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LIMIT THEOREMS FOR BIVARIATE GENERALISED ORDER STATISTICS IN STATIONARY GAUSSIAN SEQUENCES WITH RANDOM SAMPLE SIZES

Fatma Hashem Essawe[1,2], Mohamed Abd Elgawad[3,#], Haroon Mohamed Barakat[4], and Hui Zhao[1]

[1] School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, CHINA

[2] Faculty of Applied Science, Red Sea University, SUDAN

[3] Faculty of Science, Benha University13518, EGYPT

[4] Faculty of Science, Zagazig University 44519, EGYPT

[#] Corresponding author, mohamed.abdelgwad@fsc.bu.edu.eg

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In this paper, we study the limit distribution functions of the (lower-lower), (upper-upper) and (lower-upper) extreme and central-central m*-generalised order statistics (m–GOS) of stationary Gaussian sequences under an equi-correlated set up, when the random sample size is assumed to converge weakly and independent of the basic variables. Moreover, sufficient conditions for a weak convergence of generalised quasi-range with random indices are obtained.*

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INTRODUCTION

Kamps (1995) introduced the concept of generalised order statistics (GOS) as a unified approach to a variety of models of ordered random variables (RVs) with different interpretations. Ordinary order statistics (OOS), k-records (ordinary record values when $k = 1$), sequential order statistics (SOS), ordering via truncated distributions and censoring schemes can be discussed as they are special cases of the GOS. Since Kamps (1995) had introduced the unifying model of GOS, the use of such a model has been steadily growing along the years because it is more flexible in reliability theory, statistical modelling and inference.

In this work, we consider a wide subclass of GOS, known as m–GOS, which contains many important models of ordered RVs such as OOS, k-records, SOS and type II censored order statistics. Let $f(.)$ be an arbitrary continuous distribution function (DF), with the probability density function (PDF) $f(.)$ and survival function $\overline{F}(.)=1-F(.)$, then the RVs $X_{1:n}^{(m,k)} \le X_{2:n}^{(m,k)} \le .. \le X_{n:n}^{(m,k)}$ ($k > 0$, $m \ge -1$) are said to be m–GOS, if their joint PDF is given by (cf. Kamps, 1995)

$$f_{1,2,\dots,n:n}^{(m,k)}(x_1, x_2, \dots, x_n) = \left(\prod_{j=1}^{n}\gamma_j\right)\left(\prod_{j=1}^{n-1}\overline{F}^{\,m}(x_j)f(x_j)\right)\overline{F}^{\,k-1}(x_n)f(x_n),$$

where $F^{-1}(0) \le x_1 \le .. \le x_n \le F^{-1}(1)$ and $\gamma_j = k+(n-j)(m+1)$, $j=1,2,\dots,n$.

Nasri-Roudsari (1996) (see also Barakat, 2007) derived the marginal DF of the rth m–GOS, $m>-1$, in the form $\Phi_{r:n}^{(m,k)}(x) = I_{G_m(x)}(r, N-r+1)$, where $G_m(x) = 1-\overline{F}^{\,m+1}(x)$, $I_x(a,b) = \frac{1}{\beta(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}\,dt$ $a,b \ge 0$ denotes the incomplete beta ratio function and $N = \frac{k}{m+1}+n-1$. By using the well-known relation $I_x(a,b) = I_{\bar{x}}(b,a)$, where $\bar{x} = 1-x$, the marginal DF of the $r`(n)$th $m$–GOS, $m \ne -1$, is given by $\Phi_{r`(n):n}^{(m,k)}(x) = I_{G_m(x)}(N-R_r+1, R_r)$, where $r`(n) = n-r+1$ and $R_r = \frac{k}{m+1}+r-1$.

The possible non-degenerate limit distributions and the convergence rate of the upper extreme m–GOS, i.e., $r`(n)$th m–GOS for fixed r, were discussed in Nasri-Roudsari and

Cramer (1999). The asymptotic normality of intermediate and central m–GOS, which depends on the differentiability of the underlying DF F, was derived by Cramer (2003). Moreover, the necessary and sufficient conditions of the weak convergence, as $n \to \infty$, as well as the form of the possible limit DFs of extreme, intermediate and central m–GOS were derived in Barakat (2007). Finally, Barakat *et al.* (2014) obtained the limit DFs of bivariate extreme, intermediate and central m–GOS, $m \neq -1$.

Let the sequence $\{X_n\}$, $n \geq 1$, be a standard (zero-mean, unit-variance) stationary Gaussian sequence (SGS) with correlation coefficient $\rho_n = E(X_i X_j) \geq 0, i \neq j$, which depends only on the sample size. This sequence can be replaced by the sequence

$X_j = \sqrt{\rho_n} Y_0 + \sqrt{1-\rho_n} Y_j, 1 \leq j \leq n$, for the i.i.d standard normal variates $Y_0, Y_1, \ldots, Y_n$, where $X_j = Y_j$, for $\rho_n = 0$. Clearly, for the rth m–GOS in such SGS, we get

$$X_{r:n}^{(m,k)} = \sqrt{\rho_n} Y_0 + \sqrt{1-\rho_n} Y_{r:n}^{(m,k)}, 1 \leq r \leq n, \tag{1.1}$$

where $X_{r:n}^{(m,k)}$ and $Y_{r:n}^{(m,k)}$ are the rth m–GOS based on the sequences $\{X_j\}_{j=1}^{j=n}$ and $\{Y_j\}_{j=1}^{j=n}$, respectively.

In reliability theory, especially for OOS and SOS, $X_{r:n}^{(m,k)}$ represents the life length of a $(n-r+1)$-out-of-n system made up of n independent life lengths. In the typical setup used for m–GOS, the sample size n is considered fixed. On the other hand, in many life tests, biological, agriculture and some quality control problems, we usually encounter random sample sizes, where it is almost impossible to have a non-random sample size, because either some observations get lost randomly for various reasons or the size of the target population and its representative sample cannot be predetermined. In such many situations, the sample size may depend on the occurrence of some random events, which makes it random. Therefore, the sample size will be a non-negative integer-valued RV ν_n, say, and the sample itself will be described by an infinite sequence of RVs X_1, $X_2, \ldots$ independent of ν_n. Then, the rth and $r`(\nu_n)$th $m$–GOS will be denoted by $X_{r:\nu_n}^{(m,k)}$ and $X_{r`(\nu_n):\nu_n}^{(m,k)}$, respectively, based on a random sample of random size ν_n.

Vasudeva and Moridani (2010) studied the limit distribution of upper extreme of SGS (1.1) when $m = 0$, $k = 1$ (i.e., upper extreme OOS) and the sample size is itself a RV ν_n, which is independent of the basic variables. However, in the work of Vasudeva and Moridani, there is a restrictive condition that the random sequence of the correlation coefficient ρ_{ν_n} converges in probability to a positive constant, or infinity. More recently, Barakat *et al.* (2016a) got rid of this restrictive condition and obtained the parallel results for limit distributions of OOS with random indices in SGS. Moreover, Barakat *et al.* (2016b) studied the limit distributions of extreme, intermediate and central m–GOS, $m > -1$, defined by (1.1), when the random sample size is assumed to converge weakly. Recently, Barakat (2018) studied the limit joint DF of any two extreme, as well as central, m–GOS, $m > -1$, defined by (1.1), when the sample size is non-random. Moreover, Abd Elgawad *et al.* (2019a) studied the bivariate limit theorems for record values (when $m = -1$) based on random sample sizes. Finally, Barakat and Abd Elgawad (2019) and Abd Elgawad *et al.* (2019b) studied the limit distributions with fixed and random record model in SGS, respectively.

