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Controllability of some bilinear and semilinear parabolic problems

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ABSTRACT. We present in this paper a survey of recent results on the controllability of the parabolic system governed by bilinear control. We first discuss the problem of global controllability which corresponds to the question of whether the solution of the system can be driven to a given state at a some finite time by means of a control. We give some results on the global controllability of bilinear and semilinear reaction-diffusion equations. After this we introduce the case of partial controllability with the control acting on a subregion of the domain. Illustrative examples are also provided.

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1. Introduction

Parabolic problems are one of the fields of mathematics which undergoes a detailed investigation, due to the many problems which rely on this theory. Parabolic equations describe various time dependent models of many physical, chemical and biological phenomena [2, 3, 15, 19]. The goal of this paper is to give some results on the multiplicative controllability of parabolic

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system. In particular we study the multiplicative controllability of the semilinear and bilinear reaction-diffusion equation governed in a bounded domain.

The word *control* has a double meaning. First, controlling a system can be understood simply as testing or checking that its behavior is satisfactory. In a deeper sense, to control is also to act, to put things in order to guarantee that the system behaves as desired. The multiplicative (bilinear) controls are essential in modeling reaction-diffusion convection processes controlled by means of so-called catalysts that can accelerate or decelerate the reaction at hand, e.g., various chemical or biological chain reactions. The concept of controllability is a principal component of control theory which was brought to life in the 1950's by numerous applications in engineering, and has received the most significant attention both from the engineering and the mathematical communities since then.

Controllability theory was originally developed in the 1960's for the linear ordinary differential equations, governed by the additive controls. Later, since the 1970's it became the subject of keen interest for the researchers working in the area of partial differential equations as well. The controllability theory of linear pde's with additive controls has received a significant attention in the literature in the last four decades. Many powerful mathematical methods were introduced and adopted to tackle a variety of controllability problems along the above-outlined duality approach, including harmonic and nonharmonic analysis, the multiplier method, unique continuation, Carleman's estimates and microlocal analysis.

The duality approach developed to study controllability properties of the linear system does not apply to the bilinear system. This gives rise to the need for a new methodology to study controllability of pde's, governed by multiplicative controls.

Historically, the first result concerning the bilinear control in infinite dimension is of particular importance. In 1982 Ball J.M., Marsden J.E., Slemrod M [1] have demonstrated that for an abstract bilinear control system the set of states attainable from any initial state is of empty interior in the state space. The approach is based on the generalized inverse function and the concepts of nonharmonic Fourier series and Riesz bases. Because of this result the bilinear systems have long time been considered uncontrollable. Several interpretations were then given to this negative result of [1]. Indeed it can be explained by an inappropriate choice of control space and desired states. However this negative result, although interesting does not close the question of controllability. Indeed several works have been achieved in the context different type of equations.

An extension which is very important in practical applications is that of partial controllability which consists on controlling only a part of the state. For example in an industrial furnace, it may be that the control is only required to maintain the temperature at a certain level in a prescribed subregion of the furnace. The notion of partial controllability (also called regional controllability) has been introduced firstly for linear systems by El Jai et al [5, 6].

The remainder of this paper is organized as follows. Section 2 is devoted to the approximate controllability of the semilinear system, the first result are concerned with the global nonnegative approximate controllability property of a rather general semilinear heat equation with superlinear term, governed in a bounded domain by a multiplicative (bilinear), after we present some result of approximate controllability properties of a one dimensional semilinear reaction-diffusion equation governed via the coefficient of the reaction term. It is assumed

that both the initial and target states admit no more than finitely many changes of sign. Section 3 deals with the global controllability of the bilinear system, we beginning this section with the global approximate controllability of the multidimensional bilinear reaction diffusion system we consider the case when the points of change of sign of the given initial state y_0 coincide with those of the target state, then we consider the case when the initial and target states are allowed to have the same signs and an infinite number of points of sign change. Then after these results we consider the problem of exact controllability of the bilinear system, and we will show that the steering time can be arbitrary small and uniform for all initial and reached states. Section 4 present new result of the exact multiplicative controllability of semilinear reaction-diffusion is considered in several space dimensions. Section 5 deals with the partial controllability of the bilinear system governed by a locally supported control. This consists on achieving a desired state only on a privileged subregion of the geometrical domain of the system at hand. In Section 6 we give the some examples and simulation one the partial controllability of the bilinear system and the exact controllability of the semilinear system.

