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Boyd and Wong Type Fixed Point Theorems in Partial Metric Spaces

FAUSTINE NZIKU¹ AND SANTOSH KUMAR²

ABSTRACT. In this paper, we present fixed point results for Boyd and Wong type [3] generalized contractive condition in partial metric spaces. In particular, we generalize the fixed point results due to Akkouchi [1] in complete partial metric spaces in which the continuity requirement for a mapping is relaxed to obtain the results. In addition to that we present a common fixed point theorem for a pair of maps. An illustrative example is also constructed to exhibit the results.

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1. Introduction and Preliminaries

Banach fixed-point theorem is an important tool in the theory of metric spaces. It guarantees the existence and uniqueness of fixed points of certain self-maps of metric spaces, and provides a constructive method to find those fixed points. It has been extensively studied by many researchers to address different problems in analysis. This theorem [2] states that, if X is a non-empty set and d is a metric on X such that, $d : X \times X \rightarrow [0, \infty]$ and if T is a self mapping on a complete metric space (X, d) which satisfies:

$$d(Tx, Ty) \leq cd(x, y) \quad (1.1)$$

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^{1,2} *Department of Mathematics, College of Natural and Applied Sciences, University of Dar es Salaam, Tanzania*

e-mail: ¹*faustinenziku@gmail.com*

e-mail: ²*drsengar2002@gmail.com* .

for some $0 \leq c < 1$ for all $x, y \in X$, then T has a unique fixed point x_0 and a sequence $\{T^n x\}$ converges to x_0 for $x \in X$.

Boyd and Wong [3] gave an important generalization of this theorem due to Banach [2] in metric spaces. The theorem states as follows:

Theorem 1. [3] *Let (M, d) be a complete metric space and let $P := \{d(x, y) : x, y \in M\}$. Let $T : M \rightarrow M$ be a self mapping satisfying:*

$$d(Tx, Ty) \leq \chi(d(x, y)), \forall x, y \in M, \quad (1.2)$$

where $\chi : \bar{P} \rightarrow [0, \infty)$ is upper semi continuous from the right on $\bar{P}$ and satisfies $\chi(t) < t$, $\forall t \in \bar{P} \setminus \{0\}$, ($\bar{P}$ denotes the closure of P). Then T has a unique fixed point z and $d(T^n x, z) \rightarrow 0$, $\forall x \in M$.

Rakotch [9] proved that if $\alpha(t)$ is a decreasing function where $t = d(x, y)$, with the property that $\alpha(t) < 1$ for $t > 0$, then the mapping T satisfying (1.2) by taking $\chi(t) = \alpha(t)t$ has a unique fixed point. It is easy to show that if $\alpha(t)$ is an increasing function and $\alpha(t) < 1$ for $t \geq 0$, Banach's Contraction Principle (BCP) still holds. Boyd and Wong proved that the only requirement to hold the BCP is that $\chi(t)$ to be semi-continuous and $\chi(t) < t$ for $t > 0$.

Akkouchi [1] observed that the contractive condition (1.2) above signifies that T must be continuous. He generalized this condition in such a way that no information on continuity of T is needed to derive the results.

Matthews [7] introduced partial metric spaces as the foundation to study the denotational semantics of programming languages whereby a non-Hausdorff topological model is constructed as a system of logic for a programming language. Partial metric is a metric based tools for program verification in which the notion of the size of a data object in a domain is used in quantifying how the data object is well defined in the domain.

The following definitions are due to Matthews [7].

Definition 1.1. [7] Let X be a non-empty set. A function $p : X \times X \rightarrow [0, \infty)$ is called a partial metric on X if it satisfies the followings:

- (P0): $0 \leq p(x, x) \leq p(x, y)$ (non-negativity and small self-distance).
- (P1): $p(x, y) = p(x, x) = p(y, y) \Rightarrow x = y$ (indistancy implies equality).
- (P2): $p(x, y) = p(y, x)$ (symmetric).
- (P3): $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$ (triangularity), for all $x, y, z \in X$.

A pair (X, p) is called a partial metric space.

Note that $p(x, y) = 0$ implies $x = y$ by (P0-P1). In general the converse is not true. Therefore, a metric space is a partial metric space with all self-distances zero.