In this paper, we will extend the recent work of Barakat (2018) to the case when the sample size is assumed to be a positive integer-valued RV independent of the basic variables. As an application of this result, sufficient conditions for the weak convergence of random generalised quasi-range $R_{r:\nu_n}^{(m,k)} = X_{r_1(\nu_n):\nu_n}^{(m,k)} - X_{r_2:\nu_n}^{(m,k)}$ are obtained. It is worth mentioning that the results of this paper contribute not only to a critical assessment of existing statistical methodology, but also help to address their limitations within different contexts.

Throughout this paper, we will adopt some notations and abbreviations. For numerical vectors $\bar{x} = (x_1, x_2)$, the components are signified by a subscript. Basic arithmetical operations are always meant componentwise. Thus $\bar{x} \leq \bar{y}$ means $x_i \leq y_i, i = 1,2$, $\bar{x} \pm c = (x_1 \pm c, x_2 \pm c)$ and $\pm\bar{\infty} = (\pm\infty, \pm\infty)$. In addition, the symbols $(\xrightarrow[n]{})$, $(\xrightarrow[n]{w})$ and $(\xrightarrow[n]{p})$ stand for convergence, the weak convergence and convergence in probability, as $n \to \infty$. Moreover, for every $r, x \geq 0, \Gamma_r(x) = \frac{1}{\Gamma(r)} \int_0^x t^{r-1} e^{-t} dt$ denotes the incomplete gamma ratio function and $\overline{\Gamma}_r(x) = 1 - \Gamma_r(x)$. Also, $\Phi(x)$ denotes the standard normal DF, $\phi(x)$ its PDF, $\Phi_{\mu,\sigma^2}(x)$ denotes the normal DF with μ mean and σ^2 variance, while $\mathcal{N}_{\mathcal{R}}(x, y)$ denotes bivariate standard normal DF with correlation $\mathcal{R}$. Finally, the symbol "*" denotes the convolution operation and $X_n \overset{w}{\underset{n}{=}} Y_n$ means that the RVs X_n and Y_n have the same limit DF.

THE JOINT DF OF *m*–GOS BASED ON SGS, WITH RANDOM INDICES

In this section, we study the limit DFs of the (upper-upper), (lower-lower) and (lower-upper) extremes and central-central m–GOS, $m > -1$, defined by (1.1), as well as the generalised quasi-range, when the random sample size ν_n is independent of the basic variables.

The joint df of extreme *m*–GOS and generalized quasi-range with random indices in SGS. In this subsection, Theorem 2.1 characterises the weak convergence of (upper-upper) extreme m–GOS, $X_{r`_1(\nu_n):\nu_n}^{(m,k)}$ and $X_{r`_2(\nu_n):\nu_n}^{(m,k)}$ where $1 \leq r_2 < r_1$ (remember that $r_1 < n$ and $r_2 < n$ are two positive integers independent on n and $r`_i(n) = n - r_i + 1$, $i = 1,2$). Moreover, Theorem 2.2, characterises the weak convergence of (lower-lower) extreme m–GOS, $X_{r_1:\nu_n}^{(m,k)}$ and $X_{r_2:\nu_n}^{(m,k)}$, where $1 \leq r_1 < r_2$. In addition, Theorem 2.3 characterises the weak convergence of (lower-upper) extreme m-GOS, $X_{r_1:\nu_n}^{(m,k)}$ and $X_{r`_2(\nu_n):\nu_n}^{(m,k)}$, where $1 \leq r_1, r_2$. Finally, Theorem 2.4 characterises the weak convergence of the random generalised quasi-range.

Theorem 2.1. Let $m > -1$, $a_{n,m} = b_{n,m}^{-1} - \frac{b_{n,m}}{2}(\log\log n^{\frac{1}{m+1}} + \log 4\pi)$, $b_{n,m} = (\frac{2\log n}{m+1})^{\frac{-1}{2}}$, ν_n be a sequence of a positive integer-valued RVs independent of the basic variables and $P(\nu_n \le x) = A_n(x)$. Furthermore, let

$(A_1): A_n(nx) = P(\nu_n \le nx) \xrightarrow[n]{w} A(x)$, where $A(+0) = 0$ and $A(x)$ is a non-degenerate DF. Then,

$$(B_1): \Psi^{(m,k)}_{\overline{r\grave{}}(\nu_n):\nu_n}(\overline{x}_n) = P\left(X^{(m,k)}_{r\grave{}_1(\nu_n):\nu_n} \le x_{1;n}, X^{(m,k)}_{r\grave{}_2(\nu_n):\nu_n} \le x_{2;n}\right) \xrightarrow[n]{w} \Psi^{(m,k)}(\overline{x}) = \int_0^\infty H^{(m,k)}(\overline{x};\tau,z)\,dA(z),$$

where

$$H^{(m,k)}(\overline{x};\tau,z) = \begin{cases} \overline{\Gamma}_{R_{r_2}}(ze^{-(m+1)x_2-\tau}) * \Phi_{0,\frac{2\tau r}{m+1}}(x_2), & x_1 \ge x_2; \\ \overline{\Gamma}_{R_{r_1}}(ze^{-(m+1)x_1-\tau}) * \Phi_{0,\frac{2\tau}{m+1}}(x_1) - \frac{1}{\Gamma(R_{r_1})}\int_{-\infty}^{\infty}\int_{ze^{-(m+1)(x_1-w)-\tau}}^{\infty} & \\ \times I_{\frac{ze^{-(m+1)(x_2-w)-\tau}}{t}} R_{r_2}, R_{r_1} - R_{r_2})t^{R_{r_1}-1}e^{-t}\,dt\,d\Phi(\sqrt{\frac{m+1}{2\tau}}w), & x_1 < x_2, \end{cases} \tag{2.1}$$

$\overline{r\grave{}}(\nu_n) = (r\grave{}_1(\nu_n), r\grave{}_2(\nu_n))$, $\overline{x} = (x_1, x_2)$, $\overline{x}_n = (x_{1;n}, x_{2;n})$ *and* $x_{i:n} = b_{n,m}x_i + a_{n,m}$, $i = 1,2$, *if*

$(C_1): \rho_n \log n \xrightarrow[n]{} \tau,\ 0 \le \tau < \infty.$

Moreover, under Condition (A_1), we get

$$(B_2): \Psi^{(m,k)}_{\overline{r\grave{}}(\nu_n):\nu_n}(\overline{x^*_n}) = P\left(X^{(m,k)}_{r\grave{}_1(\nu_n):\nu_n} \le x^*_{1;n}, X^{(m,k)}_{r\grave{}_2(\nu_n):\nu_n} \le x^*_{2;n}\right) \xrightarrow[n]{w} \Phi(\min(\overline{x})), \tag{2.2}$$

where $\overline{x^*_2} = (x^*_{1;n}, x^*_{2;n})$ *and* $x^*_{i;n} = \sqrt{\rho_n}x_i + a_{n,m}$, $i = 1,2$, if

$(C_2): \rho_n \log n \xrightarrow[n]{} \infty$, and ρ_n is a regularly varying function of n (see, de Hann 1970), i.e., for every $\theta > 0$, we get $\frac{\rho_{n\theta}}{\rho_n} \xrightarrow[n]{} \theta$.