2. Multiplicative controllability of the semilinear system

2.1. Approximate controllability. The following semilinear system has been considered by khapalov in [14].

$$\begin{cases} y_t = \Delta y + vy - f(x, t, y, \nabla y) & \text{in } Q_T = \Omega \times (0, T), T > 0 \\ y(0, t) = 0, & \text{on } \Sigma_T = \partial\Omega \times (0, T), \\ y|_{t=0} = y_0(x) \in L^2(\Omega), & \text{in } \Omega \end{cases} \quad (2.1)$$

where Ω is a bounded domain of $\mathbb{R}^n$ and $v \in L^\infty(Q_T)$ is a multiplicative (bilinear) control. It is also assumed that the function f satisfies the following conditions:

- $f(x, t, q, p)$ is Lebesgue's measurable in x, t, q, p is continuous in q, p for almost all $(x, t) \in Q_T$;
- there exist a non-negative function ψ in $L^{1+n/(n+4)}(\Omega)$, $\beta > 0$, and

$$r_1 \in \left[0, 1 + \frac{4}{n}\right), \quad r_2 \in \left[0, 1 + \frac{2}{n+2}\right) \quad (2.2)$$

such that a.e. in Q_T for $q \in \mathbb{R}$, $p \in \mathbb{R}^n$

$$|f(x, t, q, p)| \leq \psi(x, t) + \beta|q|^{r_1} + \beta\|p\|_{\mathbb{R}^n}^{r_2}; \quad (2.3)$$

- there exist $\rho > 0$ and $\nu > 0$ such that

$$\int_{\Omega} f(x, t, \phi, \nabla \phi) \phi dx \geq (\nu - 1) \int_{\Omega} \|\nabla \phi\|_{\mathbb{R}^n}^2 dx - \rho \int_{\Omega} (1 + \phi^2) dx \quad \forall \phi \in H_0^1(\Omega). \quad (2.4)$$

Then, it has been shown that any nonnegative target state in $L^2(\Omega)$ can approximately be reached from any nonnegative, nonzero initial state by applying at most three static bilinear $L^\infty(\Omega)$ -controls subsequently in time. More precisely, the following result has been established:

Theorem 2.1. [14] Let $T > 0$ be given and the pair of the initial target states $y_0 \in H_0^1(\Omega)$ and $y_d \in L^2(\Omega)$ be such that

$$\frac{y_d}{y_0} \in H^2(\Omega), \quad \nabla \left(\frac{y_d}{y_0} \right) \in [L^\infty(\Omega)]^n, \quad \Delta \left(\frac{y_d}{y_0} \right) \in L^\infty(\Omega), \quad (2.5)$$

and

$$0 < c_1 \leq \frac{y_d(x)}{y_0(x)} \leq c_2 < 1 \quad \text{a.e. in } \Omega, \quad (2.6)$$

where c_1 and c_2 are some positive constants. Then for every $\varepsilon > 0$ there is a $T_* \in (0, T)$ such that for all, i.e., possibly multiple solutions to (2.1), (2.2)-(2.4) generated by control

$$v(x) = \frac{1}{T_*} \ln \left(\frac{y_d(x)}{y_0(x)} \right), \quad (2.7)$$

we have

$$\|y(\cdot, T_*) - y_d\|_{L^2(\Omega)} \leq \varepsilon. \quad (2.8)$$

Remark 2.1. The approach used in [14] is based on asymptotic technique and consists on suppressing the effect of a (general) nonlinear term and then make use of the pure diffusion and/or pure reaction parts of the dynamics of the system at hand.