In computer science applications $p(x, x)$ referred to as the size or weight of x . This quantity structure is used to describe the amount of information contained in x . We say that the data object x is more defined if the value of $p(x, x)$ is smaller and it is totally defined if $p(x, x) = 0$.

Definition 1.2. [7] Let $\{x_n\}$ be a sequence in a partial metric space (X, p) , then,

- (i) A sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ if and only if $p(x, x) = \lim_{n \rightarrow \infty} p(x, x_n) = \lim_{n \rightarrow \infty} p(x_n, x_n)$.

- (ii) A sequence $\{x_n\}$ is called a Cauchy sequence if there exists $a \geq 0$ such that for each $\epsilon > 0$ there exists N such that for all $n, m > N$, $|p(x_n, x_m) - a| < \epsilon$.

In other words, the sequence $\{x_n\}$ is Cauchy if the numbers $p(x_n, x_m)$ converges to some a as n, m approach infinity, that is, if $\lim_{n, m \rightarrow +\infty} p(x_n, x_m) = a$, where a is a finite number.

- (iii) A partial metric space (X, p) is complete if every Cauchy sequence $\{x_n\}$ converges to a point $x \in X$ such that $p(x, x) = \lim_{n, m \rightarrow +\infty} p(x_n, x_m)$.

Definition 1.3. [7] A contraction on a partial metric space X is a function $f : X \rightarrow X$ such that there exist a constant $0 \leq c < 1$ for all $x, y \in X$ satisfies that;

$$p(f(x), f(y)) \leq c(p(x, y)). \quad (1.3)$$

The following lemma is due to Matthews [7],

Lemma 2. [7] Let X be a non-empty set and p be a partial metric on X . Then, the function $p^s : X \times X \rightarrow [0, \infty)$ defined as,

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \quad (1.4)$$

for all $x, y \in X$ is a metric on X .

We take note that, in a partial metric space (X, p) , a convergent sequence may not be a Cauchy sequence. For instance consider Example (1.4) in the work of Nazir and Abbas [8].

The following definition was given by Akkouchi [1].

Definition 1.4. [1] Let Φ be the set of continuous functions $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfying,

- (i) $\phi(t) = 0$ if and only if $t = 0$ and
- (ii) For all sequences $\{t_n\}$ of elements in $[0, \infty)$, if $\{\phi(t_n)\}$ is a decreasing sequence then the sequence $\{t_n\}$ is bounded, that is $\sup_n t_n < \infty$.

We observe that if $\phi : [0, \infty) \rightarrow [0, \infty)$ is a continuous function satisfying either of the following then it must belong to Φ ;

- (iii) ϕ is non decreasing in $[0, \infty)$
- (iv) $\phi(t) \geq Mt^a$ for all $t > 0$ where $M, a > 0$.

The following theorem is due to Akkouchi [1];

Theorem 3. [1] Let (M, d) be a complete metric space and let $P := \{d(x, y) : x, y \in M\}$. Let $\phi \in \Phi$ and $\chi : \overline{\phi(P)} \rightarrow [0, \infty)$ be an upper semi continuous from the right on $\overline{\phi(P)}$ and satisfies $\chi(t) < t$ for all $t \in \overline{\phi(P)} \setminus \{0\}$ where $\overline{\phi(P)}$ denotes the closure of $\phi(P)$. Let T be the mapping of M satisfying the following contractive condition,

$$\phi(d(Tx, Ty)) \leq \chi(\phi(d(x, y))), \forall x, y \in M. \quad (1.5)$$

Then T has a unique fixed point z and $d(T^n x, z) \rightarrow 0$ for all $x \in M$ as $n \rightarrow \infty$.

2. Main Results

We now state and prove the generalization of Theorem 3 and then provide an illustrative example thereafter.

Theorem 4. *Let (X, p) be a complete partial metric space and let $P := \{p(x, y) : x, y \in X\}$. Let $\phi \in \Phi$ and $\chi : \overline{\phi(P)} \rightarrow [0, \infty)$ be an upper semi continuous from the right on $\overline{\phi(P)}$ and let $\chi(t) < t$ for all $t \in \overline{\phi(P)} \setminus \{0\}$ where $\overline{\phi(P)}$ denotes the closure of $\phi(P)$. Let T be a self mapping of X such that*

$$\phi(p(Tx, Ty)) \leq \chi(\phi(p(x, y))) \text{ for all } x, y \in X. \quad (2.1)$$

Then T has a unique fixed point z such that $Tz = z$ where $p(z, z) = 0$.