Conversely, if (B_1) and (C_1) (with $\tau = 0$) hold, then (A_1) will be satisfied.

Proof. Firstly, let (A_1) be satisfied. Then, by using the representation (1.1), the continuous version of the total probability rule and the independence between Y_0 and $X^{(m,k)}_{r:n}$, $r = 1,2,\ldots,n$, we get

$$\Psi^{(m,k)}_{\overline{r\grave{}}(\nu_n):\nu_n}(\overline{x}_n) =$$

$$P\left(X^{(m,k)}_{r\grave{}_1(\nu_n):\nu_n} \le x_{1;n}, X^{(m,k)}_{r\grave{}_2(\nu_n):\nu_n} \le x_{2;n}\right) = \int_0^\infty \Psi^{(m,k)}_{\overline{r\grave{}}(t):t}(\overline{x}_n)\,dA_n(t)$$

$$= \int_0^\infty\int_{-\infty}^\infty P\left(X^{(m,k)}_{r\grave{}_1(t):t} \le x_{1;n}, X^{(m,k)}_{r\grave{}_2(t):t} \le x_{2;n} \middle| Y_0 = y\right)\phi(y)\,dy\,dA_n(t)$$

$$= \int_0^\infty\int_{-\infty}^\infty P\left(Y^{(m,k)}_{r\grave{}_1(nz):nz} \le x_{1;nz}(y), Y^{(m,k)}_{r\grave{}_2(nz):nz} \le x_{2;nz}(y)\right)$$

$$\times\phi(y)\,dy\,dA_n(nz) = \int_0^\infty\int_{-\infty}^\infty \Phi^{(m,k)}_{\overline{r\grave{}}(nz):nz}(\overline{x}_{nz}(y))\phi(y)\,dy\,dA_n(nz), \tag{2.3}$$

(we use the transformation $t = nz$ in the outer integration on the second line of (2.3)), where

$\overline{x}_{nz}(y) = \left(x_{1;nz}(y), x_{2;nz}(y)\right)$, $x_{i;nz}(y) = B_{nz,m}x_i + A_{nz,m}$, $i = 1,2$, $B_{nz,m} = \frac{b_{n,m}}{\sqrt{1-\rho_{nz}}}$, $A_{nz,m} = \frac{a_{n,m} - \sqrt{\rho_{nz}}y}{\sqrt{1-\rho_{nz}}}$ and $\Phi^{(m,k)}_{\overline{r\grave{}}(nz):nz}(\overline{x}_{nz}(y))$

$$= P\left(Y^{(m,k)}_{r\grave{}_1(nz):nz} \le x_{1;nz}(y), Y^{(m,k)}_{r\grave{}_2(nz):nz} \le x_{2;nz}(y)\right).$$

On the other hand, by using Theorem 2.1, the relation (2.7) in Barakat *et al.* (2014) and by applying Theorem 2.1 in Barakat (2007) on the normal upper extreme m–GOS, we get

$$\Phi^{(m,k)}_{\overline{r\grave{}}(nz):nz}(\overline{x}_{nz}) \xrightarrow[n]{w} \begin{cases} \overline{\Gamma}_{R_{r_2}}(e^{-(m+1)x_2}), & x_1 \ge x_2, \\ \overline{\Gamma}_{R_{r_1}}(e^{-(m+1)x_1}) - \frac{1}{\Gamma_{R_{r_1}}}\int_{e^{-(m+1)x_1}}^{\infty} I_{\frac{-(m+1)x_2}{t}}(R_{r_2}, R_{r_1} - R_{r_2})t^{R_{r_1}-1}e^{-t}\,dt, & x_1 < x_2, \end{cases} \tag{2.4}$$

where $\overline{x}_{nz} = (x_{1;nz}, x_{2;nz})$ and $x_{i,nz} = b_{nz,m}x_i + a_{nz,m}$, $i = 1,2$. Therefore, in view Khnichine's types theorem and by using the relations (2.3) and (2.4), we get (2.1), if we showed that $\frac{A_{nz,m} - a_{nz,m}}{b_{nz,m}} \xrightarrow[n]{} \frac{\tau - \log z}{m+1} - \sqrt{\frac{2\tau}{m+1}}y$ and $\frac{B_{nz,m}}{b_{nz,m}} \xrightarrow[n]{} 1$. The latter is evident from the assumption that $\rho_n \log n \xrightarrow[n]{} \tau$, $0 \le \tau < \infty$ (this assumption yields $\rho_n \xrightarrow[n]{} 0$ i.e., $\rho_{nz} \xrightarrow[n]{} 0$). Hence, only the first relation needs proof. Applying that

$\sqrt{1-\rho_{nz}} = 1 - \frac{1}{2}\rho_{nz}(1+o(1))$, $\sqrt{\frac{2\log(nz)}{m+1}} = \sqrt{\frac{2\log n}{m+1}} + \frac{\log z}{\sqrt{2(m+1)\log n}}(1+o(1))$

and $\log\log(nz)^{\frac{1}{m+1}} = \log\log n^{\frac{1}{m+1}} + \frac{1}{m+1}\log(1+\frac{\log z}{\log n})$

and bearing in mind that $\frac{\log\log n}{\log n} \xrightarrow[n]{} 0$, we can verify that

$$\frac{A_{nz,m} - a_{nz,m}}{b_{nz,m}} = (1+\rho_{nz}(1+o(1)))\left[\frac{2\log n}{m+1} + \frac{\log(z(1+o(1))}{m+1} - \frac{1}{2}(\log\log n^{\frac{1}{m+1}} + \log 4\pi) - \frac{\log(z(1+o(1)))}{4\log n}(\log\log n^{\frac{1}{m+1}} + \log 4\pi) - \sqrt{\frac{2\rho_{nz}\log(nz)}{m+1}}y\right] - \frac{2\log n}{m+1} - \frac{2\log z}{m+1}$$
$$+ \frac{1}{2}\left[\log\log n^{\frac{1}{m+1}} + \log(1+\frac{\log z}{\log n})^{\frac{1}{m+1}} + \log 4\pi\right] \xrightarrow[n]{} \frac{\tau - \log z}{m+1} - \sqrt{\frac{2\tau}{m+1}}y.$$

By using (A_1), (2.3) and Theorem 2.1 of Barakat *et al.* (2014), we get

$$\Psi^{(m,k)}_{\overline{r\grave{}}(t):t}(\overline{x}_n) \xrightarrow[n]{w} H^{(m,k)}(\overline{x};\tau,z), \tag{2.5}$$

where the convergence is uniform with respect to $\overline{x}$ over any finite interval of z. Now, let ξ be a continuity point of $A(z)$ such that $1 - A(\xi) < \varepsilon$ (ε is an arbitrary small value). Then, we have

$$\int_\xi^\infty H^{(m,k)}(\overline{x};\tau,z)\,dA(z) \le \int_\xi^\infty dA(z) = 1 - A(\xi) < \varepsilon. \tag{2.6}$$

Moreover, for sufficiently large n due to (2.6) and the condition (A_1) we get

$$\int_\xi^\infty \Psi^{(m,k)}_{\overline{r\grave{}}(t):t}(\overline{x}_n)\,dA_n(t) \le 1 - A_n(n\xi) \le 2(1 - A(\xi)) < 2\varepsilon. \tag{2.7}$$