In the above result, the initial and target states admit no more than finitely many changes of sign. In [4] Cannarsa, Floridia, and Khapalov study the global approximate controllability properties of a one dimensional semilinear Dirichlet boundary value problem:

$$\begin{cases} y_t = y_{xx} + v(x, t)y - f(y), & \text{in } Q_T = (0, 1) \times (0, T), T > 0, \\ y(0, t) = y(1, t) = 0, & t \in (0, T), \\ y|_{t=0} = y_0(x) \in H_0^1(0, 1). \end{cases} \quad (2.9)$$

Here $v \in L^\infty(Q_T)$ is a bilinear control.

The nonlinear term $f : \mathbb{R} \rightarrow \mathbb{R}$ is such that:

- $f(0) = 0$, differentiable at 0.
- f is supposed to be a Lipschitz and L will denote a Lipschitz constant for

$$|f(y') - f(y)| \leq L|y' - y| \quad \forall y, y' \in \mathbb{R}.$$

It is well-known that (2.9) can be linked to various applied reaction-diffusion models such as chemical reactions, nuclear chain reactions, and biomedical models. More generally, reaction-diffusion equations or systems describe how the concentration of one or more substances changes under the influence of some processes such as local chemical reactions, where substances are transformed into each other, and diffusion which causes substances to spread out in space (see e.g. [4, 7, 13, 14]).

Remark 2.2. The system (2.9) cannot be steered anywhere from the origin. Moreover, if $y_0(x) \geq 0$ in $(0, 1)$, then the strong maximum principle demands that the respective solution to (2.9) remains nonnegative at any moment of time, regardless of the choice of v . This means that system (2.9) cannot be steered from any such y_0 to any target state which is negative on a nonzero measure set in the space domain. We remark that the strong maximum principle for linear parabolic PDEs can be extend to

semi-linear parabolic system (2.9). Indeed, since $f(0) = 0$ and $f(y)$ is differentiable at 0, the term $f(y(x, t))$ in (2.9) can be represented as $\beta(x, t)y(x, t)$, where $\beta(y) = \frac{f(y)}{y} \in L^\infty(Q_T)$.

Here we present the following result for system (2.9).

Theorem 2.2. [4] Let $y_0 \in H_0^1(0, 1)$. Assume that y_0 has finitely many points of sign change. Consider any $y_d \in H_0^1(0, 1)$ which has exactly as many points of sign change in the same order as y_0 . Then, for any $\eta > 0$ there are a $T = T(\eta, y_0, y_d) > 0$ and a piecewise static multiplicative control $v = v(\eta, y_0, y_d) \in L^\infty(Q_T)$ such that for the respective solution y to (2.9) the following inequality holds

$$\|y(\cdot, T) - y_d\|_{L^2(0,1)} \leq \eta.$$

Figure 1 : Control of two points of sign change.

Remark 2.3. Note that in the works above, the time of steering depends on the given initial and target states.

Corollary 2.1. [4] Let y_0 and y^* be given in $L^2(0, 1)$. Then for any $\eta > 0$ there exists $y_0^\eta \in H_0^1(0, 1)$ such that $\|y_0^\eta - y_0\|_{L^2(0,1)} < \eta$ and there exist $T = T(\eta, y_0, y^*) > 0$ and a piecewise static multiplicative control $v = v(\eta, y_0, y^*) \in L^\infty(Q_T)$ such that the solution y to

$$\begin{cases} y_t = y_{xx} + v(x, t)y + f(y), & \text{in } Q_T = (0, 1) \times (0, T), T > 0, \\ y(0, t) = y(1, t) = 0, & t \in (0, T), \\ y(\cdot, 0) = y_0^\eta(x) \in H_0^1(0, 1). \end{cases} \quad (2.10)$$

satisfy the following inequality $\|y(\cdot, T) - y^*\|_{L^2(0,1)} \leq \eta$.

