, v) = K'(x, v) + x^3 v^3 + x^4(v^3 + 2v^4)$, and the result follows.

Lemma 2.8 We have $\sum_{n \geq 2} a(n; n)x^n = L(x)$, where $L(x)$ is given in Lemma 2.5.

Proof. Since $n\pi' \in S_n$ avoids $T$ if and only if $\pi'$ avoids $312, 3421, 3214$, the result follows from Lemma 2.5.

Lemma 2.9 Let $3 \leq i \leq n - 1$. Then

$$a(n; i) = a(n; i, 1) + a(n; i, 2) + a(n; i, n) + \sum_{j=i+1}^{n} a(n; i, j).$$

Proof. Let $\pi = ijn' \in S_n(T)$ with $3 \leq j < i \leq n - 1$. Since $\pi$ avoids $4312$, we see that $\pi$ contains the subsequence $ij21$. Since $\pi$ avoids $3214$, we see that $\pi$ contains the subsequence $ijn21$, and $jn21$ is order isomorphic to $3421$. Thus $a(n; i, j) = 0$ for all $j$ with $3 \leq j < i \leq n - 1$, and the lemma follows.

Lemma 2.10 Let $3 \leq i < j \leq n - 1$. Then

$$a(n; i, j) = a(n - 1; i - 1, j - 1) + a(n - 1; j - 1) - a(n - 1; j - 1, 1) - a(n - 1; j - 1, 2).$$
Proof. Let $\pi = ij\pi' \in S_n(T)$ with $3 \leq i < j \leq n - 1$. By considering the third letter in $\pi$, we see that

$$a(n; i, j) = a(n - 1; i - 1, j - 1) + a(n - 1; j - 1, i) + a(n - 1; j - 1, j) + \cdots + a(n - 1; j - 1, n - 1).$$

Note that

$$a(n - 1; j - 1) = \sum_{\ell = 1}^{n - 1} a(n - 1; j - 1, \ell) = a(n - 1; j - 1, 1) + a(n - 1; j - 1, 2) + \sum_{\ell = j}^{n - 1} a(n - 1; j - 1, \ell).$$

Therefore,

$$a(n; i, j) = a(n - 1; i - 1, j - 1) + a(n - 1; j - 1, i) + a(n - 1; j - 1, j) - a(n - 1; j - 1, 1) - a(n - 1; j - 1, 2),$$

as claimed. □

Theorem 2.11 Let $T = \{3421, 3214, 4312\}$. Then

$$F_T(x) = \frac{(1 - x)(1 - 2x + 2x^2)(1 - 2x + x^3 + x^5)C(x) - x(1 - 2x + x^3 + x^4 - 2x^5 + 2x^6)}{(1 - x)^3(1 - 2x)(1 - x - x^2)},$$

Proof. Note that $a(n; k, 1) = a(n - 1; k - 1)$ for $2 \leq k \leq n$ (a permutation $k1\pi' \in S_n$ avoids $T$ if and only if $k\pi'$ avoids $T$). This fact will be used repeatedly. Let $3 \leq i \leq n - 1$. Then

$$a(n; i) = (a(n; i, 1) + a(n; i, 2) + a(n; i, n)) = \sum_{j = i + 1}^{n - 1} a(n; i, j)$$

$$= \sum_{j = i + 1}^{n - 1} a(n - 1; i - 1, j - 1) + a(n - 1; j - 1, i) + a(n - 1; j - 1, j) - a(n - 1; j - 1, 1) - a(n - 1; j - 1, 2)$$

$$= \sum_{j = i}^{n - 2} a(n - 1; i - 1, j) + \sum_{j = i}^{n - 2} a(n - 1; j) - \sum_{j = i - 1}^{n - 3} a(n - 2; j) - \sum_{j = i}^{n - 2} a(n - 1; j, 2)$$

$$= a(n - 1; i - 1) - \sum_{j = i - 1}^{n - 3} a(n - 2; j) - \sum_{j = i}^{n - 2} a(n - 1; j, 2),$$

the first equality by Lemma 2.9, the second equality by Lemma 2.10, the third equality by reindexing and the fact that $a(n; k, 1) = a(n - 1; k - 1)$, and the last equality by Lemma 2.9 again.

