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Common coupled fixed point theorems for two pairs of weakly compatible mappings in Menger metric spaces

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ABSTRACT. In this paper, we prove some common coupled fixed point theorems for contractive mappings in Menger metric spaces under geometrically convergent t-norms. Also, we prove common fixed point theorems for pairs of weakly compatible mappings, which generalize the results of Jian-Zhong Xiao and al (2011). The main results is supported by a suitable examples.

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1. Introduction

Many common coupled fixed point theorems for contractions in probabilistic metric spaces under either a t-norm of Hadzic-type or the t-norm $\Delta_p = \text{prod}$ can be found in the recent literature, see, e.g, [3, 4, 5, 17, 20, 21, 22]. The aim of this paper is to obtain similar results in a larger class of Menger metric spaces, namely in Menger metric spaces endowed with geometrically convergent t-norms.

We assume that the reader is familiar with the basic concepts and terminology of the theory of Menger metric spaces. We only recall that a t-norm Δ is said to be of Hadzic-type (denoted

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$\Delta \in H$) if the family $\{\Delta^n(t)\}_{n=1}^\infty$ defined by

$$\Delta^1(t) = t, \quad \Delta^{n+1}(t) = \Delta(t, \Delta^n(t)) \quad n = 1, 2, \dots \quad t \in [0, 1]$$

is equi-continuous at $t = 1$, and that a t-norm Δ is called geometrically convergent (or g-convergent) [7] if for all $q \in (0, 1)$,

$$\lim_{n \rightarrow \infty} \Delta_{I=n}^\infty (1 - q^i) = 1.$$

It is worth noting (see e.g, [7]) that if for a t-norm there exists $q_0 \in (0, 1)$ such that

$$\lim_{n \rightarrow \infty} \Delta_{i=n}^\infty (1 - q_0^i) = 1$$

then

$$\lim_{n \rightarrow \infty} \Delta_{i=1}^\infty (1 - q^i) = 1.$$

The well known t-norms $\Delta_M = \min$, $\Delta_p = \text{prod}$ and Δ_L (Lukasiewicz t-norm) are g-convergent.

Also, every member of Dombi family $(\Delta_\lambda^D)_{\lambda \in (0; \infty)}$, Aczel-Alsina family $(\Delta_\lambda^{AA})_{\lambda \in (0; \infty)}$ and Sugeno-Weber family $(\Delta_\lambda^{SW})_{\lambda \in (-1; \infty)}$ is g-convergent [7].

A large classe of g-convergent t-norms, in terms of the generators of strict t-norms is described in [7] (also see [6], Ch.1.8).

Bhaskar and Lakshmikantham [2], Lakshmikantham and Ćirić [9] gave some coupled fixed point theorems. Coupled fixed point theorem under contraction conditions given by Sedghi and al [17] are of great importance in the theory of fixed points in fuzzy metric spaces. Fang [5] proved a result for compatible and weakly compatible mappings under φ -contractive conditions in fuzzy metric spaces which provide a tool to Xin-Qi Hu [20] to prove a result, which is actually a generalization of the result of Sedghi [17].

In this paper, firstly we prove some common coupled fixed point theorems for contractive mappings in Menger metric spaces under geometrically convergent t-norms. At the end, we prove common fixed point theorems for pairs of weakly compatible mappings, which generalize the results of Jian-Zhong Xiao and al (2011). The main result is supported by a suitable example.

2. Preliminaries

We now state some basic concepts and results which will be used. In the standard notation, suppose that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{Z}^+$ be the set of positive integers.

A function $F : \overline{\mathbb{R}} \rightarrow [0, 1]$ is called a distribution function if it is non decreasing and left continuous with $F(-\infty) = 0$ and $F(+\infty) = 1$.

The class of all distribution functions is denoted by D_∞ .

Suppose that $D = \{F \in D_\infty : \inf F D_\infty^+(t) = 0, \sup F(t) = 1\}$, $D_\infty^+ = \{F \in D_\infty : F(0) = 0\}$ and $D^+ = D \cap D_\infty^+$ (cf [1, 17]).

A special element of D^+ is the heaviside function H defined by:

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases}$$

Definition 2.1. ([12]). A function $\Delta : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a triangular norm (for short, a t -norm) if the following conditions are satisfied for any $a, b, c, d \in [0, 1]$;

(Δ -1) $\Delta(a, 1) = a$

(Δ -2) $\Delta(a, b) = \Delta(b, a)$;

(Δ -3) $\Delta(a, b) \geq \Delta(c, d)$ for $a \geq c, b \geq d$;

(Δ -4) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c))$.

Two examples of t -norm are $\Delta_M(a, b) = \min\{a, b\}$ and $\Delta_P(a, b) = ab$.

It is evident that, as regards the point wise ordering, $\Delta \leq \Delta_M$ for each t -norm Δ .

Definition 2.2. ([11, 12]). A triplet (X, F, Δ) is called a generalised Menger probabilistic metric space if X is a non-empty set, Δ is t -norm and F is a mapping from $X \times X$ into D_∞^+ satisfying the following condition ($F(x, y)$ for $x, y \in X$ is denoted by $F_{x,y}$):

(MS-1) $F_{x,y}(t) = H(t)$ for all $t \in \mathbb{R}$ if and only if $x = y$;

(MS-2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X$ and $t \in \mathbb{R}$

(MS-3) $F_{x,y}(t+s) \geq \Delta(F_{x,z}(t), F_{z,y}(s))$ for all $x, y, z \in X$ and $t, s \in \mathbb{R}^+$. A Menger probabilistic metric space (for short, a Menger PM-space) is a generalized Menger space with $F(X \times X) \in D^+$.

Schweizer and al [15, 16] point out that if the t -norm Δ of a Menger PM-space satisfies the condition $\sup_{0 < a < 1} \Delta(a, a) = 1$, then (X, F, Δ) is a first countable hausdorff topological space in the (ε, λ) -topology τ , i.e, the family of sets

$$\{U_x(\varepsilon, \lambda) : \varepsilon > 0, \lambda \in [0, 1], (x \in X)\}$$

is the base of neighborhoods of point x for τ , where $U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}$.

By virtue of this topology τ a sequence $\{x_n\}$ in (X, F, Δ) is said to be convergent to x (we write $x_n \rightarrow x$ or $\lim_{n \rightarrow \infty} x_n = x$) if $\lim_{n \rightarrow \infty} F_{x_n, x}(t) = 1$ for all $t > 0$; $\{x_n\}$ is called a Cauchy sequences in (X, F, Δ) if for any given $\varepsilon > 0$ and $\lambda \in [0, 1]$, there exists $N = N(\varepsilon, \lambda) \in \mathbb{Z}^+$ such that $F_{x_n, x_m}(\varepsilon) > 1 - \lambda$, whenever $n, m \geq N$, (X, F, Δ) is said to be complete if each Cauchy sequence in X is convergent to some point in X . In the sequel, we will always assume that (X, F, Δ) is a Menger space with the (ε, λ) -topology.

Lemma 2.1. Let (X, d) be a usual metric space. Define a mapping $F : X \times X \rightarrow D^+$ by

$$F_{x,y}(t) = H(t - d(x, y)), \text{ for } x, y \in X \text{ and } t > 0.$$

Then (X, F, Δ_m) is a Menger PM-space; it is called the induced Menger PM-space by (X, d) and it is complete if (X, d) is complete.

An arbitrary t -norme can be extended (by(Δ -3)) in a unique way to an n -ary operation. For $(a_1, a_2, \dots, a_n) \in [0, 1]^n$ ($n \in \mathbb{Z}^+$), the valued $\Delta^n(a_1, a_2, \dots, a_n)$ is defined by $\Delta^1(a_1) = a_1$ and $\Delta^n(a_1, a_2, \dots, a_n) = \Delta(\Delta^{n-1}(a_1, a_2, \dots, a_{n-1}), a_n)$.

For each $a \in [0, 1]$, the sequence $\{\Delta^n(a)\}_{n=1}^\infty$ is defined by $\Delta^1(a) = a$ and $\Delta^n(a) = \Delta(\Delta^{n-1}(a), a)$.

Definition 2.3. A t -norm Δ is said to be of H-type if the sequence of functions $\{\Delta^n(a)\}_{n=1}^\infty$ is equi-continuous at $a = 1$.

The t -norm Δ_m is a trivial example of a t -norm of H-type, but there are t -norms Δ of H-type with $\Delta \neq \Delta_m$. It is easy to see that if Δ is of H-type, then Δ satisfies $\sup_{0 < a < 1} \Delta(a, a) = 1$.

Lemma 2.2. Let (X, F, Δ) be a Menger PM-space. For each $\lambda \in (0, 1]$, define a function $d_\lambda : X \times X \rightarrow \mathbb{R}^+$ by

$$d_\lambda(x, y) = \inf \{t > 0 : F_{x,y}(t) > 1 - \lambda\}.$$

Then the following statements hold:

- (1) $d_\lambda(x, y) < t$ if and only if $F_{x,y}(t) > 1 - \lambda$;
- (2) $d_\lambda(x, y) = d_\lambda(y, x)$ for all $x, y \in X$ and $\lambda \in (0, 1]$;
- (3) $d_\lambda(x, y) = 0$ for all $\lambda \in (0, 1]$ if and only if $x = y$.

Lemma 2.3. Let (X, F, Δ) be a Menger PM-space, and let $\{d_\lambda\}_{\lambda \in (0,1]}$ be a family of pseudo-metrics on X defined by (1.1).

If Δ is a t -norm of H-type, then for each $\lambda \in (0, 1]$, there exists $\mu \in (0, \lambda]$ such that for each $m \in \mathbb{Z}^+$,

$$d_\lambda(x_0, x_m) \leq \sum_{i=0}^{m-1} d_\mu(x_i, x_{i+1}), \text{ for all } x_0, x_1, \dots, x_m \in X.$$

Lemma 2.4. Suppose that $F \in D^+$. For each $n \in \mathbb{Z}^+$, let $F_n : \mathbb{R} \rightarrow [0, 1]$ be non-decreasing, and $g_n : (0, +\infty) \rightarrow (0, +\infty)$ satisfy $\lim_{n \rightarrow +\infty} g_n(t) = 0$ for any $t > 0$. If

$$F_n(g_n(t)) \geq F(t), \forall t > 0$$

then $\lim_{n \rightarrow +\infty} F_n(t) = 1$, for any $t > 0$.