2.2. Exact controllability. In [12] we consider the following semilinear reaction-diffusion equation

$$\begin{cases} y_t = \Delta y + q(x, t)y + f(y), & \text{in } Q_T \ (T > 0) \\ y(0, t) = 0, & \text{on } \Sigma_T \\ y(x, 0) = y_0(x), & \text{in } \Omega \end{cases} \quad (2.11)$$

where Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with a boundary $\partial\Omega$, $Q_T = \Omega \times (0, T)$ and $\Sigma_T = \partial\Omega \times (0, T)$. Here, $q \in L^\infty(Q_T)$ is a control function with corresponding solution $y = y(x, t)$. The nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be a Lipschitz function such that $f(0) = 0$.

In this part, we are interested in the exact multiplicative controllability of the semilinear reaction diffusion system (2.11) at a priori given time, when the initial and target states have the

same sign at almost every $x \in \Omega$ and satisfy $\ln \frac{y_d}{y_0} \in L^\infty(\Omega)$. Our exact controllability result for semilinear case is as follows:

Theorem 2.3. [12] Let $y_0 \in L^p(\Omega)$, ($p \geq 2$ and $p > \frac{n}{2}$) and let $y_d \in H^2(\Omega)$ such that

(i) for a.e. $x \in \Omega$, $y_0(x)y_d(x) \geq 0$ and $y_d(x) = 0 \Leftrightarrow y_0(x) = 0$,

(ii) $\ln(\frac{y_d}{y_0})\mathbf{1}_{E_{y_0}} \in L^\infty(\Omega)$, where $\mathbf{1}_{E_{y_0}}$ denotes the characteristic function of the set $E_{y_0} = \{x \in \Omega / y_0(x) \neq 0\}$,

(iii) $\frac{\Delta y_d}{y_d}\mathbf{1}_{E_{y_d}} \in L^\infty(\Omega)$ and $|y_d| \geq \alpha > 0$, a.e. on some open subset $\omega \subset \Omega$.

Then there exists a control $q(\cdot, \cdot) \in L^\infty(Q_T)$ for which the respective solution to (2.11) is such that $y(T) = y_d$.

Remark 2.4.

- (1) The exact steering (for a given couple of initial and desirable states) at arbitrary small control time which can be fixed in advance.
- (2) The approach is constructive and consists in using a set of three controls applied subsequently in time.

3. Global controllability of the bilinear system

In the case we have the following bilinear system:

$$\begin{cases} y_t = \Delta y + v(x, t)y, & \text{in } Q_T = \Omega \times (0, T_0), T_0 > 0 \\ y(0, t) = 0, & \text{on } \Sigma_T \\ y(x, 0) = y_0(x), & \text{in } \Omega \end{cases} \quad (3.1)$$

The standard problem of controllability of system like (3.1) consists on finding a suitable available control v such that the solution of the system, starting from the initial state y_0 and provided with this control function reaches the state y_d at a time $T > 0$.

3.1. Global approximate controllability. In the case of this heat equation with the homogeneous Dirichlet boundary conditions, governed via the coefficient in the reaction term (3.1), the maximum principle implies that if the initial condition is nonnegative, then its solution will remain nonnegative, regardless of the choice of control.

If, for example, $y_0(x) \geq 0$ in Ω , then the classical maximum principle demands that the respective solution to (3.1) remains nonnegative at any moment of time, regardless of the choice of v . This means that system (3.1) cannot be steered from any such y_0 to any target state which is negative on a nonzero measure set in the system's space domain.

In [3] Cannarsa and Khapalov studied the global properties of the approximate controllability of the system (3.1) in the case where the initial and target states satisfy certain sign conditions in relation to the principle of the maximum.

The following result proves the global approximate controllability for the reaction-diffusion in the case when both the initial and target states admit no more than finitely many changes of sign.