Proof. (a) First we will construct a sequence of points $x_1, x_2, x_3, \dots$ as follows:

Let a point $x_0 \in X$, and set $x_n = Tx_{n-1}$ for all integers $n \geq 1$.

Also consider, $a_n = p(x_n, x_{n+1})$ for all $x_n \in X$ and all integers $n \geq 0$.

For all $n \geq 0$ we have,

$$\phi(a_{n+1}) = \phi(p(x_{n+1}, x_{n+2})).$$

By considering (2.1) we can have,

$$\phi(p(x_{n+1}, x_{n+2})) = \phi(p(Tx_n, Tx_{n+1})) \leq \chi(\phi(p(x_n, x_{n+1}))). \quad (2.2)$$

Hence, $\phi(a_{n+1}) \leq \chi(\phi(a_n)) \leq \phi(a_n)$, for all integers $n \geq 0$.

This implies that $\{\phi(a_n)\}$ is a decreasing sequence for all integers $n \geq 0$.

Now consider, a be the limit point for the sequence $\{\phi(a_n)\}$, it is obvious that $a \in [0, \infty)$. We claim that $a = 0$. Suppose in contrary that $a > 0$, then from (2.1) we have,

$$\phi(p(x_{n+1}, x_{n+2})) = \phi(p(Tx_n, Tx_{n+1})) \leq \chi(\phi(p(x_n, x_{n+1}))).$$

Thus,

$$\phi(a_{n+1}) \leq \chi(\phi(a_n))$$

so that;

$$0 < a \leq \limsup_{n \rightarrow \infty} \chi(\phi(a_n)) \leq \chi(a), \quad (2.3)$$

which is a contradiction on the assumption of the function χ , hence

$$\lim_{n \rightarrow \infty} \phi(a_n) = a = 0.$$

Since the sequence $\{\phi(a_n)\}$ is decreasing and satisfies (ii) of Definition 1.4 such that $\lim_{n \rightarrow \infty} a_n < \infty$, then the sequence $\{a_n\}$ is bounded for all integers $n \geq 0$.

Since the sequence $\{a_n\}$ is bounded, then it contains a convergent subsequence say $\{a_{n_k}\}$ such that $\lim_{k \rightarrow \infty} a_{n_k} = b$.

Since ϕ is continuous it follows that,

$$\lim_{k \rightarrow \infty} \phi(a_{n_k}) = \phi(b) = 0.$$

From (i) of Definition 1.4 we have, $b = 0$.

Since we choose the sequence $\{a_{n_k}\}$ arbitrarily, then we can conclude that any convergent subsequence $\{a_{n_k}\}$ of the sequence $\{a_n\}$ converges to zero, hence,

$$\lim_{n \rightarrow \infty} a_n = 0. \quad (2.4)$$

We have proved that the sequence $\{a_n\}$ is bounded and converges to zero.

(b) Now consider the sequence $\{x_n\} \in X$. We want to show that the sequence $\{x_n\}$ is a Cauchy sequence.

In contrary suppose that $\{x_n\}$ is not a Cauchy sequence, therefore, $\exists \epsilon > 0$ and sequences of integers $\{m(r)\}, \{n(r)\}$ such that;

$$p(x_{n(r)}, x_{m(r)}) \geq \epsilon \quad (2.5)$$

for all $n(r) > m(r) \geq r$ for some $r \geq 0$.

Moreover, suppose that $m(r)$ is chosen as the smallest integer in such a way that inequality (2.5) above holds so that we have,

$$p(x_{n(r)}, x_{m(r)-1}) < \epsilon. \quad (2.6)$$

Now for all positive integers $n(r) > m(r)$ we have the following triangle inequality;

$$\begin{aligned} P_r &= p(x_{n(r)}, x_{m(r)}) \\ &\leq p(x_{n(r)}, x_{m(r)-1}) + p(x_{m(r)-1}, x_{m(r)}) - p(x_{m(r)-1}, x_{m(r)-1}) \\ &\leq p(x_{n(r)}, x_{m(r)-1}) + p(x_{m(r)-1}, x_{m(r)}). \end{aligned} \quad (2.7)$$

Considering (2.6) we obtain,

$$P_r = p(x_{n(r)}, x_{m(r)}) \leq \epsilon + a_{m(r)}. \quad (2.8)$$

As $r \rightarrow \infty$ in (2.8) and considering (2.4) we see that $a_{m(r)} \rightarrow 0$, hence, $P_r \rightarrow \epsilon$.