On the other hand, by the triangle inequality, we get

$$\left|\int_0^{\xi}\Psi^{(m,k)}_{\bar{r}(t):t}(\bar{x}_n)dA_n(t)-\int_0^{\xi}H^{(m,k)}(\bar{x};\tau,z)dA(z)\right|$$
$$\leq\left|\int_0^{\xi}\Psi^{(m,k)}_{\bar{r}(t):t}(\bar{x}_n)dA_n(t)-\int_0^{\xi}H^{(m,k)}(\bar{x};\tau,z)dA_n(t)\right|$$
$$+\left[\int_0^{\xi}H^{(m,k)}(\bar{x};\tau,z)dA_n(t)\right.$$
$$\left.-\int_0^{\xi}H^{(m,k)}(\bar{x};\tau,z)dA(z)\right]. \qquad (2.8)$$

Since, the convergence in (2.5) is uniform over the finite interval $z\in[0,\xi]$, then, for arbitrary $\varepsilon>0$ and for sufficiently large n we have

$$\left|\int_0^{\xi}\left[\Psi^{(m,k)}_{\bar{r}(t):t}(\bar{x}_n)-H^{(m,k)}(\bar{x};\tau,z)\right]dA_n(t)\right|$$
$$\leq\varepsilon(A_n(n\xi)-A_n(0))\leq\varepsilon. \qquad (2.9)$$

In order to estimate the third difference in (2.8), we construct Riemann sums that are close to the integral there. Let M be a fixed number and $0=\xi_0<\xi_1<\ldots<\xi_M=\xi$ be continuity points of $A(z)$. Furthermore, let M and ξ_j (j = 1,2, ...,M) be such that

$$\left|\int_0^{\xi}H^{(m,k)}(\bar{x};\tau,z)dA_n(t)\right.$$
$$\left.-\sum_{j=1}^{M}H^{(m,k)}(\bar{x};\tau,\xi_j)(A_n(n\xi_j)-A_n(n\xi_{j-1}))\right|<\varepsilon$$

and

$$\left|\int_0^{\xi}H^{(m,k)}(\bar{x};\tau,z)dA(z)\right.$$
$$\left.-\sum_{j=1}^{M}H^{(m,k)}(\bar{x};\tau,\xi_j)(A(\xi_j)-A(\xi_{j-1}))\right|<\varepsilon.$$

Since, by the assumption $A_n(n\xi_j)\xrightarrow[n]{w}A(\xi_j),\ 0\leq j\leq M,$ the two Riemann sums are closer to each other than ε for all n sufficiently large. Thus, once again by the triangle inequality, the absolute value of the difference of the integrals is smaller than 3ε. Combining this fact with (2.9), the left hand side of (2.8) becomes smaller than 4ε for all large n. Therefore, in view of (2.6), (2.7) and (2.5), we have

$$\left|\Psi^{(m,k)}_{\bar{r}(\nu_n):\nu_n}(\bar{x}_n)-\Psi^{(m,k)}(\bar{x})\right|\leq\left|\int_0^{\xi}\Psi^{(m,k)}_{\bar{r}(t):t}(\bar{x}_n)dA_n(t)\right.$$
$$\left.-\int_0^{\xi}H^{(m,k)}(\bar{x};\tau,z)dA(z)\right|+\int_0^{\xi}\Psi^{(m,k)}_{\bar{r}(t):t}(\bar{x}_n)dA_n(t)$$
$$+\int_0^{\xi}H^{(m,k)}(\bar{x};\tau,z)dA(z)\leq 7\varepsilon.$$

This completes the proof of the first part of theorem.

Turning now to the case $\rho_n\log n\xrightarrow[n]{}\infty$, for which we start with relation (2.3), with $x^*_{i,nz}(y)=\sqrt{\frac{\rho_n}{1-\rho_{nz}}}(x_i-y)+\frac{a_{n,m}}{\sqrt{\rho_n}}$, $i=1,2$. Now, for every $\varepsilon>0$ we have

$$P\left(\sqrt{\tfrac{1-\rho_{nz}}{\rho_n}}\left|Y^{(m,k)}_{r_i(nz):nz}-\tfrac{a_{n,m}}{\sqrt{1-\rho_{nz}}}\right|\geq\varepsilon\right)\leq$$
$$P\left(\sqrt{\tfrac{1-\rho_{nz}}{\rho_n}}\left|Y^{(m,k)}_{r_i(nz):nz}-a_{n,m}\right|\geq\varepsilon\right)\leq$$
$$P\left(\left|Y^{(m,k)}_{r_i(nz):nz}-a_{n,m}\right|\geq\sqrt{\rho_n}\,\varepsilon\right)=$$
$$P\left(\left|\tfrac{Y^{(m,k)}_{r_i(nz):nz}-a_{nz,m}}{b_{nz,m}}\right|\geq(\varepsilon-T_n)\tfrac{\sqrt{\rho_{nz}}}{b_{nz,m}}\times\sqrt{\tfrac{\rho_n}{\rho_{nz}}}\right)\xrightarrow[n]{}0,\ i=1,2,$$

where $T_n=\frac{a_{nz,m}-a_{n,m}}{\sqrt{\rho_n}}\xrightarrow[n]{}0$, from the relation (14) in Barakat *et al.* (2016b), since $\frac{\rho_{nz}}{\rho_n}\xrightarrow[n]{}1$ and $\frac{\sqrt{\rho_{nz}}}{b_{nz,m}}=\sqrt{\frac{2\rho_{nz}\log(nz)}{m+1}}\xrightarrow[n]{}\infty$. Thus, each of the DFs

$$P\left(\sqrt{\frac{1-\rho_{nz}}{\rho_n}}\left|Y^{(m,k)}_{r_i(nz):nz}-a_{n,m}\right|\leq x_i-y\right),\ i=1,2,$$

has a degenerate limit DF at zero, i.e., has the limit

$$\varepsilon(x_i-y)=\begin{cases}1, & x_i-y\geq 0;\\ 0, & x_i-y\leq 0,\quad i=1,2.\end{cases}$$

Consequently, by using the transformation $w=-y$ we get

$$\Psi^{(m,k)}_{\bar{r}(\nu_n):\nu_n}(\overline{x^*_n})\xrightarrow[n]{w}\int_0^{\infty}\int_{-\infty}^{\infty}\varepsilon(\bar{x}+w)\phi(w)dwdA(z)=$$
$$\int_{-\infty}^{\infty}\varepsilon(\bar{x}+w)\phi(w)dw, \qquad (2.10)$$

$$\varepsilon(\bar{x}+w)=\begin{cases}1, & w\geq\max(-\bar{x})=-\min(\bar{x});\\ 0, & w<\max(-\bar{x})=-\min(\bar{x}).\end{cases} \qquad (2.11)$$

The required relation (2.2) is now followed by combining (2.10) with (2.11). The remaining part of this case follows exactly as the proof of the case $\rho_n\log n\xrightarrow[n]{}\tau,\ 0\leq\tau<\infty$.