Theorem 3.1. [3] Let $\Omega = (0,1)$, $y_0 \in H_0^1(0,1)$. Assume that $y_0(x)$ has finitely many points of change of sign in $(0,1)$. Consider any $y_d \in H_0^1(0,1)$ which has exactly as many points of change of sign in the same order as $y_0(x)$. Then for any $\varepsilon > 0$ there are a $T > 0$ and a multiplicative control $v \in L^\infty(Q_T)$ such that for the respective solution to (3.1) the following inequality holds:

$$\|y(\cdot, T) - y_d\|_{L^2(0,1)} < \varepsilon. \quad (3.2)$$

The multiplicative control v , delivering (3.2), can be selected piecewise static.

Remark 3.1.

- (1) The main results of this paper do not follow from those of [14]. The methods we apply below principally deviate from the ones used in [14] in the technical aspect, though they employ a similar approach, focusing on the study of the qualitative nature of the solutions to (3.1).
- (2) The approach used in [3] are based on the idea to attempt to extract needed qualitative dynamics from system (3.1) by using:
 - either zero-controls, which convert (3.1) into a pure diffusion process, or
 - non-zero static controls acting during suitably "small" time-intervals, which ensure that the reaction dynamics in (3.1) will prevail over the diffusion one.

Ouzahra [20] proved the approximate and exact global controllability of the bilinear reaction-diffusion equation by static control (does not depend on time), using an approaches based on the theory of semi-groups and the null controllability of linear systems.

The following result is more general than the result of [3], this result are established the approximate controllability of the n-dimensional of system (3.1) in the case when the initial and target states admit the same sign and the same points of change of sign that can be in finite or infinite number

Theorem 3.2. [20] Let $y_0, y_d \in L^2(\Omega)$ be such that:

- (i) for a.e. $x \in \Omega$, $y_d y_0 \geq 0$.
- (ii) for a.e. $x \in \Omega$, $y_0(x) = 0 \iff y_d(x) = 0$
- (iii) $a = \ln(\frac{y_d}{y_0}) \mathbf{1}_\Lambda \in W^{2,\infty}(\Omega)$, where $\mathbf{1}_\Lambda$ indicates the characteristic function of Λ .

Then for any $\varepsilon > 0$, there are a time $0 < T = T(T_0, y_0, y_d, \varepsilon) < T_0$ and a static control $v \in W^{2,\infty}(\Omega)$ such that for the respective solution to (3.1) the following inequality holds:

$$\|y(T) - y_d\| < \varepsilon.$$

Remark 3.2. (1) Notice that in Theorem 3.2, assumptions (i) and (ii) are necessary for the assumption (iii) to make sense.

- (2) The time of approximate controllability can be chosen arbitrary small.
- (3) The work [20] introduce a new technique of proof consists in finding a control v that realizes the approximate controllability for (1), and then combine the L^∞ -estimation of the heat-semigroup theory with the null controllability of linear heat equation with small initial states to build the final steering control.

3.2. Global exact controllability. We have the following results for the homogeneous system defined by the equation (3.1).

Theorem 3.3. [20] Assume that $n = 1$. Let $y_0 \in L^2(\Omega)$ and $y_d \in L^2(\Omega)$ satisfy assumptions (i)-(iii) of Theorem 3.2. We further assume that: (iv) $\frac{\Delta y_d}{y_d} \mathbf{1}_\Omega \in L^\infty(\Omega)$ and (v) $|y_d| > 0$, a.e. on some nonempty open subset O of Ω .

Then there are a time $0 < T_* = T_*(T_0, y_0, y_d) < T_0$ and a control $v \in L^\infty(\Omega) \times (0, T_*)$ for which $y(T_*) = y_d$ a.e. in Ω .

Remark 3.3. The result of Theorem 3.3 may be extended to other situations including initial and desirable states in $L^p(\Omega)$ with p not necessarily equal to 2. In this case, we can derive exact controllability results for multidimensional systems, that is follow the same method of the proof of Theorem 3.3 and use an estimate in $L^p(\Omega)$ for $\Omega \subset \mathbb{R}^n$ with $1 < n < 2p$.

We introduce the following general result.