But now, for all positive integers $n(r) > m(r)$ we have,

$$\begin{aligned} P_r &= p(x_{n(r)}, x_{m(r)}) \\ &\leq p(x_{n(r)}, x_{n(r)+1}) + p(x_{n(r)+1}, x_{m(r)+1}) + p(x_{m(r)+1}, x_{m(r)}) \\ &\quad - p(x_{n(r)+1}, x_{n(r)+1}) \\ &\leq p(x_{n(r)}, x_{n(r)+1}) + p(x_{n(r)+1}, x_{m(r)+1}) + p(x_{m(r)+1}, x_{m(r)}). \end{aligned} \quad (2.9)$$

Now,

$$\begin{aligned} P_r &= p(x_{n(r)}, x_{m(r)}) \leq a_{n(r)} + p(x_{n(r)+1}, x_{m(r)+1}) + a_{m(r)} \\ &\leq a_{n(r)} + a_{m(r)} + \chi(p(x_{n(r)+1}, x_{m(r)+1})). \end{aligned} \quad (2.10)$$

As $r \rightarrow \infty$ in (2.10) and considering (2.4) we see that $a_{m(r)} \rightarrow 0$ and $a_{n(r)} \rightarrow 0$, hence we obtain, $\epsilon \leq \chi(\epsilon)$, which is a contradiction for $\epsilon > 0$.

Therefore, the sequence $\{x_n\}$ is a Cauchy sequence. Consequently,

$$\lim_{n, m \rightarrow \infty} p(x_n, x_m) = 0. \quad (2.11)$$

(c) Since X is complete then we can choose a point $z \in X$ such that z is the limit of the sequence $\{x_n\}$ and $\lim_{n \rightarrow \infty} p(x_n, z) = 0$. Now we prove that z is the fixed point of T and $p(z, z) = 0$.

By considering (2.1), for all integers $n \geq 0$ we have,

$$\phi(p(x_{n+1}, Tz)) = \phi(p(Tx_n, Tz)) \leq \chi(\phi(p(x_n, z))) \quad (2.12)$$

Letting $n \rightarrow \infty$ in (2.12) above we get;

$$\begin{aligned} \phi(p(z, Tz)) &= \lim_{n \rightarrow \infty} \phi(p(x_{n+1}, Tz)) \\ &\leq \limsup_{n \rightarrow \infty} \chi(\phi(p(x_n, z))) \\ &= \chi(\phi(0)) \\ &\leq \phi(0) = 0. \end{aligned} \quad (2.13)$$

From (2.13) above we have,

$$\phi(p(z, Tz)) = 0.$$

By using (i) of Definition 1.4 we see that,

$$p(z, Tz) = 0. \quad (2.14)$$

Hence by (P0) and (P1) we conclude that $z = Tz$, hence z is a fixed point of T and $p(z, z) = 0$.

(d) We now show that z is the unique fixed point of T . Suppose that there exists another fixed point $z_0 \neq z$ of T . From our contractive condition (2.1), we have,

$$\begin{aligned} 0 < \phi(p(z_0, z)) &= \phi(p(Tz_0, Tz)) \\ &\leq \chi(\phi(p(z_0, z))) \\ &< \phi(p(z_0, z)) \end{aligned} \quad (2.15)$$

which is a contradiction, hence z is a unique fixed point of T .