Turning now to prove the converse part that (B_1) and (C_1) imply (A_1). Starting with the relation (2.3) by the compactness of DFs, we can select a subsequence $\{n'\}$ of $\{n\}$ for which $A_{n'}(n'z)$ converges weakly to an extended DF $A'(z)$ (i.e., $A'(\infty)-A'(0)\leq 1$). Then, by repeating the first part of the theorem for the subsequence $\{n'\}$ with the exception that we choose ξ so that $d(\xi,\infty)=A'(\infty)-A'(\xi)<\varepsilon$, we get $\Psi^{(m,k)}(\bar{x})=\int_0^{\infty}H^{(m,k)}(\bar{x};\tau,z)dA'(z)$. Since the functions $\Psi^{(m,k)}(\bar{x})$ and $H^{(m,k)}(\bar{x};\tau,z)$ are DFs, we get $\Psi^{(m,k)}(\bar{\infty})=1=\int_0^{\infty}dA'(z)=d(0,\infty)$, which implies that $A'(z)$ is a DF. Now, if $A_n(nz)$ did not converge weakly, then we could select another subsequences $\{n''\}$ such that $A_{n''}(n''z)\xrightarrow[n'']{w}A''(z)$. In this case, we get

$$\Psi^{(m,k)}(\bar{x})=\int_0^{\infty}H^{(m,k)}(\bar{x};\tau,z)dA'(z)$$
$$=\int_0^{\infty}H^{(m,k)}(\bar{x};\tau,z)dA''(z). \qquad (2.12)$$

Thus, $x_2\to\infty$, we get

$$\Psi^{(m,k)}(x_1)=\int_0^{\infty}\bar{\Gamma}_{R_{r_1}}(ze^{-(m+1)x_1})dA'(z)$$
$$=\int_0^{\infty}\bar{\Gamma}_{R_{r_1}}(ze^{-(m+1)x_1})dA''(z).$$

Let $L'(v)=\int_0^{\infty}\bar{\Gamma}_{R_{r_1}}(zv)dA'(z)$ and $L''(v)=\int_0^{\infty}\bar{\Gamma}_{R_{r_1}}(zv)dA''(z)$. Evidently $L'(v)$ and $L''(v)$ are analytic functions on the region $D=\{v:0<|v|<\infty\}$. In view of (2.12), L' and L'' coincide on some interval contained in D. Thus, by the uniqueness theory of analytic functions, we deduce that L' and L'' coincide on the region D which implies $A'(z)=A''(z)$. This completes proof of the theorem.

Corollary 2.1. As a simple consequence of Theorem 2.1 is

$$P\left(\frac{X^{(m,k)}_{r`(\nu_n):\nu_n} - a_{n,m}}{b_{n,m}} \le x\right) \xrightarrow[n]{w} \int_0^\infty \overline{\Gamma}_{R_r}(ze^{-(m+1)x-\tau})$$

$$* \Phi_{0,\frac{2\tau}{m+1}}(x)dA(z)$$

if $\rho_n \log n \xrightarrow[n]{} \tau$, $0 \le \tau < \infty$. Moreover, if $\rho_n \log n \xrightarrow[n]{} \infty$, and ρ_n is regularly varying function of n we get

$$P\left(\frac{X^{(m,k)}_{r`(\nu_n):\nu_n} - a_{n,m}}{\sqrt{\rho_n}} \le x\right) \xrightarrow[n]{w} \Phi(x).$$

Theorem 2.2. Let $m > -1$, $\tilde{a}_{n,m} = -\tilde{b}^{-1}_{n,m} + \frac{\tilde{b}_{n,m}}{2}(\log\log(n(m+1))) + \log 4\pi)$ and $\tilde{b}_{n,m} = (2\log(n(m+1)))^{\frac{-1}{2}}$. Furthermore, let $\nu_n, A(z)$ and the condition (A_1) be defined as Theorem 2.1. Then, under the condition (A_1) we get

$$(B_1): \Psi^{(m,k)}_{\bar{r}:\nu_n}(\overline{\tilde{x}_n})$$
$$= P\left(X^{(m,k)}_{r_1:\nu_n} < \tilde{x}_{1;n}, X^{(m,k)}_{r_2:\nu_n} < \tilde{x}_{2;n}\right) \xrightarrow[n]{w} \Psi^{(m,k)}(\bar{x})$$
$$= \int_0^\infty H^{(m,k)}(\bar{x};\tau,z)dA(z),$$

where

$$H^{(m,k)}(\bar{x};\tau,z)$$
$$= \begin{cases} \Gamma_{r_2}(ze^{x_2-\tau}) * \Phi_{0,2\tau}(x_2), & x_1 \ge x_2; \\ \frac{1}{(r_1-1)!}\int_{-\infty}^{\infty}\int_0^{ze^{x_1-\tau-w}} \Gamma_{r_2-r_1}(ze^{x_2-\tau-w} - t) & \\ \qquad \times t^{r_1-1}e^{-t}dtd\Phi(\frac{w}{\sqrt{2\tau}}) & x_1 < x_2, \end{cases}$$

$\bar{r} = (r_1, r_2)$, $\overline{\tilde{x}_n} = (\tilde{x}_{1;n}, \tilde{x}_{2;n})$ and $\tilde{x}_{i:n} = \tilde{b}_{n,m}x_i + \tilde{a}_{n,m}$, if

$(C_1): \rho_n \log n \xrightarrow[n]{} \tau$, $0 \le \tau < \infty$.

Moreover, under Condition (A_1) we get

$$(B_2): \Psi^{(m,k)}_{\bar{r}:\nu_n}(\overline{\tilde{x}^*_n})$$
$$= P\left(X^{(m,k)}_{r_1:\nu_n} \le \tilde{x}^*_{1:n}, X^{(m,k)}_{r_2:\nu_n} \le \tilde{x}^*_{2:n}\right) \xrightarrow[n]{w} \Phi(\min(\bar{x})),$$

where $\tilde{x}^*_{i;n} = \sqrt{\rho_n}x_i + \tilde{a}_{n,m}$, $i = 1,2$, if

$(C_2): \rho_n \log n \xrightarrow[n]{} \infty$ and ρ_n is a regularly varying function of n.

Conversely, if (B_1) and (C_1) (with $\tau = 0$) hold, then the relation (A_1) will be satisfied.

Proof. The proof of Theorem 2.2 is similar to the proof of Theorem 2.1 with only the exception of obvious changes. This completes the proof of Theorem 2.2.

Corollary 2.2. A simple consequence of Theorem 2.2 is

$$P\left(\frac{X^{(m,k)}_{r:\nu_n} - \tilde{a}_{n,m}}{\tilde{b}_{n,m}} \le x\right) \xrightarrow[n]{w} \int_0^\infty \Gamma_r(ze^{x-\tau}) * \Phi_{0,2\tau}(x)dA(z),$$

if $\rho_n \log n \xrightarrow[n]{} \tau$, $0 \le \tau < \infty$. Moreover, if $\rho_n \log n \xrightarrow[n]{} \infty$, and ρ_n is a regularly varying function of n, we get

$$P\left(\frac{X^{(m,k)}_{r:\nu_n} - \tilde{a}_{n,m}}{\sqrt{\rho_n}} \le x\right) \xrightarrow[n]{w} \Phi(x).$$

Theorem 2.3. Let $\tilde{x}_{1;n}, \tilde{x}^*_{1;n}, \tilde{x}_{2;n}, \tilde{x}^*_{2;n}$, be defined as in Theorems 2.1 and 2.2. Furthermore, let $\nu_n, A(z)$ and the condition (A_1) be defined as Theorem 2.1. Also, let