Theorem 3.4. [12] Let $y_0 \in L^p(\Omega)$, ($p \geq 2$ and $p > \frac{n}{2}$) and let $y_d \in H^2(\Omega)$ such that

(i) for a.e. $x \in \Omega$, $y_0(x)y_d(x) \geq 0$ and $y_d(x) = 0 \Leftrightarrow y_0(x) = 0$,

(ii) $\ln(\frac{y_d}{y_0}) \mathbf{1}_{E_{y_0}} \in L^\infty(\Omega)$, where $\mathbf{1}_{E_{y_0}}$ denotes the characteristic function of the set $E_{y_0} = \{x \in \Omega / y_0(x) \neq 0\}$,

(iii) $\frac{\Delta y_d}{y_d} \mathbf{1}_{E_{y_d}} \in L^\infty(\Omega)$ and $|y_d| \geq \alpha > 0$, a.e. on some open subset $\omega \subset \Omega$.

Then for any $T > 0$, there exists a control $v \in L^\infty(Q_T)$ such that the respective solution to (3.1) satisfies $y(T) = y_d$, a.e. in Ω .

Remark 3.4. The result of Theorem 3.4 improves the results from the literature in terms of the steering time which is here independent of initial and target states.

In [18] Lin, Zhou and Gao given a positive result on the exact controllability of the system parabolic controlled by a bilinear control for a class of states desired.

The heat equation with nonhomogeneous Dirichlet boundary conditions using bilinear controls given by the following system:

$$\begin{cases} y_t = \Delta y + v(x, t)y, & \text{in } Q_T = \Omega \times (0, T), T > 0 \\ y(0, t) = g, & \text{on } \Sigma_T \\ y(x, 0) = y_0(x), & \text{in } \Omega \end{cases} \quad (3.3)$$

where $g \in C(\overline{\Omega})$, $y_0 \in L^2(\Omega)$ is the initial data, $v \in L^\infty(\Omega)$ is the control.

The following Theorem give the exact controllability for a some particular class of the target state.

Theorem 3.5. [18] Assume that $\theta \in W^{2,\infty}(\Omega)$, $\theta > 0$ in $\overline{\Omega}$, and $\Delta \theta \geq 0$ a.e. in Ω , $g \in C(\overline{\Sigma_T})$, $g = \theta$ on Σ_T , then there exists a $T(\theta) > 0$ such that for any $y_0 \in L^2(\Omega)$, there exists a bilinear control $u \in L^\infty(Q_T)$ such that the corresponding solution to (3.3) in $C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ satisfies

$$y(x, T) = \theta(x) \text{ in } \Omega, \text{ for all } T \geq T(\theta).$$

Remark 3.5. It appears that $\theta \neq 0$ is essential in Theorem 3.5. Indeed, if $\{x \in \Omega / \theta(x) = 0\}$ has a non-zero measurement, the zero-state becomes the fixed point of the solution mapping, regardless of the choice of u , then by the backward uniqueness of the parabolic equation, for any $y \in L^2(\Omega)$, $\text{meas}\{x / y_0(x) \neq 0\} > 0$, the system (3.3) can not be steered to 0.

In [21] Ouzahra, Tsouli and Boutoulout established the exact controllability of the bilinear equation with inhomogeneous boundary conditions for a target state $y(T) = \theta \in H^2(\Omega)$ satisfying the following assumptions:

(H_0): $\theta = g$ on $\partial\Omega$.

(H_1): $\theta\Delta\theta \geq 0$ in Ω .

(H_2): $\frac{\Delta\theta}{\theta}\mathbf{1}_\Lambda \in L^\infty(\Omega)$, where $\Lambda = \{x \in \Omega / \theta(x) \neq 0\}$.

(H_3): $\theta \in W^{2,\infty}(\omega_0)$ and $|\theta| > 0$ in $\overline{\omega_0}$, where ω_0 is a nonempty open subset of Ω .

Remark 3.6. Note that the assumptions (H_0) – (H_3) imply that $g \neq 0$.

We introduce the following result concerns the exact controllability of the system (3.3).