Remark 5. By setting the function ϕ to be the identity mapping in Theorem 4, then (2.1) which is,

$$\phi(p(Tx, Ty)) \leq \chi(\phi(p(x, y)))$$

becomes,

$$p(Tx, Ty) \leq \chi(p(x, y)),$$

which leads to the following corollary:

Corollary 6. Let (X, p) is a complete partial metric space. Let $\chi : [0, \infty) \rightarrow [0, \infty)$ be an upper semi continuous mapping from the right on $[0, \infty)$ such that $\chi(t) < t$ for all $t > 0$. Let $T : X \rightarrow X$ be a mapping satisfying,

$$p(Tx, Ty) \leq \chi(p(x, y))$$

for all $x, y \in X$. Then there exists a unique fixed point $x_0 \in X$ for mapping T where $p(x_0, x_0) = 0$.

We observe that, Corollary 6 above is the generalization of Boyd and Wong [3] fixed point theorem in partial metric spaces which is the work of Romaguera [10].

Remark 7. By setting the function ϕ to be an identity mapping in Theorem 4 and letting $\chi(\alpha) = k\alpha$ for $k \in [0, 1)$ and $\alpha \geq 0$, then (2.1) which is,

$$\phi(p(Tx, Ty)) \leq \chi(\phi(p(x, y)))$$

becomes,

$$p(Tx, Ty) \leq kp(x, y)$$

which leads to the following corollary:

Corollary 8. Let (X, p) be a complete partial metric space and let $T : X \rightarrow X$ be a mapping such that,

$$p(Tx, Ty) \leq kp(x, y)$$

for all $x, y \in X$ and $k \in [0, 1)$. Then T has a unique fixed point z such that $Tz = z$ and $p(z, z) = 0$.

Which is the Banach fixed point theorem in partial metric space by Matthews [7].

Now, we present an example to show that our results improves the existing results in the literature as there is no any information given on the continuity of mapping T in the process of deriving the results.

Example 2.1. Let $X = [0, \infty)$ be a set with partial metric $p : X \times X \rightarrow [0, \infty)$ which is defined by,

$$p(x, y) = \max\{x, y\} = t$$

for all $x, y \in X$.

Using $(P0 - P3)$ it is easy to check that (X, p) is partial metric space. It is easy to check that the space (X, p) is a complete partial metric space. In fact p^* from Lemma 2 is the Euclidean metric on X .

Define,

$$Tx = \begin{cases} x - \frac{1}{4}x^2, & x \in [0, 1), \\ x - 1, & x \in [1, \infty). \end{cases}$$

Also we define,

$$\phi(t) = \begin{cases} t - \frac{1}{2}t^2, & t \in [0, 1), \\ t - \frac{1}{2}, & t \in [1, \infty). \end{cases}$$

We can observe that, $\phi(t)$ is a continuous and non decreasing function, then from Definition 1.4, ϕ belongs to Φ .

Thus for all $x, y \in [0, 1)$ we have,

$$\begin{aligned} \phi(p(Tx, Ty)) &= \phi(\max\{Tx, Ty\}) \\ &= \phi(\max\{x - \frac{1}{4}x^2, y - \frac{1}{4}y^2\}) \\ &\leq \phi(\max\{x, y\}) \\ &= \phi(p(x, y)) \\ &= \phi(t) = t - \frac{1}{2}t^2. \end{aligned} \tag{2.16}$$

Similarly for $x, y \in [1, \infty)$ we have,

$$\begin{aligned}
 \phi(p((Tx, Ty))) &= \phi(\max\{Tx, Ty\}) \\
 &= \phi(\max\{x-1, y-1\}) \\
 &\leq \phi(\max\{x, y\}) \\
 &= \phi(p(x, y)) \\
 &= \phi(t) = t - \frac{1}{2}.
 \end{aligned} \tag{2.17}$$

Thus if we define our mapping χ by,

$$\chi(\phi(t)) = \begin{cases} t - \frac{1}{2}t^2, & t \in [0, 1), \\ t - \frac{1}{4}, & t \in [1, \infty). \end{cases}$$

Then $\chi(\phi(t))$ is upper semi-continuous from right on $[0, \infty) = \overline{\phi(P)}$, $\chi(t) < t$, for all $t \in \overline{\phi(P)} \setminus \{0\}$ and condition (2.1) holds. Therefore our mapping T has a unique fixed point $x = 0$ such that $T(0) = 0$ and $p(0, 0) = \max\{0, 0\} = 0$.

In the following theorem we obtain a unique common fixed point for a pair of maps by using generalized Boyd and Wong type contractive condition on complete partial metric spaces. This result is the consequence of Theorem 4, besides here we employ a pair of maps instead of one map.