By Lemma 2.6, we see that $a(n; i, n) = a(n - 1; i - 1, n - 1)$ for all $3 \leq i \leq n - 1$. The preceding identities thus simplify to

$$a(n; i) = a(n - 1; i - 1) + \sum_{j = i - 1}^{n - 2} a(n - 1; j) - \sum_{j = i - 2}^{n - 3} a(n - 2; j)$$

$$+ a(n; i, 2) - a(n - 1; i - 1, 2) - \sum_{j = i}^{n - 2} a(n - 1; j, 2).$$
Define $A_n(v) = \sum_{i=1}^n \lambda(n;i)v^{i-1}$ Thus $A_n(1) = |S_n(T)|$. Define $B_n(v) = \sum_{i=3}^n \lambda(n;i,2)v^i$ and $\ell_n = a(n;n)$. Note that $a(n;1) = a(n;2) = a(n-1)$, where $a(n) = |S_n(T)|$.

Multiplying the recurrence for $a(n;\ell)$ by $v^{-1}$ and summing over $\ell = 3,4,\ldots,n-1$, we obtain

$$\sum_{i=3}^n \lambda(n;i,2)v^i = \sum_{i=1}^n \lambda(n;i)v^{i-1} - \lambda(2,2)v^2 + v \sum_{i=1}^n \lambda(n;i)v^{i-1} - \lambda(1,1)v.$$ 

This equation for $A_n(v)$ can be solved by the kernel method, taking $v = C(x)$ and using the expressions for $L(x)$ and $K(x,v)$ from Lemmas 2.5 and 2.7. After simplification $A_n(1)$, which coincides with $F_T(x)$, agrees with the stated expression.

2.3 Case 207: $T = \{1243, 2134, 1423\}$.

Let $a(n; i_1, i_2, \ldots, i_m)$ be the number of permutations in $\pi = i_1i_2\cdots i_m \pi'$ in $S_n(T)$ and $a_n = |S_n(T)|$. Thus $|S_n(T)| = \sum_{i=1}^n a(n;i)$. 

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Lemma 2.12 We have

\[
a(n; i, j) = \begin{cases} 
  a(n - 1; i, j) + \sum_{k=1}^{j-1} a(n - 1; j, k), & 1 \leq j < i \leq n - 2, \\
  a(n - 1; i, j), & 1 \leq i < j \leq n - 2, \\
  \sum_{k=1}^{i} a(n - 1; k, n - 1), & 1 \leq i \leq j - 2 = n - 2, \\
  a(n - 1; i, n - 2) + \sum_{k=1}^{i} a(n - 1; k, n - 1), & 1 \leq i \leq j - 2 = n - 3,
\end{cases}
\]

with \( a(n; n) = a(n; n - 1) = a_{n-1}, \) \( a(n; i, 1) = 1 \) for all \( i = 2, 3, \ldots, n - 2, \) and \( a(n; n - 2, n - 1) = a(n; n - 2, n) = a_{n-2}. \)

Proof. It is not hard to check the initial conditions. Let \( 1 \leq j < i \leq n - 2, \) then

\[
a(n; i, j) = a(n; i, j, n) + \sum_{k=1}^{j-1} a(n; i, j, k)
\]

\[
= a(n - 1; i, j) + \sum_{k=1}^{j-1} a(n - 1; j, k)
\]

with \( a(n; i, 1) = 1 \) (by definitions). For \( 1 \leq i < j \leq n - 2, \) we have \( a(n; i, j) = a(n; i, j, j + 1) = a(n - 1, i, j). \)

For all \( 1 \leq i \leq j - 2 = n - 2, \) we have

\[
a(n; i, n) = a(n; i, n, 1) + \cdots + a(n; i, n, i - 1) + a(n; i, n, n - 1)
\]

\[
= a(n - 1; 1, n - 1) + \cdots + a(n - 1, i - 1, n - 1) + a(n - 1; i, n - 1).
\]

Similarly, for all \( 1 \leq i \leq j - 2 = n - 3, \)

\[
a(n; i, n - 1) = a(n - 1; i, n - 2) + a(n - 1; i, n - 1) + a(n - 1; i - 1, n - 1) + \cdots + a(n - 1; 1, n - 1),
\]

which completes the proof. \( \square \)

Corollary 2.13 Define \( b(n; i) = a(n; i, n) \) and \( c(n; i) = a(n; i, n - 1). \) Then \( b(n; i) = \sum_{j=1}^{i} b(n - 1; j) \) and \( c(n; i) = c(n - 1; i) + b(n; i) \) with \( b(n; n) = c(n; n - 1) = 0, \) \( b(n; n - 1) = b(n; n - 2) = a_{n-2} \) and \( c(n; n) = c(n; n - 2) = a_{n-2}. \)