Theorem 3.6. [21] Let $y_0 \in L^2(\Omega) - \{0\}$ and let θ satisfy the assumptions (H_0) – (H_3). Then, there exists a time $T = T(\theta) > 0$, which is independent of y_0 , for which there exists a bilinear control $v \in L^\infty(Q_T)$ such that the corresponding solution to system (3.3) satisfies $y(\cdot, T) = \theta$ a.e., in Ω .

Remark 3.7. If $g = 0$, then based on the maximum principle, we deduce that it is not possible to steer (3.3) from $y_0 \leq 0$ to $\theta > 0$.

4. Partial controllability

This section deals with the approximate and exact partial controllability of a bilinear reaction-diffusion equation governed by a locally supported control. This consists on achieving a desired state only on a privileged subregion of the geometrical domain of the system at hand. We consider the following bilinear system

$$\begin{cases} y_t = \Delta y + v(x, t)\mathbf{1}_O y, & \text{in } Q = \Omega \times (0, +\infty) \\ y = 0, & \text{in } \Sigma = \partial\Omega \times (0, +\infty) \\ y(0) = y_0, & \text{in } \Omega \end{cases} \quad (4.1)$$

with the control bilinear $v \in L^\infty(Q_T)$ is applied only in the open subset O of Ω . For a given initial state the following result provide sufficient conditions on the desirable state to be reached. The steering control is explicitly expressed

4.1. Partial approximate controllability. The following theorem give sufficient conditions for partial approximate controllability.

Theorem 4.1. [11] Let ω be an open subset of O , and let $y_0, y_d \in L^2(\Omega)$ be such that:

(i) for a.e. $x \in \omega$, $y_0(x)y_d(x) \geq 0$,

(ii) for a.e. $x \in \omega$, $y_0(x) = 0 \Leftrightarrow y_d(x) = 0$,

(iii) the function defined by:

$$g = \begin{cases} \ln\left(\frac{y_d}{y_0}\right), & x \in E \cap \omega \\ 0, & x \notin E \cap \omega \end{cases}$$

lies in $L^\infty(\Omega)$, where $E = \{x \in \Omega / y_0(x) \neq 0\}$.

Then for all $\varepsilon > 0$, there exist a time $T > 0$ and a time-independent control $v_T \in L^\infty(\Omega)$ such that the respective solution of (4.1) satisfies:

$$\|y(T)|_\omega - y_d|_\omega\|_{L^2(\omega)} < \varepsilon.$$

where $\|y(T)|_\omega - y_d|_\omega\|_{L^2(\omega)} = (\int_\omega |y(x, T) - y_d(x)|^2 dx)^{\frac{1}{2}}$.

Remark 4.1. For $\omega = O = \Omega$, we retrieve the result of [20].

4.2. Partial exact controllability. In the section, we identify a set of desirable states which are partially exactly accessible from y_0 in a subregion $\omega \subset \Omega$, at a suitable time $T > 0$.

Theorem 4.2. [11] Let $p \in \mathbb{R}$; $p \geq 2$ and $1 \leq n < 2p$ and let ω be an open subset of O . Let $y_0 \in L^p(\Omega)$ and $y_d \in L^p(\omega)$. Assume that y_d admits an extension $\hat{y}_d \in H^2(\Omega)$ such that:

- (i) $\frac{\Delta \hat{y}_d}{\hat{y}_d} \mathbf{1}_{\hat{E}} \in L^\infty(\Omega)$, $\hat{E} = \{x \in \Omega; \hat{y}_d(x) \neq 0\}$,
- (ii) for a.e. $x \in \Omega$, $y_0(x)\hat{y}_d(x) \geq 0$ and $y_0(x) = 0 \Leftrightarrow \hat{y}_d(x) = 0$,
- (iii) $\hat{g} = \ln(\frac{\hat{y}_d}{y_0}) \mathbf{1}_E \in L^\infty(\Omega)$,
- (iv) $|\hat{y}_d| \geq \alpha > 0$, a.e. in ω_0 , for some open subset ω_0 of O .