Definition 2.4. ([5]). An element $x \in X$ is called a common fixed point of the mappings $f : X \times X \rightarrow X$ and $g : X \rightarrow X$ if

$$x = f(x, x) = g(x)$$

Definition 2.5. ([10]). An element $(x, y) \in X \times X$ is called

(i) a coupled fixed point of the mapping $f : X \times X \rightarrow X$ if

$$f(x, y) = x; \quad f(y, x) = y.$$

(ii) a coupled coincidence point of the mappings $f : X \times X \rightarrow X$ and $g : X \rightarrow X$ if

$$f(x, y) = g(x); \quad f(y, x) = g(y)$$

(iii) a common coupled fixed point of the mappings $f : X \times X \rightarrow X$ and $g : X \rightarrow X$ if

$$x = f(x, y) = g(x); \quad y = f(y, x) = g(y)$$

In [1], Abbas and al introduced the concept of weakly compatible mappings, Here we give a similar concept in Menger metric spaces as follows:

Definition 2.6. Let (X, F, Δ) be a Menger metric space and let $f : X \times X \rightarrow X$ and $g : X \rightarrow X$ be two mappings. f and g are said to be weakly compatible (or w -compatible) if they commute at their coupled coincidence points, i.e. if (x, y) is a coupled coincidence point of f and g , then

$$g(f(x, y)) = f(g(x), g(y)), \quad g(f(y, x)) = f(g(y), g(x))$$

Definition 2.7. ([8]) Let $A : X \times X \rightarrow X$, $B : X \times X \rightarrow X$, $T : X \rightarrow X$, $S : X \rightarrow X$ be four mappings. Then, the pairs (B, S) and (A, T) are said to have common coupled coincidence point if there exists a, b in X such that

$$B(a, b) = S(a) = T(a) = A(a, b) \text{ and } B(b, a) = S(b) = T(b) = A(b, a).$$

3. Main Results

We now give our main result which provides a common coupled fixed point theorems for contractive mappings in Menger metric spaces under geometrically convergent t-norms.

Theorem 3.1. Let (X, F, Δ) be a Menger metric space with Δ is a g-convergent t-norm. Let $A : X \times X \rightarrow X$, $B : X \times X \rightarrow X$, $T : X \rightarrow X$ and $S : X \rightarrow X$ be four mappings satisfying the following condition

- (1) $A(X \times X) \subseteq T(X)$, $B(X \times X) \subseteq S(X)$
- (2) There exists $k \in (0, 1)$ such that

$$F_{A(x,y), B(u,v)}(kt) \geq \text{Min}(F_{Sx, Tu}(t), F_{Sy, Tv}(t)) \quad (3.1)$$

for all $x, y, u, v \in X$ and $t > 0$

- (3) The pairs (A, S) and (B, T) are weakly compatible.
 - (4) One of the subspaces $A(X \times X)$ or $T(X)$ and one of $B(X \times X)$ or $S(X)$ are complete.
- If there exists $\alpha > 0$ and $x_0, y_0 \in X$ such that

$$\sup_{t>0} t^\alpha \left(1 - F_{Sx_0, A(x_0, y_0)}(t) \right) < \infty,$$

and

$$\sup_{t>0} t^\alpha \left(1 - F_{Sy_0, A(y_0, x_0)}(t) \right) < \infty.$$

Then there exists a unique point a in X such that $A(a, a) = S(a) = T(a) = B(a, a) = a$.

Proof. For arbitrary x_0, y_0 in X , by (1), we can choose x_1, y_1 in X such that $T(x_1) = A(x_0, y_0)$, $T(y_1) = A(y_0, x_0)$.

Again by (1), we can choose x_2, y_2 in X such that $S(x_2) = B(x_1, y_1)$, $S(y_2) = A(y_1, x_1)$.

Continuing in this way, we can construct two sequences $\{Z_n\}$ and $\{Z'_n\}$ in X such that

$$\begin{cases} Z_{2n+1} = A(x_{2n}, y_{2n}) = T(x_{2n+1}) \\ Z_{2n+2} = B(x_{2n+1}, y_{2n+1}) = S(x_{2n+2}) \\ Z'_{2n+1} = A(y_{2n}, x_{2n}) = T(y_{2n+1}) \\ Z'_{2n+2} = B(y_{2n+1}, x_{2n+1}) = S(y_{2n+2}) \end{cases}$$

We divide the proof into 6 steps.

Step 1 : We show that $\{Z_n\}$ and $\{Z'_n\}$ are Cauchy sequences.

Indeed, let $\beta > 0$. be such that

$$t^\alpha \left(1 - F_{Sx_0, A(x_0, y_0)}(t) \right) \leq \beta$$

and

$$t^\alpha \left(1 - F_{Sy_0, A(y_0, x_0)}(t) \right) \leq \beta$$

for all $t > 0$. Then

$$F_{Sx_0, Tx_1} \left(\frac{1}{t^n} \right) > 1 - \beta (t^\alpha)^n \text{ and } F_{Sy_0, Ty_1} \left(\frac{1}{t^n} \right) > 1 - \beta (t^\alpha)^n \text{ for every } t > 0 \text{ and } n \in N.$$

If $t > 0$ and $\varepsilon \in (0, 1)$ are given, we choose μ in interval $(k, 1)$ such that

$$\Delta_{i=n+1}^\infty \left(1 - (\mu^\alpha)^i \right) > 1 - \varepsilon \text{ and } \delta = \frac{k}{\mu}.$$

As $\delta \in (0, 1)$, we can find $n_1 (= n_1(t))$ such that $\sum_{n=n_1}^\infty \delta^n < t$.

Condition (3.1) implies that, for all $s > 0$,

$$F_{Tx_1, Sx_2}(ks) = F_{A(x_0, y_0), B(x_1, y_1)}(ks) \geq \text{Min} \left(F_{Sx_0, Tx_1}(s), F_{Sy_0, Ty_1}(s) \right),$$

and

$$F_{Ty_1, Sy_2}(ks) = F_{A(y_0, x_0), B(y_1, x_1)}(ks) \geq \text{Min} \left(F_{Sx_0, Tx_1}(s), F_{Sy_0, Ty_1}(s) \right).$$

It follows by induction that

$$F_{Z_n, Z_{n+1}}(k^n s) \geq \text{Min} \left(F_{Sx_0, Tx_1}(s), F_{Sy_0, Ty_1}(s) \right)$$

and

$$F_{Z_n, Z_{n+1}}(k^n s) \geq \text{Min} \left(F_{Sx_0, Tx_1}(s), F_{Sy_0, Ty_1}(s) \right)$$

for all $n \in N$. Then for all $n \geq n_1$ and $m \in N$ we obtain

$$\begin{aligned} F_{Z_n, Z_{n+m}}(t) &\geq F_{Z_n, Z_{n+m}} \left(\sum_{i=n_1}^\infty \delta^i \right) \\ &\geq F_{Z_n, Z_{n+m}} \left(\sum_{i=n}^{n+m-1} \delta^i \right) \\ &\geq \Delta_{i=n}^{n+m-1} \left(F_{Z_i, Z_{i+1}}(\delta^i) \right) \\ &\geq \Delta_{i=n}^{n+m-1} \left(\Delta \left(F_{Sx_0, Tx_1} \left(\frac{1}{\mu^i} \right), F_{Sy_0, Ty_1} \left(\frac{1}{\mu^i} \right) \right) \right) \\ &\geq \Delta_{i=n}^{n+m-1} \left(1 - \beta \mu^{\alpha i} \right). \end{aligned}$$

If we choose $l_0 \in N$ such that $\beta \mu^{\alpha l_0} \leq \mu^\alpha$, then $1 - \beta (\mu^\alpha)^{n+l_0} \geq 1 - (\mu^\alpha)^{n+1}$, for all n . Thus,

$$F_{Z_{n+l_0}, Z_{n+l_0+m}}(t) \geq \Delta_{i=n+1}^\infty \left(1 - (\mu^\alpha)^i \right) > 1 - \varepsilon,$$

for every $n \geq n_1$ and $m \in N$, hence $\{Z_{2n+1}\}$ is a Cauchy sequence. Similarly one can show that $\{Z_n\}$ is a Cauchy sequence.

Step 2 : We show that

$$T(a) = B(a, b), T(b) = B(a, b) \text{ and } S(a) = A(a, b), S(b) = A(b, a).$$

Without loss of generality, assume that $T(X)$ and $S(X)$ are complete. Now, $\{Z_{2n+1}\}, \{Z_{2n+2}\}$ and $\{Z'_{2n+1}\}, \{Z'_{2n+2}\}$ being respective subsequences of Cauchy sequences $\{Z_n\}$ and $\{Z'_n\}$ are also Cauchy.

Since $T(X)$ is complete, so there exists a, b in X such that:

$$\{Z_{2n+1}\} \rightarrow a \text{ and } \{Z'_{2n+1}\} \rightarrow b$$

Again convergence of the subsequences $\{Z_{2n+1}\}$ and $\{Z'_{2n+1}\}$ implies the convergence of original Cauchy sequences $\{Z_n\}$ and $\{Z'_n\}$ respectively such that

$$\{Z_n\} \rightarrow a \text{ and } \{Z'_n\} \rightarrow b$$

It follows that the sequences $\{Z_n\}, \{Z_{2n+1}\}, \{Z_{2n+2}\}$ converge to a and $\{Z'_n\}, \{Z'_{2n+1}\}, \{Z'_{2n+2}\}$ converge to b .