Before stating our theorem, we give the following lemma which results from Definition 1.1.

Lemma 9. Let (X, p) be a complete partial metric space,

- (i) If $p(x, y) = 0$, then $x = y$.
- (ii) If $x \neq y$, then $p(x, y) > 0$.

Note that the proof of the above Lemma follows from Definition 1.1.

Theorem 10. Let (X, p) be a complete partial metric space and let $P := \{p(x, y) : x, y \in X\}$. Let $\phi \in \Phi$ and $\chi : \overline{\phi(P)} \rightarrow [0, \infty)$ be an upper semi continuous from the right on $\overline{\phi(P)}$ and let $\chi(t) < t$ for all $t \in \overline{\phi(P)} \setminus \{0\}$ where $\overline{\phi(P)}$ denotes the closure of $\phi(P)$. Let T and F be self mappings of X such that:

$$\phi(p(Tx, Fy)) \leq \chi(\phi(S(x, y))), \tag{2.18}$$

for all $x, y \in X$, where $S(x, y) = \max\{p(x, y), p(x, Tx), p(y, Fy), \frac{p(x, Fy) + p(y, Tx)}{2}\}$. Then T and F have a unique common fixed point $z \in X$ such that $\text{Fix } T = \text{Fix } F = \{z\}$.

Proof. Let $x_0 \in X$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ and $x_{n+2} = Fx_{n+1}$, for all $n = 0, 1, 2, 3, \dots$

Now, if there exists a positive integer N such that, $x_N = x_{N+1}$, then x_N is a fixed point of T . Similarly, if there exists an integer $n \geq 0$ such that $x_{n+1} = x_{n+2}$, then x_{n+1} is a fixed point of F . Here, we suppose that $x_n \neq x_{n+1}$, for all integers $n \geq 0$. Let $a_n = p(x_n, x_{n+1})$, for all $\{x_n\} \in X$. For all integers $n \geq 0$, $\phi(a_{n+1}) = \phi(p(x_{n+1}, x_{n+2}))$. Considering (2.18) we have,

$$\begin{aligned}
 &\phi(a_{n+1}) \\
 &= \phi(p(x_{n+1}, x_{n+2})) = \phi(p(Tx_n, Fx_{n+1})) \leq \chi(\phi(S(x_n, x_{n+1})))
 \end{aligned}$$

$$\begin{aligned}
&= \chi(\phi(\max\{p(x_n, x_{n+1}), p(x_n, Tx_n), p(x_{n+1}, Fx_{n+1}), \frac{p(x_n, Fx_{n+1}) + p(x_{n+1}, Tx_n)}{2}\})) \\
&= \chi(\phi(\max\{p(x_n, x_{n+1}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \frac{p(x_n, x_{n+2}) + p(x_{n+1}, x_{n+1})}{2}\})). \quad (2.19)
\end{aligned}$$

Since,

$$\begin{aligned}
&\frac{p(x_n, x_{n+2}) + p(x_{n+1}, x_{n+1})}{2} \\
&\leq \frac{p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) - p(x_{n+1}, x_{n+1}) + p(x_{n+1}, x_{n+1})}{2} \\
&= \frac{p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})}{2},
\end{aligned}$$

then,

$$\begin{aligned}
\phi(p(x_{n+1}, x_{n+2})) &= \phi(p(Tx_n, Fx_{n+1})) \\
&\leq \chi(\phi(\max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\})) \quad (2.20)
\end{aligned}$$

If $\max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} = p(x_{n+1}, x_{n+2})$, then,

$$\phi(p(x_{n+1}, x_{n+2})) = \phi(p(Tx_n, Fx_{n+1})) \leq \chi(\phi(p(x_{n+1}, x_{n+2}))) < \phi(p(x_{n+1}, x_{n+2})), \quad (2.21)$$

which is a contradiction. Hence $\max\{p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2})\} = p(x_n, x_{n+1})$. Thus,

$$\phi(p(x_{n+1}, x_{n+2})) = \phi(p(Tx_n, Fx_{n+1})) \leq \chi(\phi(p(x_n, x_{n+1}))) < \phi(p(x_n, x_{n+1})), \quad (2.22)$$

which implies that the sequence $\{\phi(a_n)\}$ is a decreasing sequence in $[0, \infty)$, for all integers $n \geq 0$ and it is bounded below.