Then there exist a time $T > 0$ and a control $v \in L^\infty(\Omega \times (0, T))$ such that the respective solution of (4.1) satisfies:

$$y(T)|_\omega = y_d.$$

5. Examples and simulations

Example 5.1. (Exact controllability of the semilinear reaction-diffusion equation)

Let us consider the system (2.11) with $f(y) = |y|$, and let us take $y_0 = x(1-x)$ and $y_d = y_0 e^x$ in $\Omega = (0, 1)$.

Thus, we have $a(x) = \frac{1}{T} \ln \frac{y_d}{y_0} = \frac{x}{T}$ and $g(x) = -\frac{\Delta y_d}{y_d} = -1 + \frac{4}{1-x}$.

According to Theorem 2.3, we deduce that for every $T > 0$, there are positive real numbers T_1 and T_2 which are small enough for which the control:

$$q(x, t) = \begin{cases} \frac{x}{T_1} - \frac{|\varphi|}{\varphi} \mathbf{1}_{E_\varphi}, & (0, T_1) \\ -1 + \frac{4}{1-x} - \frac{|\varphi|}{\varphi} \mathbf{1}_{E_\varphi}, & (T_1, T_2), \\ -1 + \frac{4}{1-x} + \frac{u(x, t)}{\psi(x, t) + y_d} - \frac{|\varphi|}{\varphi} \mathbf{1}_{E_\varphi}, & (T_2, T) \end{cases}$$

($E_\varphi = \{(x, t) : \varphi(x, t) \neq 0\}$) achieves the exact steering of system (2.11) from y_0 to y_d at T , where φ solves the following system

$$\begin{cases} \varphi_t = \Delta \varphi + v(x, t)\varphi, & \text{in } Q_T \\ \varphi(0, t) = 0, & \text{on } \Sigma_T \\ \varphi(x, 0) = y_0(x), & \text{in } \Omega \end{cases} \quad (5.1)$$

with

$$v(x, t) = \begin{cases} \frac{x}{T_1}, & (0, T_1) \\ -1 + \frac{4}{1-x}, & (T_1, T_2) \\ -1 + \frac{4}{1-x} + \frac{u(x, t)}{\psi(x, t) + y_d}, & (T_2, T) \end{cases}$$

and $u(x, t)$ is the control of null-controllability of the linear system:

$$\begin{cases} \psi_t = \Delta\psi - \psi + u(x, t), & \text{in } \Omega \times (T_2, T) \\ \psi(0, t) = 0, & \text{on } \partial\Omega \times (T_2, T) \\ \psi(T_2) = y(T_2) - y_d, & \text{in } \Omega \end{cases}$$

and ψ is the corresponding solution.

Example 5.2. (Partial controllability of the bilinear reaction-diffusion equation)

Let us consider the system (4.1) with $\Omega = (0, 1) = O$, and let us $y_0, y_d \in L^2(\Omega)$.

Let $\omega = (0.35, 0.95)$, $y_0 = x \cos(\frac{\pi}{2}x)$ and

$$y_d(x) = \begin{cases} x(\sqrt{3} + 20(x - 1/3)), & x \in (0, 1/3) \\ 2x \cos(\frac{\pi}{2}x), & x \in (1/3, 1) \end{cases}$$

Notice that, according to maximum principle, it is not possible to attain y_d from y_0 all over the geometrical domain $\Omega = (0, 1)$.

According to Theorem 4.1 the control defined by:

$$v(x) = \begin{cases} 0, & x \in (0, 1/3) \\ \frac{1}{T} \ln\left(\frac{y_d(x)}{y_0(x)}\right), & x \in (1/3, 1) \end{cases}$$

realized the partial approximate controllability in $\omega = (0.35, 0.95)$ with $T \rightarrow 0^+$.

Below, the following figure corresponding to the partial approximate controllability (trajectory $y(T)$) at $T = 0.01$.

Figure: The evolution of the state at T .

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