Now $a, b \in T(X)$ implies the existence of $p, q \in X$ such that $T(p) = a, T(q) = b$, so that we have

$$\begin{aligned} \lim_{n \rightarrow \infty} Z_{2n+1} &= \lim_{n \rightarrow \infty} A(x_{2n}, y_{2n}) = \lim_{n \rightarrow \infty} T(x_{2n+1}) = a = T(p) \\ \lim_{n \rightarrow \infty} Z_{2n+2} &= \lim_{n \rightarrow \infty} B(x_{2n+1}, y_{2n+1}) = \lim_{n \rightarrow \infty} S(x_{2n+2}) = a = T(p), \end{aligned}$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} Z'_{2n+1} &= \lim_{n \rightarrow \infty} A(y_{2n}, x_{2n}) = \lim_{n \rightarrow \infty} T(y_{2n+1}) = b = T(q) \\ \lim_{n \rightarrow \infty} Z'_{2n+2} &= \lim_{n \rightarrow \infty} B(y_{2n+1}, x_{2n+1}) = \lim_{n \rightarrow \infty} S(y_{2n+2}) = b = T(q), \end{aligned}$$

From condition (2),

$$F_{A(x_{2n}, y_{2n}), B(p, q)}(kt) \geq \text{Min} \left(F_{S(x_{2n}), T(p)}(t), F_{S(y_{2n}), T(q)}(t) \right).$$

Letting $n \rightarrow \infty$, we get

$$F_{T(p), B(p, q)}(kt) \geq 1 \text{ this implies } T(p) = B(p, q) = a.$$

Similarly $T(q) = B(q, p) = b$.

Since the pair (B, T) are weakly compatible, so that $T(p) = B(p, q) = a$ implies $T(a) = B(a, b)$, similarly, $T(b) = B(b, a)$.

Again, since $S(X)$ is complete, so that $a, b \in S(X)$, which implies the existence of r, s in X , so that: $S(r) = a, S(s) = b$.

from inequality (3.1), we get

$$F_{A(r, s), B(x_{2n+1}, y_{2n+1})}(kt) \geq \text{Min} \left(F_{S(r), T(x_{2n+1})}(t), F_{S(s), T(y_{2n+1})}(t) \right)$$

Letting $n \rightarrow \infty$, we get

$$F_{A(r, s), a}(kt) \geq 1 \text{ implies } A(r, s) = a = S(r)$$

Similarly, $A(s, r) = b = S(s)$.

Since the pair (A, S) are weakly compatible, it follows that:

$$A(a, b) = S(a) \text{ and } A(b, a) = S(b).$$

Step 3 : Next we show $S(a) = T(a)$ and $S(b) = T(b)$.

From inequality (3.1):

$$F_{A(a,b),B(a,b)}(kt) \geq \text{Min} \left(F_{S(a),T(a)}(t), F_{S(b),T(b)}(t) \right)$$

and similarly

$$F_{A(b,a),B(b,a)}(kt) \geq \text{Min} \left(F_{S(a),T(a)}(t), F_{S(b),T(b)}(t) \right)$$

Thus

$$\text{Min} \left\{ F_{S(a),T(a)}(t), F_{S(a),T(a)}(t) \right\} \geq \text{Min} \left\{ F_{S(a),T(a)}\left(\frac{t}{k^n}\right), F_{S(b),T(b)}\left(\frac{t}{k^n}\right) \right\}$$

for all $n \in \mathbb{N}$ implying

$$\text{Min} \left\{ F_{S(a),T(a)}(t), F_{S(b),T(b)}(t) \right\} = 1.$$

It follows that

$$F_{S(a),T(a)}(t) = 1 = F_{S(b),T(b)}(t) \text{ for all } t > 0.$$

Whence $S(a) = T(a)$ and $S(b) = T(b)$, as claimed.

Therefore,

$$S(a) = A(a,b) = B(a,b) = T(a) \text{ and } S(b) = A(b,a) = B(b,a) = T(b).$$

Step 4 : We next show that

$$S(a) = a \text{ and } S(b) = b.$$

Indeed, letting $n \rightarrow \infty$ in the inequality

$$F_{A(x_{2n},y_{2n}),B(a,b)}(kt) \geq \text{Min} \left(F_{S(x_{2n}),T(a)}(t), F_{S(y_{2n}),T(b)}(t) \right).$$

We get

$$F_{a,S(a)}(kt) \geq \text{Min} \left(F_{a,S(a)}(t), F_{b,S(b)}(t) \right)$$

and similarly

$$F_{b,S(b)}(kt) \geq \text{Min} \left(F_{a,S(a)}(t), F_{b,S(b)}(t) \right).$$

Thus

$$\text{Min} \left\{ F_{a,S(a)}(t), F_{b,S(b)}(t) \right\} \geq \text{Min} \left\{ F_{a,S(a)}\left(\frac{t}{k^n}\right), F_{b,S(b)}\left(\frac{t}{k^n}\right) \right\},$$

for all $n \in \mathbb{N}$, implies

$$\text{Min} \left\{ F_{a,S(a)}(t), F_{b,S(b)}(t) \right\} = 1.$$

It follows that

$$F_{a,S(a)}(t) = 1 = F_{b,S(b)}(t) \text{ for all } t > 0.$$

Whence $S(a) = a$ and $S(b) = b$. There are :

$$B(a,b) = S(a) = a = T(a) = A(a,b) \text{ and } B(b,a) = S(b) = b = T(b) = A(b,a).$$

Step 5 : We shall that $a = b$. Indeed, letting $n \rightarrow \infty$ in the inequality

$$F_{A(x_{2n},y_{2n}),B(y_{2n+1},x_{2n+1})}(kt) \geq \text{Min} \left(F_{S(x_{2n}),T(y_{2n+1})}(t), F_{S(y_{2n}),T(x_{2n+1})}(t) \right)$$

We get

$$F_{a,b}(kt) \geq \min(F_{a,b}(t), F_{b,a}(t))$$

It follows that $F_{a,b}(kt) \geq F_{a,b}(t)$ for all $t > 0$, and so $a = b$.

Hence:

$$B(a, a) = S(a) = a = T(a) = A(a, a).$$

Step 6: We show that the fixed point is unique.

Let z, w be common fixed points for A, B, S and T , then from (3.1), we obtain

$$F_{A(z,z), B(w,w)}(kt) \geq \min(F_{Sz, Tw}(t), F_{S_z, T_w}(t)).$$

That is $F_{z,w}(kt) \geq F_{z,w}(t)$ for all $t > 0$ implying $z = w$. □

If the t -norm Δ is of Hadžić-type, then the conditions

$$\sup_{t>0} t^\alpha (1 - F_{Sx_0, A(x_0, y_0)}(t)) < \infty,$$

and

$$\sup_{t>0} t^\alpha (1 - F_{Sy_0, A(y_0, x_0)}(t)) < \infty.$$

can be dropped, so we have the following result :

Corollary 3.1. Let (X, F, Δ) be a Menger metric space with Δ is a t -norm of H-type. Let $A : X \times X \rightarrow X, B : X \times X \rightarrow X, T : X \rightarrow X, S : X \rightarrow X$ be four mappings satisfying the following condition

(1) $A(X \times X) \subseteq T(X), B(X \times X) \subseteq S(X)$

(2) There exists $k \in (0, 1)$ such that

$$F_{A(x,y), B(u,v)}(kt) \geq \min(F_{Sx, Tu}(t), F_{Sy, Tv}(t))$$

for all $x, y, u, v \in X$ and $t > 0$

(3) The pairs (A, S) and (B, T) are w -compatible.

(4) One of the subspaces $A(X \times X)$ or $T(X)$ and one of $B(X \times X)$ or $S(X)$ are complete.

Then there exists a unique point a in X such that $A(a, a) = S(a) = T(a) = B(a, a) = a$.

Corollary 3.2. Let (X, F, Δ) be a Menger metric space with Δ is a t -norm of H-type. Let $A : X \times X \rightarrow X$ be a mapping, and assume that for any $t > 0$; there exists $k \in (0, 1)$ such that

$$F_{A(x,y), A(u,v)}(kt) \geq \min(F_{x,u}(t), F_{y,v}(t))$$

for all $x, y, u, v \in X$.

Suppose that $A(X \times X)$ is complete. Then A has a unique coupled fixed point $a \in X$ such that $A(a, a) = a$.

Jian-Zhong Xiao in [19] proved the following result :

Theorem 3.2. Let (X, F, Δ) be a complete Menger metric space with Δ is a t -norm of H-type and $\Delta \geq \Delta_p$. Let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for any $t > 0$. Let $A : X \times X \rightarrow X, T : X \rightarrow X$ be two mappings such that

$$F_{A(x,y), A(u,v)}(\varphi(t)) \geq [\Delta(F_{Tx, Tu}(t), F_{Ty, Tv}(t))]^{1/2}$$

for all $x, y, u, v \in X$ and $t > 0$, where $A(X \times X) \subseteq T(X)$, T is continuous and commutative with A . Then there exists a unique $u \in X$ such that $u = Tu = A(u, u)$.

Now, we prove the following result which generalizes the previous theorem for two pairs of weakly compatible mappings:

Theorem 3.3. Let (X, F, Δ) be a Menger metric space with Δ is a t -norm of H -type and $\Delta \geq \Delta_p$. Let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$ and $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$ for any $t > 0$. Let $A : X \times X \rightarrow X$, $B : X \times X \rightarrow X$, $T : X \rightarrow X$ and $S : X \rightarrow X$ be four mappings satisfying the following condition

(1) $A(X \times X) \subseteq T(X)$, $B(X \times X) \subseteq S(X)$

(2)

$$F_{A(x,y), B(u,v)}(\varphi(t)) \geq [\Delta(F_{Sx, Tu}(t), F_{Sy, Tv}(t))]^{1/2} \quad (3.2)$$

for all $x, y, u, v \in X$ and $t > 0$

(3) The pairs (A, S) and (B, T) are w -compatible.

(4) One of the subspaces $A(X \times X)$ or $T(X)$ and one of $B(X \times X)$ or $S(X)$ are complete.

Then there exists a unique point $a \in X$ such that $A(a, a) = S(a) = T(a) = B(a, a) = a$.

Proof. Step 1: We show that $\{Z_n\}$ and $\{Z'_n\}$ are Cauchy sequences.