Let a be the limit point of the sequence $\{\phi(a_n)\}$. It is obvious that $a \in [0, \infty)$.

We claim that $a = 0$. In contrary suppose that $a > 0$, applying $\limsup$ in (2.22) we obtain,

$$\begin{aligned}
0 < a &\leq \limsup_{n \rightarrow \infty} \chi(\phi(a_n)) \\
&\leq \chi(a) < a \quad (2.23)
\end{aligned}$$

which contradicts the assumption on the mapping χ , hence $a = 0$. Therefore, $\lim_{n \rightarrow \infty} \phi(a_n) =$

$$\lim_{n \rightarrow \infty} \phi(p(x_n, x_{n+1})) = 0.$$

It is clear that, $\lim_{n \rightarrow \infty} \phi(a_n) = \phi(\lim_{n \rightarrow \infty} a_n) = 0$. From the definition of ϕ we see that,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} p(x_n, x_{n+1}) = 0. \quad (2.24)$$

Now, we will show that a sequence $\{x_n\}$ is a Cauchy sequence in X . In contrary suppose that $\{x_n\}$ is not a Cauchy sequence. Therefore, there exists $\epsilon > 0$ and a sequence of integers $m(k), n(k)$ such that,

$$p(x_{n(k)}, x_{m(k)}) \geq \epsilon, \quad (2.25)$$

for all $n(k) > m(k) \geq k$ for some $k \geq 0$.

Moreover, suppose that $m(k)$ is chosen as the smallest integer in such away that (2.25) holds so that we have,

$$p(x_{(k)}, x_{m(k)-1}) < \epsilon. \quad (2.26)$$

Now, for all $n(k) > m(k)$ we have,

$$\begin{aligned} p(x_{n(k)}, x_{m(k)}) &\leq p(x_{n(k)}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) - p(x_{m(k)-1}, x_{m(k)-1}) \\ &\leq p(x_{n(k)}, x_{m(k)-1}) + p(x_{m(k)-1}, x_{m(k)}) \\ &\leq p(x_{n(k)}, x_{m(k)-1}) + a_{n(k)}. \end{aligned} \quad (2.27)$$

As $k \rightarrow \infty$ in (2.27) and considering (2.24) we see that,

$$p(x_{n(k)}, x_{m(k)}) \rightarrow \epsilon. \quad (2.28)$$

Similarly, by elementary computations we see that,

$$p(x_{n(k)-1}, x_{m(k)-1}) \rightarrow \epsilon. \quad (2.29)$$

From (2.18) we have,

$$\begin{aligned} \phi(p(x_{n(k)}, x_{m(k)})) &= \phi(p(Tx_{n(k)-1}, Fx_{m(k)-1})) \\ &\leq \chi(\phi(S(x_{n(k)-1}, x_{m(k)-1}))), \end{aligned} \quad (2.30)$$

where,

$$\begin{aligned} S(x_{n(k)-1}, x_{m(k)-1}) &= \max\{p(x_{n(k)-1}, x_{m(k)-1}), p(x_{n(k)-1}, Tx_{n(k)-1}), p(x_{m(k)-1}, Fx_{m(k)-1}), \\ &\quad \frac{p(x_{n(k)-1}, Fx_{m(k)-1}) + p(x_{m(k)-1}, Tx_{n(k)-1})}{2}\} \\ &= \max\{p(x_{n(k)-1}, x_{m(k)-1}), p(x_{n(k)-1}, x_{n(k)}), p(x_{m(k)-1}, x_{m(k)}), \\ &\quad \frac{p(x_{n(k)-1}, x_{m(k)}) + p(x_{m(k)-1}, x_{n(k)})}{2}\}, \end{aligned} \quad (2.31)$$

as $k \rightarrow \infty$ in (2.31) and considering (2.28) and (2.29), then (2.30) becomes,

$$0 < \phi(\epsilon) \leq \chi(\phi(\epsilon)) < \phi(\epsilon). \quad (2.32)$$

Since $\phi(\epsilon) \in \overline{\phi(P) \setminus \{0\}}$ and $\chi(t) < t$, for all $t > 0$, then (2.32) contains a contradiction. Therefore, the sequence $\{x_n\}$ is a Cauchy sequence and $\lim_{n,m \rightarrow \infty} p(x_n, x_m) = 0$.