$\Psi^{(m,k)}_{r_1,r`_2(\nu_n):\nu_n}(\tilde{x}_{1;n}, x_{2;n}) = P(X^{(m,k)}_{r_1:\nu_n} \le \tilde{x}_{1;n}, X^{(m,k)}_{r`_2(\nu_n):\nu_n} \le \tilde{x}_{2;n})$.
Then, under the condition (A_1) we get

$$(B_1): \Psi^{(m,k)}_{r_1,r`_2(\nu_n):\nu_n}(\tilde{x}_{1;n}, x_{2;n}) \xrightarrow[n]{w} \Psi^{(m,k)}(\bar{x})$$
$$= \int_0^\infty H^{(m,k)}(\bar{x};\tau,z)dA(z),$$

where

$$H^{(m,k)}(\bar{x};\tau,z)$$
$$= \int_{-\infty}^{\infty} \Gamma_{r_1}(ze^{x_1-\tau-w\sqrt{m+1}})\overline{\Gamma}_{R_{r_2}}(ze^{-(m+1)(x_2-w)-\tau})d\Phi_{0,\frac{2\tau}{m+1}}(w),$$

if

$$(C_1): \rho_n \log n \xrightarrow[n]{} \tau,\ 0 \le \tau < \infty. \tag{2.13}$$

Moreover, under Condition (A_1) we get

$$(B_2):\ \Psi^{(m,k)}_{r_1,r`_2(\nu_n):\nu_n}(\tilde{x}^*_{1;n}, x^*_{2;n}) \xrightarrow[n]{w} \Phi(\min(\bar{x})),\ \text{if} \tag{2.14}$$

$(C_2): \rho_n \log n \xrightarrow[n]{} \infty$ and ρ_n is a regularly varying function of n.

Conversely, if (B_1) and (C_1) (with $\tau = 0$) hold, then the relation (A_1) will be satisfied.

Proof. The proof of Theorem 2.3 is similar to the proof of Theorems 2.1, 2.2 with only the exception of obvious changes.

Remark 2.1. Clearly, Theorem 2.3 (the relations (2.13) and (2.14)) reveals that the lower and upper random extreme m–GOS are asymptotically independent only if $\rho_n \log n \xrightarrow[n]{} \tau = 0$ and $A(x)$ is degenerate at $x = 1$.

Remark 2.2. Under the condition (C_2) we get the interesting results (by virtue to the result of Barakat (2018) and Theorems 2.1–2.3):

1. $\left(X^{(m,k)}_{r`_1(\nu_n):\nu_n}, X^{(m,k)}_{r`_2(\nu_n):\nu_n}\right) \underset{n}{\overset{w}{=}} (X^{(m,k)}_{r`_1(n):n}, X^{(m,k)}_{r`_2(n):n})$, in Theorem 2.1.

2. $\left(X^{(m,k)}_{r_1:\nu_n}, X^{(m,k)}_{r_2:\nu_n}\right) \underset{n}{\overset{w}{=}} (X^{(m,k)}_{r_1:n}, X^{(m,k)}_{r_2:n})$, in Theorem 2.2.

3. $\left(X^{(m,k)}_{r_1:\nu_n}, X^{(m,k)}_{r`_2(\nu_n):\nu_n}\right) \underset{n}{\overset{w}{=}} (X^{(m,k)}_{r_1:n}, X^{(m,k)}_{r`_2(n):n})$, in Theorem 2.3.

Moreover, the above equalities hold in all cases of Theorems 2.1–2.3, if $A(x)$ is degenerate at $x = 1$.

Theorem 2.4. Let $R^{(m,k)}_{r:\nu_n} = Y^{(m,k)}_{r`_1(\nu_n):\nu_n} - Y^{(m,k)}_{r_2:\nu_n}$ and $\Omega(x)$ be a non-degenerate DF. Then,

$$P\left(\mathcal{R}^{(m,k)}_{r:\nu_n} \le C_n x + D_n\right) \xrightarrow[n]{w} \Omega(x),$$

if and only if

$$P\left(R^{(m,k)}_{r:\nu_n} \le B_n x + A_n\right) \xrightarrow[n]{w} \Omega(x),$$

where $C_n = B_n$ and $D_n = A_n$ if $\rho_n \log n \xrightarrow[n]{p} \tau$, $0 \le \tau < \infty$ (see Remark 2.3), while $C_n = B_n\sqrt{1-\rho_{\nu_n}}$ and $D_n = A_n\sqrt{1-\rho_{\nu_n}}$ if $\rho_n \log n \xrightarrow[n]{p} \infty$ (see Remark 2.3).

Proof. It is easy verifying the following equality (which is easily proved by representation (1.1)):

$$\mathcal{R}^{(m,k)}_{r:\nu_n} \overset{w}{\underset{n}{=}} R^{(m,k)}_{r:\nu_n}\sqrt{1-\rho_{\nu_n}},$$

By using Lemma 2.3 in Vasudeva and Moridani (2010) and the obvious relation $\sqrt{1-\rho_{\nu_n}} \overset{w}{\underset{n}{=}} 1$, if $\rho_{\nu_n} \log n \xrightarrow[n]{p} \tau$, $0 \le \tau < \infty$, the proof of Theorem 2.4 immediately follows.

Remark 2.3. Vasudeva and Moridani (2010) presented several examples of sequences ρ_{ν_n} that satisfy the conditions $\rho_{\nu_n} \log n \xrightarrow[n]{p} \tau$, $0 \le \tau < \infty$ and $\rho_{\nu_n} \log n \xrightarrow[n]{p} \infty$. On the other hand, it is worth mentioning that the class of possible limit DF $\Omega(x)$ when $m = 0$ and $k = 1$ (i.e., for OOS) is fully characterised by Barakat and Nigm (1991).

The joint df of central *m*–GOS with random indices in a SGS. Let $0 < \lambda_1 < \lambda_2 < 1$ and x_{oi} be such that $\Phi(x_{oi}) = \lambda_i$, $i = 1,2$. Moreover, let r_{in}, $i = 1,2$ be central rank sequences such that $\sqrt{n}(\frac{r_{in}}{n} - \lambda_i) \xrightarrow[n]{} 0$. It is known that (cf. Barakat *et al.*, 2014, Lemma 3.1 and Theorem 3.1, see also Theorem 2.2 of Barakat, 2007)

$$P\left(\frac{X^{(m,k)}_{r_{1n}:n} - x_{o1}}{c_{1:n}} \le x_1, \frac{X^{(m,k)}_{r_{2n}:n} - x_{o2}}{c_{2:n}} \le x_2\right)$$

$$\xrightarrow[n]{w} \mathcal{N}_{\mathcal{R}}\left(\frac{(m+1)c^*_{\lambda_1(m)}}{c^*_{\lambda_1}} x_1, \frac{(m+1)c^*_{\lambda_2(m)}}{c^*_{\lambda_2}} x_2\right),$$

where

$c_{i:n} = \sqrt{\frac{\lambda_i\bar{\lambda}_i}{n\phi^2(x_{oi})}}$, $= \sqrt{\frac{\lambda_1\bar{\lambda}_2}{\lambda_2\bar{\lambda}_1}}$, $c^*_{\lambda_i} = \frac{c_{\lambda_i}}{\bar{\lambda}_i}$, $c_{\lambda_i} = \sqrt{\lambda_i\bar{\lambda}_i}$, $\lambda_i(m) = 1 - \bar{\lambda}_i^{\frac{1}{m+1}}$, $\bar{\lambda}_i = 1 - \lambda_i$, $i = 1,2$. The following theorem gives the limit joint DF of the (r_{1n}, r_{2n})th central *m*–GOS of SGS (1.1), when the sample size is random.