By condition (1), we can construct two sequences $\{Z_n\}$ and $\{Z'_n\}$ in X such that

$$\begin{cases} Z_{2n+1} = A(x_{2n}, y_{2n}) = T(x_{2n+1}) \\ Z_{2n+2} = B(x_{2n+1}, y_{2n+1}) = S(x_{2n+2}) \\ Z'_{2n+1} = A(y_{2n}, x_{2n}) = T(y_{2n+1}) \\ Z'_{2n+2} = B(y_{2n+1}, x_{2n+1}) = S(y_{2n+2}) \end{cases}$$

Suppose that $t > 0$. From (3.2), we have

$$\begin{aligned} F_{Z_n, Z_{n+1}}(\varphi(t)) &= F_{Tx_n, Sx_{n+1}}(\varphi(t)) \\ &= F_{A(x_{n-1}, y_{n-1}), B(x_n, y_n)}(\varphi(t)) \\ &\geq [\Delta(F_{Sx_{n-1}, Tx_n}(t), F_{Sy_{n-1}, Ty_n}(t))]^{1/2} \\ &= [\Delta(F_{Z_{n-1}, Z_n}(t), F_{Z'_{n-1}, Z'_n}(t))]^{1/2} \end{aligned} \quad (3.3)$$

$$\begin{aligned} F_{Z'_n, Z'_{n+1}}(\varphi(t)) &= F_{Ty_n, Sy_{n+1}}(\varphi(t)) \\ &= F_{A(y_{n-1}, x_{n-1}), B(y_n, x_n)}(\varphi(t)) \\ &\geq [\Delta(F_{Sy_{n-1}, Ty_n}(t), F_{Sx_{n-1}, Tx_n}(t))]^{1/2} \\ &= [\Delta(F_{Z'_{n-1}, Z'_n}(t), F_{Z_{n-1}, Z_n}(t))]^{1/2} \end{aligned} \quad (3.4)$$

Suppose that $G_n(t) = [\Delta(F_{Z'_{n-1}, Z'_n}(t), F_{Z_{n-1}, Z_n}(t))]^{1/2}$. Then, operating by t -norm Δ on (3.3) and (3.4), from the condition $\Delta \geq \Delta_p$, we obtain

$$G_{n+1}(\varphi(t)) \geq [\Delta(G_n(t), G_n(t))]^{1/2} \geq [G_n(t) G_n(t)]^{1/2} = G_n(t) \quad (3.5)$$

Thus, it follows from (3.3) – (3.5) that

$$F_{Z_n, Z_{n+1}}(\varphi^n(t)) \geq G_n(\varphi^{n-1}(t)) \geq \dots \geq G_1(t) \quad \text{and.} \quad (3.6)$$

$$F_{Z'_n, Z'_{n+1}}(\varphi^n(t)) \geq G_n(\varphi^{n-1}(t)) \geq \dots \geq G_1(t). \quad (3.7)$$

In the next step we show that $\{Z_n\}$ is a Cauchy sequence. For each $\lambda \in (0, 1]$, suppose that

$$D_\lambda = \inf \{t > 0 : G_1(t) > 1 - \lambda\}.$$

Then, $G_1(D_\lambda + 1) > 1 - \lambda$. From (3.6) we see that $F_{Z_n, Z_{n+1}}(\varphi^n(D_\lambda + 1)) > 1 - \lambda$. By lemma 3, we have

$$d_\lambda(Z_n, Z_{n+1}) < \varphi^n(D_\lambda + 1), \quad \text{for each } \lambda \in (0, 1]. \quad (3.8)$$

By Lemma 3, for each $\lambda \in (0, 1]$ there exists $\mu \in (0, \lambda]$ such that

$$d_\lambda(Z_n, Z_m) \leq \sum_{i=0}^{m-1} d_\mu(Z_i, Z_{i+1}), \quad \text{for all } m, n \in \mathbb{Z}^+ \text{ with } m > n. \quad (3.9)$$

Suppose that $\varepsilon > 0$ and $\lambda \in (0, 1]$ are given. Since $\sum_{n=1}^{\infty} \varphi^n(D_\lambda + 1) < +\infty$, there exists $N \in \mathbb{Z}^+$ such that $\sum_{i=0}^{m-1} \varphi^i(D_\mu + 1) < \varepsilon$ for all $m > n \geq N$. Thus, by (3.8) and (3.9), we have $d_\lambda(Z_n, Z_{n+1}) < \varepsilon$. Using Lemma 3, we obtain that $F_{Z_n, Z_m}(\varepsilon) > 1 - \lambda$ for all $m > n \geq N$, i.e., $\{Z_n\}$ is a Cauchy sequence. Similarly, from (3.7) we can show that $\{Z'_n\}$ is a Cauchy sequence.

Step 2 : We show that $T(a) = B(a, b)$, $T(b) = B(b, a)$ and $S(a) = A(a, b)$, $S(b) = A(b, a)$ without loss of generality, assume that $T(X)$ and $S(X)$ are complete. Now $\{Z_{2n+1}\}$, $\{Z_{2n+2}\}$ and $\{Z'_{2n+1}\}$, $\{Z'_{2n+2}\}$ being respective subsequences of Cauchy sequences $\{Z_n\}$ and $\{Z'_n\}$ are also Cauchy.

Since $T(X)$ is complete, so there exists a, b in X such that:

$$\{Z_{2n+1}\} \rightarrow a \quad \text{and} \quad \{Z'_{2n+1}\} \rightarrow b$$

Again convergence of the subsequences $\{Z_{2n+1}\}$ and $\{Z'_{2n+1}\}$ implies the convergence of original Cauchy sequences $\{Z_n\}$ and $\{Z'_n\}$ respectively such that

$$\{Z_n\} \rightarrow a \quad \text{and} \quad \{Z'_n\} \rightarrow b.$$

It follows that the sequences $\{Z_n\}$, $\{Z_{2n+1}\}$, $\{Z_{2n+2}\}$ converge to a and $\{Z'_n\}$, $\{Z'_{2n+1}\}$, $\{Z'_{2n+2}\}$ converge to b .

Now $a, b \in T(X)$ implies the existence of $p, q \in X$ such that $T(p) = a$, $T(q) = b$, so that we have

$$\lim_{n \rightarrow \infty} Z_{2n+1} = \lim_{n \rightarrow \infty} A(x_{2n}, y_{2n}) = \lim_{n \rightarrow \infty} T(x_{2n+1}) = a = T(p) \quad (3.10)$$

$$\lim_{n \rightarrow \infty} Z_{2n+2} = \lim_{n \rightarrow \infty} B(x_{2n+1}, y_{2n+1}) = \lim_{n \rightarrow \infty} S(x_{2n+2}) = a = T(p) \quad (3.11)$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} Z'_{2n+1} &= \lim_{n \rightarrow \infty} A(y_{2n}, x_{2n}) = \lim_{n \rightarrow \infty} T(y_{2n+1}) = b = T(q) \\ \lim_{n \rightarrow \infty} Z'_{2n+2} &= \lim_{n \rightarrow \infty} B(y_{2n+1}, x_{2n+1}) = \lim_{n \rightarrow \infty} S(y_{2n+2}) = b = T(q). \end{aligned}$$

Since $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$, we have $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$, so there exists $n_0 \in \mathbb{Z}^+$ such that $\varphi^{n_0}(t) < t$. Thus, from (3.2) we have

$$\begin{aligned} F_{T(x_{2n+1}), B(p,q)}(t) &\geq F_{T(x_{2n+1}), B(p,q)}(\varphi^{n_0}(t)) = F_{A(x_{2n}, y_{2n}), B(p,q)}(\varphi^{n_0}(t)) \\ &\geq \left[\Delta \left(F_{S(x_{2n}), T(p)}(\varphi^{n_0-1}(t)), F_{S(y_{2n}), T(q)}(\varphi^{n_0-1}(t)) \right) \right]^{1/2} \\ &\geq \left[F_{S(x_{2n}), T(p)}(\varphi^{n_0-1}(t)) F_{S(y_{2n}), T(q)}(\varphi^{n_0-1}(t)) \right]^{1/2} \end{aligned} \quad (3.12)$$

Letting $n \rightarrow \infty$ in (3.12), we have $\lim_{n \rightarrow \infty} T(x_{2n+1}) = B(p, q)$. By (3.10), $T(p) = B(p, q) = a$. Similarly, we can show that $T(q) = B(q, p) = b$.

Since the pair (B, T) are weakly compatible, so that $T(p) = B(p, q) = a$ implies $T(a) = B(a, b)$, similarly, $T(b) = B(b, a)$.

Again, since $S(X)$ is complete, so that $a, b \in S(X)$, which implies the existence of r, s in X , so that: $S(r) = a$, $S(s) = b$.

Since $\sum_{n=1}^{\infty} \varphi^n(t) < +\infty$, we have $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$, and so there exists $n_0 \in \mathbb{Z}^+$ such that $\varphi^{n_0}(t) < t$. Thus, from (3.2) we get

$$\begin{aligned} F_{A(r,s), S(x_{2n+2})}(t) &\geq F_{A(r,s), S(x_{2n+2})}(\varphi^{n_0}(t)) = F_{A(r,s), B(x_{2n+1}, y_{2n+1})}(\varphi^{n_0}(t)) \\ &\geq \left[\Delta \left(F_{S(r), T(x_{2n+1})}(\varphi^{n_0-1}(t)), F_{S(s), T(y_{2n+1})}(\varphi^{n_0-1}(t)) \right) \right]^{1/2} \\ &\geq \left[F_{S(r), T(x_{2n+1})}(\varphi^{n_0-1}(t)) F_{S(s), T(y_{2n+1})}(\varphi^{n_0-1}(t)) \right]^{1/2} \end{aligned} \quad (3.13)$$

Letting $n \rightarrow \infty$ in (3.13), we have $\lim_{n \rightarrow \infty} S(x_{2n+2}) = A(r, s)$. By (3.11), $S(r) = A(r, s) = a$. Similarly, we can show that $S(s) = A(s, r) = b$.

Since the pair (A, S) are weakly compatible, it follows that: $A(a, b) = S(a)$ and $A(b, a) = S(b)$.