Since X is complete, then we can choose a point $z \in X$ such that $\lim_{n \rightarrow \infty} p(x_n, z) = 0$. Now, we prove that z is a fixed point of F . In contrary suppose that $z \neq Fz$.

From (2.18) we have,

$$\begin{aligned} \phi(p(x_{n+1}, Fz)) &= \phi(p(Tx_n, Fz)) \\ &\leq \chi(\phi(S(x_n, z))). \end{aligned} \quad (2.33)$$

where,

$$\begin{aligned} S(x_n, z) &= \max\{p(x_n, z), p(x_n, Tx_n), p(z, Fz), \frac{p(x_n, Fz) + p(z, Tx_n)}{2}\} \\ &= \max\{p(x_n, z), p(x_n, x_{n+1}), p(z, Fz), \frac{p(x_n, Fz) + p(z, x_{n+1})}{2}\}, \end{aligned} \quad (2.34)$$

taking limit as $n \rightarrow \infty$ in (2.34) we get,

$$S(x_n, z) \rightarrow p(z, Fz). \quad (2.35)$$

Applying limit as $n \rightarrow \infty$ in (2.33) we have,

$$\phi(p(z, Fz)) \leq \chi(\phi(p(z, Fz))) < \phi(p(z, Fz)) \quad (2.36)$$

which is a contradiction, hence $z = Fz$.

Now, we shall show that a point z is a unique common fixed of T and F .

In contrary, suppose that T and F have two common fixed points x_0 and y_0 in X with $x_0 \neq y_0$ respectively. Thus, from Lemma 9 we have, $p(x_0, y_0) > 0$.

From (2.18), for all $x_0, y_0 \in X$ we have,

$$\phi(p(Tx_0, Fy_0)) = \phi(p(x_0, y_0)) \leq \chi(\phi(S(x_0, y_0))) \quad (2.37)$$

where,

$$\begin{aligned} S(x_0, y_0) &= \max\{p(x_0, y_0), p(x_0, Tx_0), p(y_0, Fy_0), \frac{p(x_0, Fy_0) + p(y_0, Tx_0)}{2}\} \\ &= \max\{p(x_0, y_0), p(x_0, x_0), p(y_0, y_0), \frac{p(x_0, y_0) + p(y_0, x_0)}{2}\} \\ &= p(x_0, y_0). \end{aligned} \quad (2.38)$$

Thus, (2.37) becomes,

$$\phi(p(Tx_0, Fy_0)) = \phi(p(x_0, y_0)) \leq \chi(\phi(p(x_0, y_0))) < \phi(p(x_0, y_0)).$$

which is a contradiction. Therefore, T and F have a unique common fixed point, that is $x_0 = y_0$.

In contrary to the hypothesis of the above theorem, suppose that x_0 is a fixed point of T and $x_0 \neq Fx_0$.

From (2.18), for all $x_0 \in X$ we have,

$$\phi(p(Tx_0, Fx_0)) = \phi(p(x_0, Fx_0)) \leq \chi(\phi(S(x_0, Fx_0))) \quad (2.39)$$

where,

$$\begin{aligned} S(x_0, Fx_0) &= \max\{p(x_0, Fx_0), p(x_0, Tx_0), p(Fx_0, F^2x_0), \frac{p(x_0, F^2x_0) + p(Fx_0, Tx_0)}{2}\} \\ &= \max\{p(x_0, Fx_0), p(x_0, x_0), p(Fx_0, F^2x_0), \frac{p(x_0, F^2x_0) + p(Fx_0, x_0)}{2}\} \\ &= p(x_0, Fx_0). \end{aligned} \quad (2.40)$$

Thus, (2.39) becomes,

$$\phi(p(Tx_0, Fx_0)) = \phi(p(x_0, Fx_0)) \leq \chi(\phi(p(x_0, Fx_0))) < \phi(p(x_0, Fx_0)) \quad (2.41)$$

which is a contradiction, hence $x_0 = Fx_0$. Similarly, it is easy to show that any fixed point of F is also a fixed point of T .

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