Define \( B(n; v) = \sum_{i=1}^{n} a(n; i, i) v^{i-1} \) and \( C(n; v) = \sum_{i=1}^{n} a(n; i, i-1) v^{i-1}. \) By Corollary 2.13 we obtain

\[
B(n; v) = a_{n-2} v^{n-2} + a_{n-2} v^{n-3} + \frac{1}{1-v} (B(n-1; v) - v^{n-3} B(n-1; 1)),
\]

\[
C(n; v) = C(n-1; v) + a_{n-2} (v^{n-1} - v^{n-2}) - a_{n-3} v^{n-2} + B(n; v)
\]

with \( B(1; v) = C(1; v) = 0, \) \( B(2; v) = 1 \) and \( C(2; v) = v. \)

Define \( B(x, v) = \sum_{n \geq 1} B(n; v) x^n \) and \( C(x, v) = \sum_{n \geq 1} C(n; v) x^n. \) Note that \( F_T(x) = \sum_{n \geq 0} a_n x^n. \)

So the above recurrences can be formulated as

\[
(1 - \frac{x}{v(1-v)}) B(x/v, v) = \frac{x^2}{v^2} F_T(x) + \frac{x^2}{v^3} (F_T(x) - 1) - \frac{x}{v^3(1-v)} B(x; 1),
\]

\[
C(x, v) = x C(x; v) + x^2 (v - 1) F_T(xv) - \frac{x^3}{v} F_T(xv) + B(x; v).
\]
Using the kernel method with \( v = \frac{1}{C(x)} \), we obtain
\[
B(x; 1) = x^2F_T(x) + x^2C(x)(F_T(x) - 1).
\]
(1)

Then, by substituting \( v = 1 \) in the second equation, we obtain
\[
C(x; 1) = x^2F_T(x) + \frac{x^2}{1 - x}C(x)(F_T(x) - 1).
\]
(2)

**Lemma 2.14** For all \( 1 \leq j < i \leq n - 2 \), \( a(n; i, j) = b(n; j) \).

**Proof.** Clearly, \( b(n; 1) = b(n - 1; 1) \) for all \( n \geq 3 \). But \( b(2; 1) = 1 \), so \( b(n; 1) = 1 = a(n; i, 1) \) for all \( i = 2, 3, \ldots, n - 2 \). Assume by induction that \( a(n - 1; i, j) = b(n - 1; j) \) for all \( n - 3 \geq i > j \geq 1 \). Then by Lemma 2.12,
\[
a(n; i, j) = a(n - 1; i, j) + \sum_{k=1}^{j-1} a(n - 1; j, k) = b(n - 1; j) + \sum_{k=1}^{j-1} b(n - 1; k) = b(n; j).
\]

Now, we are ready to find an explicit formula for \( F_T(x) \). By Lemmas 2.12 and 2.14, we have
\[
a(n, i) = b(n; 1) + \cdots + b(n; i) + c(n; i) + a(n - 1; i) - b(n - 1; 1) - \cdots - b(n - 1; i)
\]
\[
= a(n - 1; i) + c(n + 1; i) - b(n, i)
\]
with \( a(n; n - 2) = b(n; 1) + \cdots + b(n; n - 2) + c(n; n - 2) = c(n + 1; n - 2) \) and \( a(n; n) = a(n; n - 1) = a_{n-1} \). Summing over \( i = 1, 2, \ldots, n - 3 \), we get that
\[
a_n = a_{n-1} + c_{n+1} - b_n
\]
with \( a_0 = a_1 = 1 \). Hence,
\[
F_T(x) = 1 + xF_T(x) + C(x; 1)/x - B(x; 1),
\]
Solving for \( F_T(x) \) and using (1) and (2), we obtain the following result. Recall that \( C(x) \) denotes the generating function for the Catalan numbers.

**Theorem 2.15** Let \( T = \{1243, 1423, 2134\} \). Then
\[
F_T(x) = \frac{1 - x(1 - x)C(x)}{(1 - x)(2 - C(x)) + x^2}.
\]

**References**


ON PERMUTATIONS AVOIDING 1243, 2134, AND ANOTHER 4-LETTER PATTERN