Step 3 : We claim that $Ta = b$, $Tb = a$ and $Sa = b$, $Sb = a$

In fact, from (3.2) we have

$$\begin{aligned} F_{T(y_{2n+1}), Ta}(\varphi(t)) &= F_{A(y_{2n}, x_{2n}), B(a,b)}(\varphi(t)) \\ &\geq \left[\Delta \left(F_{S(y_{2n}), T(a)}(t), F_{S(x_{2n}), T(b)}(t) \right) \right]^{1/2} \\ &\geq \left[F_{S(y_{2n}), T(a)}(t) F_{S(x_{2n}), T(b)}(t) \right]^{1/2}. \end{aligned} \quad (3.14)$$

Similarly, we have

$$F_{T(x_{2n+1}), Tb}(\varphi(t)) \geq \left[F_{S(x_{2n}), T(b)}(t) F_{S(y_{2n}), Ta}(t) \right]^{1/2} \quad (3.15)$$

Suppose that $Q_n(t) = F_{S(y_{2n}), T(a)}(t) F_{S(x_{2n}), T(b)}(t)$. By (3.14) and (3.15), we have $Q_n(\varphi(t)) \geq Q_{n-1}(t)$, hence

$$Q_n(\varphi^n(t)) \geq Q_{n-1}(\varphi^{n-1}(t)) \geq \dots \geq Q_0(t). \quad (3.16)$$

Furthermore, from (3.14) – (3.16) it follows that

$$F_{T(y_{2n+1}), Ta}(\varphi^n(t)) \geq [Q_0(t)]^{1/2}; \text{ and } F_{T(x_{2n+1}), Tb}(\varphi^n(t)) \geq [Q_0(t)]^{1/2} \quad (3.17)$$

It is evident that $[Q_0(t)]^{1/2} \in D^+$. Since $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$, from (3.17) and lemma 4 we have

$$\lim_{n \rightarrow \infty} T(y_{2n+1}) = Ta \quad \text{and} \quad \lim_{n \rightarrow \infty} T(x_{2n+1}) = Tb.$$

This shows that $Ta = b$ and $Tb = a$. Hence, $B(a, b) = b$ and $B(b, a) = a$.

Similarly, we can show that $Sa = b$ and $Sb = a$. Hence, $A(a, b) = b$ and $A(b, a) = a$.

Step 4: Now we prove that $a = b$.

By (3.2) we have

$$\begin{aligned} F_{a,b}(\varphi(t)) &= F_{A(b,a),B(a,b)}(\varphi(t)) \\ &\geq \left[\Delta \left(F_{S(b),T(a)}(t), F_{S(a),T(b)}(t) \right) \right]^{1/2} \\ &\geq F_{a,b}(t). \end{aligned} \quad (3.18)$$

From (3.18), we have $F_{a,b}(\varphi^n(t)) \geq F_{a,b}(t)$. Using Lemma 2.4, we have $F_{a,b}(t) = 1$, i.e., $a = b$. The uniqueness of a follows from (3.2). So, the proof of Theorem 3.3 is finished.

Theorem 3.4. Let (X, F, Δ) be a Menger metric space with Δ is a t -norm of H -type. Let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$ and $\lim_{n \rightarrow \infty} \varphi^n(t) = +\infty$ for any $t > 0$. Let $A : X \times X \rightarrow X$, $B : X \times X \rightarrow X$, $T : X \rightarrow X$, $S : X \rightarrow X$ be four mappings satisfying the following condition

- (1) $A(X \times X) \subseteq T(X)$, $B(X \times X) \subseteq S(X)$
- (2)

$$F_{A(x,y),B(u,v)}(t) \geq \min \{ F_{Sx,Tu}(\varphi(t)), F_{Sy,Tv}(\varphi(t)) \} \quad (3.19)$$

for all $x, y, u, v \in X$ and $t > 0$

(3) The pairs (A, S) and (B, T) are w -compatible.

(4) One of the subspaces $A(X \times X)$ or $T(X)$ and one of $B(X \times X)$ or $S(X)$ are complete. Then there exists a unique point $a \in X$ such that $A(a, a) = S(a) = T(a) = B(a, a) = a$.

Proof. **Step1:** We show that $\{Z_n\}$ and $\{Z'_n\}$ are Cauchy sequences.

By condition (1), we can construct two sequences $\{Z_n\}$ and $\{Z'_n\}$ in X such that

$$\begin{cases} Z_{2n+1} = A(x_{2n}, y_{2n}) = T(x_{2n+1}) \\ Z_{2n+2} = B(x_{2n+1}, y_{2n+1}) = S(x_{2n+2}) \\ Z'_{2n+1} = A(y_{2n}, x_{2n}) = T(y_{2n+1}) \\ Z'_{2n+2} = B(y_{2n+1}, x_{2n+1}) = S(y_{2n+2}) \end{cases}$$

Suppose that $t > 0$. From (3.19), we have

$$\begin{aligned} F_{Z_n, Z_{n+1}}(\varphi(t)) &= F_{Tx_n, Sx_{n+1}}(\varphi(t)) \\ &= F_{A(x_{n-1}, y_{n-1}), B(x_n, y_n)}(\varphi(t)) \\ &\geq [F_{Sx_{n-1}, Tx_n}(t) F_{Sy_{n-1}, Ty_n}(t)]^{1/2} \\ &= [F_{Z_{n-1}, Z_n}(t) F_{Z'_{n-1}, Z'_n}(t)]^{1/2} \end{aligned} \quad (3.20)$$

$$\begin{aligned}
F_{Z'_n, Z'_{n+1}}(\varphi(t)) &= F_{Ty_n, Sy_{n+1}}(\varphi(t)) \\
&= F_{A(y_{n-1}, x_{n-1}), B(y_n, x_n)}(\varphi(t)) \\
&\geq [F_{Sy_{n-1}, Ty_n}(t) F_{Sx_{n-1}, Tx_n}(t)]^{1/2} \\
&= [F_{Z'_{n-1}, Z'_n}(t) F_{Z_{n-1}, Z_n}(t)]^{1/2}
\end{aligned} \tag{3.21}$$

Suppose that $P_n(t) = [F_{Z_{n-1}, Z_n}(t) F_{Z'_{n-1}, Z'_n}(t)]^{1/2}$. Then, (3.20) and (3.21) we obtain $P_{n+1}(\varphi(t)) \geq P_n(t)$. This implies that

$$F_{Z_n, Z_{n+1}}(\varphi^n(t)) \geq P_n(\varphi^{n-1}(t)) \geq \dots \geq P_1(t) \quad \text{and.} \tag{3.22}$$

$$F_{Z'_n, Z'_{n+1}}(\varphi^n(t)) \geq P_n(\varphi^{n-1}(t)) \geq \dots \geq P_1(t). \tag{3.23}$$

Since $P_1(t) = [F_{Z_0, Z_1}(t) F_{Z'_0, Z'_1}(t)]^{1/2} \in D^+$ and $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$ for each $t > 0$, by lemma(4) we have

$$\lim_{n \rightarrow \infty} F_{Z_n, Z_{n+1}}(t) = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} F_{Z'_n, Z'_{n+1}}(t) = 1 \tag{3.24}$$

Thus, by (3.24), we have

$$\lim_{n \rightarrow \infty} P_n(t) = 1 \quad \text{for all } t > 0 \tag{3.25}$$

We claim that, for any $k \in \mathbb{Z}^+$,

$$F_{Z_n, Z_{n+k}}(t) \geq \Delta^k(P_n(t - \varphi(t))) \quad \text{and} \quad F_{Z'_n, Z'_{n+k}}(t) \geq \Delta^k(P_n(t - \varphi(t))). \tag{3.26}$$

In fact, this is obvious for $k = 1$ by (3.20) and (3.21). Assume that (3.26) holds for some k .

Since $\varphi(t) < t$. By (3.20), we have

$F_{Z_n, Z_{n+1}}(t) \geq F_{Z_n, Z_{n+1}}(\varphi(t)) \geq P_n(t)$. By (3.19) and (3.26), we have

$$\begin{aligned}
F_{Z_{n+1}, Z_{n+k+1}}(\varphi(t)) &\geq [F_{Z_n, Z_{n+k}}(t) F_{Z'_n, Z'_{n+k}}(t)]^{1/2} \\
&\geq \Delta^k(P_n(t - \varphi(t))).
\end{aligned}$$

Hence, by the monotonicity of Δ , we have

$$\begin{aligned}
F_{Z_n, Z_{n+k+1}}(t) &= F_{Z_n, Z_{n+k+1}}(t - \varphi(t) + \varphi(t)) \\
&\geq \Delta(F_{Z_n, Z_{n+1}}(t - \varphi(t)), F_{Z_{n+1}, Z_{n+k+1}}(\varphi(t))) \\
&\geq \Delta(P_n(t - \varphi(t)), \Delta^k(P_n(t - \varphi(t)))) = \Delta^{k+1}(P_n(t - \varphi(t))).
\end{aligned}$$

Similarly, we have $F_{Z'_n, Z'_{n+k+1}}(t) \geq \Delta^{k+1}(P_n(t - \varphi(t)))$. Therefore, by induction, (3.26) holds for all $k \in \mathbb{Z}^+$. Suppose that $\varepsilon > 0$ and $\lambda \in (0, 1]$ are given.

By hypothesis, Δ is a t-norm of H-type; there exists $\delta > 0$ such that

$$\Delta^k(s) > 1 - \lambda, \quad \text{for all } s \in (1 - \delta, 1] \text{ and } k \in \mathbb{Z}^+. \tag{3.27}$$

By (3.25), there exists $N \in \mathbb{Z}^+$ such that $P_n(\varepsilon - \varphi(\varepsilon)) > 1 - \delta$ for all $n \geq N$.

Hence, from (3.26) and (3.27) we get

$$F_{Z_n, Z_{n+k}}(\varepsilon) > 1 - \lambda \text{ and } F_{Z'_n, Z'_{n+k}}(\varepsilon) > 1 - \lambda, \text{ for all } n \geq N; k \in \mathbb{Z}^+.$$

Therefore, $\{Z_n\}$ and $\{Z'_n\}$ are all Cauchy sequences.