Theorem 2.5. Let the condition (A_1) in Theorems 2.1-2.3 be satisfied. Then,

$$(B_1^*):\quad \Psi^{(m,k)}_{\bar{r}_{\nu_n}:\nu_n}(\bar{x}_n)$$
$$= P\left(X^{(m,k)}_{r_{1\nu_n}:\nu_n} \le x_{1:n}, X^{(m,k)}_{r_{2\nu_n}:\nu_n} \le x_{2:n}\right) \xrightarrow[n]{w}$$
$$\Phi^{(m,k)}(\bar{x}) = \int_0^\infty L^{(m,k)}(\bar{x};\tau,z)\,dA(z),$$

where

$$L^{(m,k)}(\bar{x};\tau,z) = \int_{-\infty}^{\infty} \mathcal{N}_{\mathcal{R}}\left(\frac{(m+1)c^*_{\lambda_1(m)}}{c^*_{\lambda_1}}\left(\sqrt{z}x_1 - \sqrt{\frac{\tau\phi^2(x_{o1})}{\lambda_1\bar{\lambda}_1}}\right),\right.$$

$$\left.\frac{(m+1)c^*_{\lambda_2(m)}}{c^*_{\lambda_2}}\left(\sqrt{z}x_2 - \sqrt{\frac{\tau\phi^2(x_{o2})}{\lambda_2\bar{\lambda}_2}}\right)\right)\phi(y)\,dy, \tag{2.15}$$

$x_{i:n} = c_{i:n}x_i + x_{oi}$, $i = 1,2$, if

$(C_1^*): n\rho_n \xrightarrow[n]{} \tau$, $0 \le \tau < \infty$.

Moreover, under the condition (A_1) we get

$$(B_2^*):\quad \Psi^{(m,k)}_{\bar{r}_{\nu_n}:\nu_n}(\overline{x^*_n})$$
$$= P\left(X^{(m,k)}_{r_{1\nu_n}:\nu_n} \le x^*_{1:n}, X^{(m,k)}_{r_{2\nu_n}:\nu_n} \le x^*_{2:n}\right) \xrightarrow[n]{w} \Phi(\min(\bar{x})), \tag{2.16}$$

where $x^*_{i:n} = \sqrt{\rho_n}\,x_i + x_{oi}$, $i = 1,2$, if

$(C_2^*): n\rho_n \xrightarrow[n]{} \infty$ and ρ_n is a regularly varying function of n.

Conversely, if (B_1^*) and (C_1^*) (with $\tau = 0$) hold, then the relation (A_1) will be satisfied.

Proof. First, let (A_1) be satisfied. Then, by using the representation (1.1), the continuous version of the total probability rule and the independence between Y_0 and $X^{(m,k)}_{r_n:n}$, $r_n = 1,2,\ldots,n$, we get

$$\Psi^{(m,k)}_{\bar{r}_{\nu_n}:\nu_n}(\bar{x}_n)$$
$$= P\left(X^{(m,k)}_{r_{1\nu_n}:\nu_n} \le x_{1:n}, X^{(m,k)}_{r_{2\nu_n}:\nu_n} \le x_{2:n}\right) = \int_0^\infty \Psi^{(m,k)}_{\bar{r}_t:t}(\bar{x}_n)\,dA_n(t)$$
$$= \int_0^\infty\int_{-\infty}^\infty P\left(X^{(m,k)}_{r_{1t}:t} \le x_{1:n}, X^{(m,k)}_{r_{2t}:t} \le x_{2:n} \middle| Y_0 = y\right)\phi(y)\,dy\,dA_n(t)$$
$$= \int_0^\infty\int_{-\infty}^\infty G^{(m,k)}_{\bar{r}_{nz}:nz}(\bar{x}_{nz}(y))\phi(y)\,dy\,dA_n(nz), \tag{2.17}$$

where

$\bar{x}_{nz}(y) = (x_{1:nz}(y), x_{2:nz}(y)), x_{i,nz}(y) = C_{i:nz}x_i + D_{i:nz}(y)$,

$C_{i:nz} = \frac{C_{i:n}}{\sqrt{1-\rho_{nz}}}$, $D_{i:nz}(y) = \frac{x_{oi} - \sqrt{\rho_{nz}}\,y}{\sqrt{1-\rho_{nz}}}$, $i = 1,2$,

and $G^{(m,k)}_{\bar{r}_{nz}:nz}(\bar{x}_{nz}(y)) = P\left(Y^{(m,k)}_{r_{1nz}:nz} \le x_{1:nz}(y), Y^{(m,k)}_{r_{2nz}:nz} \le x_{2:nz}(y)\right)$. On the other hand, by applying Theorem 3.1 in Barakat *et al.* (2014) on the normal central *m*–GOS, we get

$$G^{(m,k)}_{\bar{r}_{nz}:nz}(\bar{x}_{nz})$$
$$\xrightarrow[n]{w} \mathcal{N}_{\mathcal{R}}\left(\frac{(m+1)c^*_{\lambda_1(m)}x_1}{c^*_{\lambda_1}}, \frac{(m+1)c^*_{\lambda_2(m)}x_2}{c^*_{\lambda_2}}\right), \tag{2.18}$$

where $\bar{x}_{nz} = (x_{1:nz}, x_{2:nz})$ and $x_{i,nz} = c_{i:nz}x_i + x_{oi}$, $i = 1,2$. Therefore, in view of Khnichine's types theorem and by using the relations (2.17) and (2.18) yield (2.15), we show that $\frac{D_{i:nz}(y) - x_{oi}}{C_{i:nz}} \xrightarrow[n]{} -\sqrt{\frac{\tau\phi^2(x_{oi})}{\lambda_i\bar{\lambda}_i}}\,y$ and $\frac{C_{i:nz}}{c_{i:nz}} \xrightarrow[n]{} \sqrt{z}$, $i = 1,2$. The latter is evident from the assumption that $n\rho_n \xrightarrow[n]{} \tau$, $0 \le \tau < \infty$. (this assumption yields $\rho_{nz} \xrightarrow[n]{} 0$). Hence, only the first relation needs proof. Applying that $\sqrt{1-\rho_{nz}} = 1 - \frac{1}{2}\rho_{nz}(1+o(1))$ and bearing in mind that

$\frac{\rho_{nz}}{c_{i:nz}} = \frac{nz\rho_{nz}\phi(x_{oi})}{\sqrt{nz\lambda_i\bar{\lambda}_i}} \xrightarrow[n]{} 0$, $\frac{\sqrt{\rho_{nz}}}{c_{i:nz}} \xrightarrow[n]{} \sqrt{\frac{\tau\phi^2(x_{oi})}{\lambda_i\bar{\lambda}_i}}$, we can verify that

$$\frac{D_{i;nz}(y)-x_{oi}}{c_{i;nz}}=\frac{(x_{oi}-\sqrt{\rho_{nz}}\,y)(1+\frac{1}{2}\rho_{nz}(1+o(1)))-x_{oi}}{c_{i;nz}}$$

$$=\frac{-\sqrt{\rho_{nz}}\,y+\frac{1}{2}\rho_{nz}x_{oi}(1+o(1))-\frac{1}{2}\rho_{nz}\sqrt{\rho_{nz}}\,y}{c_{i;nz}}$$

$$\xrightarrow[n]{}-\sqrt{\frac{\tau\phi^2(x_{oi})}{\lambda_i\bar{\lambda}_i}}\,y,\ i=1,2.$$

The remaining part of the proof of the theorem, under the condition $0\le\tau<\infty$ follows now by using the relations (2.17) and (2.18) exactly as the proof of Theorem 2.1, under the same condition (i.e., $\rho_n\log n\xrightarrow[n]{}\tau,\ 0\le\tau<\infty$).