Step 2: We show that $T(a) = B(a, b)$, $T(b) = B(b, a)$ and $S(a) = A(a, b)$, $S(b) = A(b, a)$ without loss of generality, assume that $T(X)$ and $S(X)$ are complete. Now $\{Z_{2n+1}\}$, $\{Z_{2n+2}\}$ and $\{Z'_{2n+1}\}$, $\{Z'_{2n+2}\}$ being respective subsequences of Cauchy sequences $\{Z_n\}$ and $\{Z'_n\}$ are also Cauchy.

Since $T(X)$ is complete, so there exists a, b in X such that:

$$\{Z_{2n+1}\} \rightarrow a \text{ and } \{Z'_{2n+1}\} \rightarrow b$$

Again convergence of the subsequences $\{Z_{2n+1}\}$ and $\{Z'_{2n+1}\}$ implies the convergence of original Cauchy sequences $\{Z_n\}$ and $\{Z'_n\}$ respectively such that

$$\{Z_n\} \rightarrow a \text{ and } \{Z'_n\} \rightarrow b.$$

It follows that the sequences $\{Z_n\}$, $\{Z_{2n+1}\}$, $\{Z_{2n+2}\}$ converge to a and $\{Z'_n\}$, $\{Z'_{2n+1}\}$, $\{Z'_{2n+2}\}$ converge to b .

Now $a, b \in T(X)$ implies the existence of $p, q \in X$ such that $T(p) = a$, $T(q) = b$, so that we have

$$\lim_{n \rightarrow \infty} Z_{2n+1} = \lim_{n \rightarrow \infty} A(x_{2n}, y_{2n}) = \lim_{n \rightarrow \infty} T(x_{2n+1}) = a = T(p) \quad (3.28)$$

$$\lim_{n \rightarrow \infty} Z_{2n+2} = \lim_{n \rightarrow \infty} B(x_{2n+1}, y_{2n+1}) = \lim_{n \rightarrow \infty} S(x_{2n+2}) = a = T(p) \quad (3.29)$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} Z'_{2n+1} &= \lim_{n \rightarrow \infty} A(y_{2n}, x_{2n}) = \lim_{n \rightarrow \infty} T(y_{2n+1}) = b = T(q) \\ \lim_{n \rightarrow \infty} Z'_{2n+2} &= \lim_{n \rightarrow \infty} B(y_{2n+1}, x_{2n+1}) = \lim_{n \rightarrow \infty} S(y_{2n+2}) = b = T(q), \end{aligned}$$

From (3.19) and $\varphi(t) < t$, we obtain

$$\begin{aligned} F_{T(x_{2n+1}), B(p, q)}(t) &\geq F_{T(x_{2n+1}), B(p, q)}(\varphi(t)) = F_{A(x_{2n}, y_{2n}), B(p, q)}(\varphi(t)) \\ &\geq \left[F_{S(x_{2n}), T(p)}(t) F_{S(y_{2n}), T(q)}(t) \right]^{1/2} \end{aligned} \quad (3.30)$$

Letting $n \rightarrow \infty$ in (3.30), we have $\lim_{n \rightarrow \infty} T(x_{2n+1}) = B(p, q)$. Hence, $T(p) = B(p, q) = a$.

Similarly, we can show that $T(q) = B(q, p) = b$.

Since the pair (B, T) are weakly compatible, so that $T(p) = B(p, q) = a$ implies $T(a) = B(a, b)$, similarly, $T(b) = B(b, a)$.

Again, since $S(X)$ is complete, so that $a, b \in S(X)$, which implies the existence of r, s in X , so that: $S(r) = a$, $S(s) = b$.

Similarly, we can show that $S(s) = A(s, r) = b$ and $S(r) = A(r, s) = a$. Since the pair (A, S) are weakly compatible, it follows that: $A(a, b) = S(a)$ and $A(b, a) = S(b)$.

Step 3: We claim that $Ta = b$, $Tb = a$ and $Sa = b$, $Sb = a$

In fact, from (3.19) we have

$$\begin{aligned} F_{T(y_{2n+1}),Ta}(\varphi(t)) &= F_{A(y_{2n},x_{2n}),B(a,b)}(\varphi(t)) \\ &\geq \left[F_{S(y_{2n}),T(a)}(t) F_{S(x_{2n}),T(b)}(t) \right]^{1/2}. \end{aligned} \quad (3.31)$$

Similarly, we have

$$F_{T(x_{2n+1}),Tb}(\varphi(t)) \geq \left[F_{S(x_{2n}),T(b)}(t) F_{S(y_{2n}),Ta}(t) \right]^{1/2} \quad (3.32)$$

Suppose that $Q_n(t) = F_{S(y_{2n}),T(a)}(t) F_{S(x_{2n}),T(b)}(t)$.

By (3.31) and (3.32), we have

$$Q_n(\varphi^n(t)) \geq Q_{n-1}(\varphi^{n-1}(t)) \geq \dots \geq Q_0(t).$$

$$F_{T(y_{2n+1}),Ta}(\varphi^n(t)) \geq [Q_0(t)]^{1/2}; \text{ and } F_{T(x_{2n+1}),Tb}(\varphi^n(t)) \geq [Q_0(t)]^{1/2}$$

Since $[Q_0(t)]^{1/2} \in D^+$. Since $\lim_{n \rightarrow \infty} \varphi^n(t) = 0$, by Lemma 2.4 we conclude that

$$\lim_{n \rightarrow \infty} T(y_{2n+1}) = Ta \text{ and } \lim_{n \rightarrow \infty} T(x_{2n+1}) = Tb.$$

This shows that $Ta = b$ and $Tb = a$. Hence, $B(a, b) = b$ and $B(b, a) = a$.

Similarly, we can show that $Sa = b$ and $Sb = a$. Hence, $A(a, b) = b$ and $A(b, a) = a$.

Step 4: Finally, we prove that $a = b$. By (3.19) we have

$$\begin{aligned} F_{a,b}(\varphi(t)) &= F_{A(b,a),B(a,b)}(\varphi(t)) \\ &\geq \left[F_{S(b),T(a)}(t) F_{S(a),T(b)}(t) \right]^{1/2} = F_{a,b}(t). \end{aligned} \quad (3.33)$$

From (3.33), we have $F_{a,b}(\varphi^n(t)) \geq F_{a,b}(t)$. Using Lemma 2.4, we have $F_{a,b}(t) = 1$, i.e., $a = b$. The uniqueness of a follows from (3.19). So, the proof of Theorem 3.4 is finished.

Theorem 3.5. Let (X, F, Δ) be a Menger metric space with Δ is a t -norm of H -type. Let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a gauge function such that $\varphi^{-1}(\{0\}) = \{0\}$, $\varphi(t) < t$ and $\lim_{n \rightarrow \infty} \varphi^n(t) = +\infty$ for any $t > 0$. Let $A : X \times X \rightarrow X$, $B : X \times X \rightarrow X$, $T : X \rightarrow X$, $S : X \rightarrow X$ be four mappings satisfying the following condition

- (1) $A(X \times X) \subseteq T(X)$, $B(X \times X) \subseteq S(X)$
- (2)

$$F_{A(x,y),B(u,v)}(t) \geq \min \{ F_{Sx,Tu}(\varphi(t)), F_{Sy,Tv}(\varphi(t)) \} \quad (3.34)$$

for all $x, y, u, v \in X$ and $t > 0$

(3) The pairs (A, S) and (B, T) are w -compatible.

(4) One of the subspaces $A(X \times X)$ or $T(X)$ and one of $B(X \times X)$ or $S(X)$ are complete. Then there exists a unique point $a \in X$ such that $A(a, a) = S(a) = T(a) = B(a, a) = a$.

Proof. **Step1:** We show that $\{Z_n\}$ and $\{Z'_n\}$ are Cauchy sequences.

By condition (1), we can construct two sequences $\{Z_n\}$ and $\{Z'_n\}$ in X such that

$$\begin{cases} Z_{2n+1} = A(x_{2n}, y_{2n}) = T(x_{2n+1}) \\ Z_{2n+2} = B(x_{2n+1}, y_{2n+1}) = S(x_{2n+2}) \\ Z'_{2n+1} = A(y_{2n}, x_{2n}) = T(y_{2n+1}) \\ Z'_{2n+2} = B(y_{2n+1}, x_{2n+1}) = S(y_{2n+2}) \end{cases}$$

Suppose that $t > 0$. From (3.34), we have

$$\begin{aligned} F_{Z_n, Z_{n+1}}(t) &= F_{Tx_n, Sx_{n+1}}(t) \\ &= F_{A(x_{n-1}, y_{n-1}), B(x_n, y_n)}(t) \\ &\geq \min \{F_{Sx_{n-1}, Tx_n}(\varphi(t)), F_{Sy_{n-1}, Ty_n}(\varphi(t))\} \\ &= \min \{F_{Z_{n-1}, Z_n}(\varphi(t)), F_{Z'_{n-1}, Z'_n}(\varphi(t))\} \end{aligned} \quad (3.35)$$

$$\begin{aligned} F_{Z'_n, Z'_{n+1}}(t) &= F_{Ty_n, Sy_{n+1}}(t) \\ &= F_{A(y_{n-1}, x_{n-1}), B(y_n, x_n)}(t) \\ &\geq \min \{F_{Sy_{n-1}, Ty_n}(\varphi(t)), F_{Sx_{n-1}, Tx_n}(\varphi(t))\} \\ &= \min \{F_{Z'_{n-1}, Z'_n}(\varphi(t)), F_{Z_{n-1}, Z_n}(\varphi(t))\} \end{aligned} \quad (3.36)$$

Suppose that $E_n(t) = \min \{F_{Z'_{n-1}, Z'_n}(\varphi(t)), F_{Z_{n-1}, Z_n}(\varphi(t))\}$. Then, (3.35) and (3.36) we obtain $E_{n+1}(t) \geq E_n(\varphi(t))$. This implies that

$$E_{n+1}(t) \geq E_n(\varphi(t)) \geq E_{n-1}(\varphi^2(t)) \geq \dots \geq E_1(\varphi^n(t)) \quad \text{and.} \quad (3.37)$$

Since $\lim_{t \rightarrow +\infty} E_1(t) = \lim_{t \rightarrow +\infty} \min \{F_{Z'_0, Z'_1}(\varphi(t)), F_{Z_0, Z_1}(\varphi(t))\} = 1$ and $\lim_{n \rightarrow \infty} \varphi^n(t) = +\infty$ for each $t > 0$, we have

$\lim_{t \rightarrow +\infty} E_1(\varphi^n(t)) = 1$. Moreover, by (3.35)-(3.37), we have $F_{Z_n, Z_{n+1}}(t) \geq E_1(\varphi^n(t))$ and $F_{Z'_n, Z'_{n+1}}(t) \geq E_1(\varphi^n(t))$. Hence, $\lim_{t \rightarrow +\infty} F_{Z_n, Z_{n+1}}(t) = 1$ and $\lim_{t \rightarrow +\infty} F_{Z'_n, Z'_{n+1}}(t) = 1$. This implies that

$$\lim_{t \rightarrow +\infty} E_n(t) = 1 \quad \text{for all } t > 0. \quad (3.38)$$

In the next we show that, for any $k \in \mathbb{Z}^+$,

$$F_{Z_n, Z_{n+k}}(\varphi(t)) \geq \Delta^k(E_n(\varphi(t) - t)) \quad \text{and} \quad F_{Z'_n, Z'_{n+k}}(\varphi(t)) \geq \Delta^k(E_n(\varphi(t) - t)) \quad (3.39)$$

In fact, this is obvious for $k = 1$ by (3.35) and (3.36). Assume that (3.39) holds for some k . Since $\varphi(t) > t$, by (3.35), we have

$F_{Z_n, Z_{n+1}}(t) \geq E_n(\varphi(t)) \geq E_n(t)$. By (3.34) and (3.39), we have

$$F_{Z_{n+1}, Z_{n+k+1}}(t) \geq \min \{F_{Z_n, Z_{n+k}}(\varphi(t)), F_{Z'_n, Z'_{n+k}}(\varphi(t))\} \geq \Delta^k(E_n(\varphi(t) - t))$$

Hence, by the monotonicity of Δ , we have

$$F_{Z_n, Z_{n+k+1}}(\varphi(t)) = F_{Z_n, Z_{n+k+1}}(\varphi(t) - t + t)$$

$$\begin{aligned} &\geq \Delta (F_{Z_n, Z_{n+1}} (\varphi(t) - t), F_{Z_{n+1}, Z_{n+k+1}} (t)) \\ &\geq \Delta (E_n (\varphi(t) - t), \Delta^k (E_n (\varphi(t) - t))) = \Delta^{k+1} (E_n (\varphi(t) - t)) \end{aligned}$$

Similarly, we have $F_{Z'_n, Z'_{n+k+1}} (\varphi(t)) \geq \Delta^{k+1} (E_n (\varphi(t) - t))$. Therefore, by induction, (3.39) holds for all $k \in \mathbb{Z}^+$. Furthermore, by (3.34) and (3.39) we have

$$F_{Z_n, Z_{n+k}} (\varphi(t)) \geq \Delta^k (E_{n-1} (\varphi(t) - t)) \text{ and } F_{Z'_n, Z'_{n+k}} (\varphi(t)) \geq \Delta^k (E_{n-1} (\varphi(t) - t)) \quad (3.40)$$

Suppose that $\varepsilon > 0$ and $\lambda \in (0, 1]$ are given. Since Δ is a t-norm of H-type, there exists $\delta > 0$ such that

$$\Delta^k (s) > 1 - \lambda, \text{ for all } s \in (1 - \delta, 1] \text{ and } k \in \mathbb{Z}^+. \quad (3.41)$$

By (3.38), there exists $N \in \mathbb{Z}^+$ such that $E_{n-1} (\varphi(\varepsilon) - \varepsilon) > 1 - \delta$ for all $n \geq N$.

Hence, from (3.40) and (3.41) we get

$$F_{Z_n, Z_{n+k}} (\varepsilon) > 1 - \lambda \text{ and } F_{Z'_n, Z'_{n+k}} (\varepsilon) > 1 - \lambda, \text{ for all } n \geq N \text{ and } k \in \mathbb{Z}^+. \text{ This shows that, } \{Z_n\}$$

and $\{Z'_n\}$ are all Cauchy sequences.

Step 2: We show that $T(a) = B(a, b)$, $T(b) = B(b, a)$ and $S(a) = A(a, b)$, $S(b) = A(b, a)$ without loss of generality, assume that $T(X)$ and $S(X)$ are complete. Now $\{Z_{2n+1}\}$, $\{Z_{2n+2}\}$ and $\{Z'_{2n+1}\}$, $\{Z'_{2n+2}\}$ being respective subsequences of Cauchy sequences $\{Z_n\}$ and $\{Z'_n\}$ are also Cauchy.

Since $T(X)$ is complete, so there exists a, b in X such that:

$$\{Z_{2n+1}\} \rightarrow a \text{ and } \{Z'_{2n+1}\} \rightarrow b$$

Again convergence of the subsequences $\{Z_{2n+1}\}$ and $\{Z'_{2n+1}\}$ implies the convergence of original Cauchy sequences $\{Z_n\}$ and $\{Z'_n\}$ respectively such that

$$\{Z_n\} \rightarrow a \text{ and } \{Z'_n\} \rightarrow b$$

It follows that the sequences $\{Z_n\}$, $\{Z_{2n+1}\}$, $\{Z_{2n+2}\}$ converge to a and $\{Z'_n\}$, $\{Z'_{2n+1}\}$, $\{Z'_{2n+2}\}$ converge to b .

Now $a, b \in T(X)$ implies the existence of $p, q \in X$ such that $T(p) = a$, $T(q) = b$, so that we have

$$\lim_{n \rightarrow \infty} Z_{2n+1} = \lim_{n \rightarrow \infty} A(x_{2n}, y_{2n}) = \lim_{n \rightarrow \infty} T(x_{2n+1}) = a = T(p) \quad (3.42)$$

$$\lim_{n \rightarrow \infty} Z_{2n+2} = \lim_{n \rightarrow \infty} B(x_{2n+1}, y_{2n+1}) = \lim_{n \rightarrow \infty} S(x_{2n+2}) = a = T(p) \quad (3.43)$$

and

$$\begin{aligned} \lim_{n \rightarrow \infty} Z'_{2n+1} &= \lim_{n \rightarrow \infty} A(y_{2n}, x_{2n}) = \lim_{n \rightarrow \infty} T(y_{2n+1}) = b = T(q) \\ \lim_{n \rightarrow \infty} Z'_{2n+2} &= \lim_{n \rightarrow \infty} B(y_{2n+1}, x_{2n+1}) = \lim_{n \rightarrow \infty} S(y_{2n+2}) = b = T(q), \end{aligned}$$

From (3.34), we obtain

$$F_{T(x_{2n+1}), B(p, q)} (t) = F_{A(x_{2n}, y_{2n}), B(p, q)} (t) \geq \min \left\{ F_{S(x_{2n}), T(p)} (\varphi(t)), F_{S(y_{2n}), T(q)} (\varphi(t)) \right\} \quad (3.44)$$

Letting $n \rightarrow \infty$ in (3.42), we have $\lim_{n \rightarrow \infty} T(x_{2n+1}) = B(p, q)$. Hence, $T(p) = B(p, q) = a$. In the same manner we can show that $T(q) = B(q, p) = b$.

Since the pair (B, T) are weakly compatible, so that $T(p) = B(p, q) = a$ implies $T(a) = B(a, b)$, similarly, $T(b) = B(b, a)$.

Again, since $S(X)$ is complete, so that $a, b \in S(X)$, which implies the existence of r, s in X , so that: $S(r) = a, S(s) = b$.

Similarly, we can show that $S(s) = A(s, r) = b$ and $S(r) = A(r, s) = a$.

Since the pair (A, S) are weakly compatible, it follows that: $A(a, b) = S(a)$ and $A(b, a) = S(b)$.

Step 3: We claim that $Ta = b, Tb = a$ and $Sa = b, Sb = a$

In fact, from (3.34) we have

$$\begin{aligned} F_{T(y_{2n}), Ta}(t) &= F_{A(y_{2n}, x_{2n}), B(a, b)}(t) \\ &\geq \min \left\{ F_{S(y_{2n-1}), T(a)}(\varphi(t)), F_{S(x_{2n-1}), T(b)}(\varphi(t)) \right\} \end{aligned} \quad (3.45)$$

Similarly, we have

$$F_{T(x_{2n}), Tb}(t) \geq \min \left\{ F_{S(x_{2n-1}), T(b)}(t), F_{S(y_{2n-1}), Ta}(t) \right\} \quad (3.46)$$

Suppose that $M_n(t) = \min \left\{ F_{T(x_{2n}), Tb}(t), F_{T(y_{2n}), Ta}(t) \right\}$. From (3.43) and (3.44) it follows that

$$M_n(t) \geq M_{n-1}(\varphi(t)) \geq \dots \geq M_0(\varphi^n(t)).$$

Since $\lim_{n \rightarrow \infty} \varphi^n(t) = +\infty$, we have

$$M_0(\varphi^n(t)) = \min \left\{ F_{T(x_0), Tb}(t), F_{T(y_0), Ta}(t) \right\} \rightarrow 1 \text{ as } n \rightarrow \infty$$

This shows that $M_n(t) \rightarrow 1$ as $n \rightarrow \infty$, and so

$$\lim_{n \rightarrow \infty} T(y_{2n+1}) = Ta \text{ and } \lim_{n \rightarrow \infty} T(x_{2n+1}) = Tb.$$

Hence, $Ta = b$ and $Tb = a$.

Similarly, we can show that $Sa = b$ and $Sb = a$. Hence, $A(a, b) = b$ and $A(b, a) = a$.

Step 4: Finally, we prove that $a = b$.