Turning now to the case $n\rho_n\xrightarrow[n]{}\infty$, for which we start with relation (2.17), with $x^*_{i;nz}(y)=\sqrt{\frac{\rho_n}{1-\rho_{nz}}}(x_i-y_i)+\frac{x_{oi}}{\sqrt{\rho_n}}$, $i=1,2$. Now, for every $\varepsilon>0$ we have

$$P\left(\sqrt{\frac{1-\rho_{nz}}{\rho_n}}\left|Y^{(m,k)}_{r_{inz}:nz}-\frac{x_{oi}}{\sqrt{1-\rho_{nz}}}\right|\ge\varepsilon\right)$$

$$\le P\left(\sqrt{\frac{1-\rho_{nz}}{\rho_n}}\left|Y^{(m,k)}_{r_{inz}:nz}-x_{oi}\right|\ge\varepsilon\right)\le P\left(\left|Y^{(m,k)}_{r_{inz}:nz}-x_{oi}\right|\ge\sqrt{\rho_n}\,\varepsilon\right)$$

$$=\left(\left|\frac{Y^{(m,k)}_{r_{inz}:nz}-x_{oi}}{c_{i;nz}}\right|\ge\varepsilon\sqrt{\frac{nz\rho_n\phi^2-x_{oi}}{\lambda_i\bar{\lambda}_i}}\right)\xrightarrow[n]{}0,\ i=1,2.$$

Thus, each of the DFs $P\left(\sqrt{\frac{1-\rho_{nz}}{\rho_n}}\left|Y^{(m,k)}_{r_{inz}:nz}-x_{oi}\right|\le x_i-y\right)$, $i=1,2$, has a degenerate limit DF at zero. Consequently, by using the transformation $w=-y$ we get

$$\Psi^{(m,k)}_{\bar{r}_{v_n}:v_n}(\overline{x^*_n})\xrightarrow[n]{w}\int_{-\infty}^{\infty}\varepsilon'(\bar{x}+w)\phi(w)dw, \tag{2.19}$$

$$\varepsilon'(\bar{x}+w)=\begin{cases}1, & w\ge\max(-\bar{x})=-\min(\bar{x});\\ 0, & w<\max(-\bar{x})=-\min(\bar{x}).\end{cases} \tag{2.20}$$

The required relation (2.16) is now followed by combining (2.19) with (2.20). The remaining part of this case follows exactly as the proof of the case $n\rho_n\xrightarrow[n]{}\tau,\ 0\le\tau<\infty$.

We turning now to prove the converse part that (B_1^*) and (C_1^*) imply (A_1). Starting with the relation (2.17) by the compactness of DFs, we can select a subsequence $\{n'\}$ of $\{n\}$ for which $A_{n'}(n'z)$ converges weakly to an extended DF $A'(z)$. Then, by repeating the first part of the theorem for the subsequence $\{n'\}$ with the exception that we choose e so that $d(e,\infty)<\varepsilon$, we get $\Phi^{(m,k)}(\bar{x})=\int_0^\infty L^{(m,k)}(\bar{x};\tau,z)dA'(z)$. Since the two limits $\Phi^{(m,k)}(\bar{x})$ and $L^{(m,k)}(\bar{x};\tau,z)$ are DFs, we get $\Phi^{(m,k)}(\overline{\infty})=1=\int_0^\infty dA'(z)=d(0,\infty)$, which implies that $A'(z)$ is a DF. Now, if $A_n(nz)$ did not converge weakly, then we could select another subsequence $\{n''\}$ for which $A_{n''}(n''z)\xrightarrow[n'']{w}A''(z)$. In this case, we get

$$\Phi^{(m,k)}(\bar{x})=\int_0^\infty L^{(m,k)}(\bar{x};\tau,z)dA'(z)=\int_0^\infty L^{(m,k)}(\bar{x};\tau,z)dA''(z).$$

Thus, let $x_2\to\infty$, we get $\Phi^{(m,k)}(x_1)=\int_0^\infty\Phi(\sigma\sqrt{z}x_1)dA'(z)=\int_0^\infty\Phi(\sigma\sqrt{z}x_1)dA''(z)$, where $\sigma=\frac{(m+1)c^*_{\lambda_1(m)}}{c^*_{\lambda_1}}\sqrt{\frac{\lambda_1\bar{\lambda}_1}{\tau\phi^2(x_{oi})+\lambda_1\bar{\lambda}_1}}$. Let $G(u)=\int_0^\infty\Phi(\sqrt{z}u)dA'(z)$ and $G'(u)=\int_0^\infty\Phi(\sqrt{z}u)dA''(z)$. By differentiating $G(u)$ and $G'(u)$ with respect to u, we get $\int_0^\infty e^{-\frac{zu^2}{2}}(\sqrt{z}dA'(z))=\int_0^\infty e^{-\frac{zu^2}{2}}(\sqrt{z}dA''(z))$. Since, the Laplace transformations with respect to the measures $\{\sqrt{z}dA'(z)\}$ and $\{\sqrt{z}dA''(z)\}$ have the same effect, then we deduce that $A'(z)=A''(z)$. This completes the proof of the theorem.

APPLICATION

Theorems 2.1–2.5 reduce the limitations of some statistical methodologies of survival analysis and clinical trials, in different contexts. For example, we consider the $(n-r+1)$-out-of-n system, where the life-length distribution of the remaining components may change after each failure of the components. In literature, one of the most important models that describe such a system is the SOS model, where we may take the original DF of the ith component $(i=1,2,\ldots,n)$, before beginning the test, as $F_i(x)=1-(1-\Phi(x))^{\alpha_i}$, $\alpha_i>0$. Clearly, this model is a m–GOS model, with $k=\alpha_n$, $m_i=(n-i+1)\alpha_i-(n-1)\alpha_{i+1}+1$, $i=1,2,\ldots,n-1$, and $m_1=m_2=\ldots=m_{n-r+1}=m$ (cf. Kamps, 1995). On the other hand, Theorem 2.1 reveals the asymptotic dependence structure between the components in the (v_n-r+1)-out-of-v_n, system, when the components constitute a SGS with the inter-correlation between the different components depends only on their total number.

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DIVDIMENSIONĀLAS VISPĀRINĀTAS SAKĀRTOJUMA STATISTIKAS ROBEŽTEORĒMAS STACIONĀRAJĀS GAUSA VIRKNĒS AR MAZĀM IZLASĒM

Tika pētītas robežsadalījuma funkcijas ekstremālajām un centrālajām vispārinātām sakārtojuma statistikām stacionārajās Gausa virknēs. Ir atrasti pietiekamie nosacījumi, kas garantē kvazi-ranga vāju konverģenci ar gadījuma indeksiem.