By (3.34) we have

$$F_{a, b}(t) = F_{A(b, a), B(a, b)}(t) \geq \min \left\{ F_{S(b), T(a)}(\varphi(t)), F_{S(a), T(b)}(\varphi(t)) \right\} = F_{a, b}(\varphi(t)). \quad (3.47)$$

From (3.45), we have $F_{a, b}(t) \geq F_{a, b}(\varphi^n(t))$.

Letting $n \rightarrow \infty$, we have $F_{a, b}(t) = 1$, i.e., $a = b$. Since the uniqueness of a follows from (3.34), the proof of Theorem 3.5 is completed.

We give an example to illustrate the validity of Theorem 5.

Example 3.1. Let $X = [0, 1)$ and

$$F_{x, y}(t) = \frac{t}{t + |x - y|}.$$

Let $A : X \times X \rightarrow X$, $B : X \times X \rightarrow X$, $T : X \rightarrow X$ and $S : X \rightarrow X$ mappings, such that:

$$A(x, y) = \begin{cases} \frac{x^2 - y^2}{6} & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases}$$

$$B(x, y) = \begin{cases} \frac{x - y}{6} & \text{if } x \geq y \\ 0 & \text{if } x < y \end{cases}$$

and

$$S(x) = \frac{x^2}{2}, \quad T(x) = \frac{x}{2}.$$

Clearly, $A(X \times X) \subseteq T(X)$ and $B(X \times X) \subseteq S(X)$ and $S(X)$ and $T(X)$ are complete subspace of X .

Next we show that our results can be used for this case .

Let us prove that the pairs (A, S) and (B, T) are w -compatible. It is obtained that:

$$A(x, y) = S(x) \text{ and } A(y, x) = S(y) \text{ if and only if } x = y = 0.$$

Since $A(S(0), S(0)) = S(A(0, 0))$, the mappings A and S are weakly compatible. And

$$B(x, y) = T(x) \text{ and } B(y, x) = T(y) \text{ if and only if } x = y = 0.$$

Since $B(T(0), T(0)) = T(B(0, 0))$.

Finally, we prove that for $x, y, u, v \in X$,

$$F_{A(x,y), B(u,v)}(\varphi(t)) \geq \Delta(F_{Sx, Tu}(t), F_{Sy, Tv}(t)).$$

Let $\varphi : (0, \infty) \rightarrow (0, \infty)$ by $\varphi(t) = \frac{3}{2}t$, Then $\lim_{n \rightarrow +\infty} \varphi^n(t) = +\infty$ for any $t > 0$. we distinguish the following cases:

Cas 1: $x \geq y$ and $u \geq v$

$$\begin{aligned} F_{A(x,y), B(u,v)}(t) &= \frac{t}{t + \left| \frac{x^2 - y^2}{6} - \frac{u - v}{6} \right|} \\ &= \frac{3t}{3t + \left| \left(\frac{x^2}{2} - \frac{u}{2} \right) - \left(\frac{y^2}{2} - \frac{v}{2} \right) \right|} \\ &\geq \text{Min} \left\{ \frac{\frac{3}{2}t}{\frac{3}{2}t + |x^2 - u|}, \frac{\frac{3}{2}t}{\frac{3}{2}t + |y^2 - v|} \right\} \\ &= \text{Min} \{ F_{Sx, Tu}(\varphi(t)), F_{Sy, Tv}(\varphi(t)) \}. \end{aligned}$$

Cas 2: $x \geq y$, and $u \leq v$

$$\begin{aligned} F_{A(x,y), B(u,v)}(t) &= \frac{t}{t + \left| \frac{x^2 - y^2}{6} - 0 \right|} \\ &= \frac{3t}{3t + \left| \frac{x^2 - y^2}{2} \right|} \end{aligned}$$

$$\begin{aligned}
&= \frac{3t}{3t + \left| \left(\frac{x^2}{2} - \frac{u}{2} \right) - \left(\frac{y^2}{2} - \frac{u}{2} \right) \right|} \\
&\geq \frac{\frac{3}{2}t}{\frac{3}{2}t + \left| \frac{x^2}{2} - \frac{u}{2} \right|} \\
&\geq \text{Min} \{ F_{Sx,Tu}(\varphi(t)), F_{Sy,Tv}(\varphi(t)) \}
\end{aligned}$$

Cas 3: $x \leq y$, and $u \geq v$

$$\begin{aligned}
F_{A(x,y),B(u,v)}(t) &= \frac{t}{t + \left| 0 - \frac{u-v}{6} \right|} \\
&= \frac{3t}{3t + \left| \left(\frac{x^2}{2} - \frac{u}{2} \right) - \left(\frac{v}{2} - \frac{x^2}{2} \right) \right|} \\
&\geq \frac{\frac{3}{2}t}{\frac{3}{2}t + \left| \frac{x^2}{2} - \frac{u}{2} \right|} \\
&\geq \text{Min} \{ F_{Sx,Tu}(\varphi(t)), F_{Sy,Tv}(\varphi(t)) \}
\end{aligned}$$

Cas 4: $x \leq y$, and $u \leq v$

$$\begin{aligned}
F_{A(x,y),B(u,v)}(t) &= \frac{t}{t + 0} \\
&= 1 \\
&\geq \text{Min} \{ F_{Sx,Tu}(\varphi(t)), F_{Sy,Tv}(\varphi(t)) \}
\end{aligned}$$

Hence, all the hypotheses of Theorem 3.5 hold. Clearly $(0,0)$ is the unique common coupled fixed point of A, B, S and T .

4. Application to integral equations

As an application of the coupled fixed point theorems established in section 3 of our paper, we study the existence and uniqueness of the solution to a Fredholm nonlinear integral equation.

We shall consider the following integral equation,

$$x(p) = \int_a^b (K_1(p,q) + K_2(p,q)) [f(q, x(q)) + g(q, x(q))] dq + h(p), \quad (4.1)$$

for all $p \in I = [a, b]$.

Let Θ denote the set of all functions $\theta : [0, 1] \rightarrow [0, 1]$ satisfying

- (i) _{θ} θ is non-decreasing,
- (ii) _{θ} $\theta(p) \leq p$.

We assume that the functions K_1, K_2, f, g fulfill the following conditions:

Assumption 4.1.

- (i) $K_1(p, q) \geq 0$ and $K_2(p, q) \leq 0$ for all $p, q \in I$,
(ii) There exists $\theta \in \Theta$ such that for all $x, y \in \mathbb{R}$ with $x \geq y$, the following conditions hold:

$$0 \leq f(q, x) - f(q, y) \leq \lambda \theta(x - y) \quad (4.2)$$

and

$$-\mu \theta(x - y) \leq g(q, x) - g(q, y) \leq 0, \quad (4.3)$$

(iii)

$$\max\{\lambda, \mu\} \sup_{p \in I} \int_a^b [K_1(p, q) - K_2(p, q)] dq \leq \frac{1}{4} \quad (4.4)$$

Consider the integral equation (4.1) with $K_1, K_2 \in C(I \times I, \mathbb{R})$ and $h \in C(I, \mathbb{R})$. Suppose that Assumption 4.1 is satisfied. Then the integral equation (4.1) has a unique solution in $C(I, \mathbb{R})$.

Proof. Consider $X = C(I, \mathbb{R})$. It is easy to check that $(X, F, L, *, \diamond)$ is a complete Menger metric space with respect to the distribution distance

$$F_{x,y}(t) = \frac{t}{t + |x - y|}, \text{ for all } x, y \in X \text{ and } t > 0 \text{ with } x * y = \min(x, y) \text{ for all } x, y \in X.$$

Define now the mapping $T : X \times X \rightarrow X$ by

$$\begin{aligned} T(x, y)(p) &= \int_a^b K_1(p, q) [f(q, x(q)) + g(q, y(q))] dq \\ &\quad + \int_a^b K_2(p, q) [f(q, y(q)) + g(q, x(q))] dq + h(p) \end{aligned} \quad (4.5)$$

for all $p \in I$ and $k = \frac{1}{2}$ for all $t > 0$. Now, for all $x, y, u, v \in X$, using (4.2) and (4.3), we have

$$\begin{aligned} &T(x, y)(p) - T(u, v)(p) \\ &= \int_a^b K_1(p, q) [f(q, x(q)) + g(q, y(q))] dq \\ &\quad + \int_a^b K_2(p, q) [f(q, y(q)) + g(q, x(q))] dq \\ &\quad - \int_a^b K_1(p, q) [f(q, u(q)) + g(q, v(q))] dq \\ &\quad - \int_a^b K_2(p, q) [f(q, v(q)) + g(q, u(q))] dq \\ &= \int_a^b K_1(p, q) [f(q, x(q)) - f(q, u(q)) + g(q, y(q)) - g(q, v(q))] dq \\ &\quad + \int_a^b K_2(p, q) [f(q, y(q)) - f(q, v(q)) + g(q, x(q)) - g(q, u(q))] dq \\ &= \int_a^b K_1(p, q) [(f(q, x(q)) - f(q, u(q))) - (g(q, v(q)) - g(q, y(q)))] dq \end{aligned} \quad (4.6)$$

$$\begin{aligned}
& - \int_a^b K_2(p, q) [(f(q, v(q)) - f(q, y(q))) - (g(q, x(q)) - g(q, u(q)))] dq \\
\leq & \int_a^b K_1(p, q) [\lambda \theta(x(q) - u(q)) + \mu \theta(v(q) - y(q))] dq \\
& - \int_a^b K_2(p, q) [\lambda \theta(v(q) - y(q)) + \mu \theta(x(q) - u(q))] dq
\end{aligned}$$

Since the function θ is non-decreasing and so we have

$$\theta(x(q) - u(q)) \leq \theta(|x(q) - u(q)|)$$

and

$$\theta(v(q) - y(q)) \leq \theta(|v(q) - y(q)|),$$

hence by (4.6), in view of the fact $K_2(p, q) \leq 0$, we get

$$\begin{aligned}
& |T(x, y)(p) - T(u, v)(p)| \tag{4.7} \\
\leq & \int_a^b K_1(p, q) [\lambda \theta(|x(q) - u(q)|) + \mu \theta(|v(q) - y(q)|)] dq \\
& - \int_a^b K_2(p, q) [\lambda \theta(|v(q) - y(q)|) + \mu \theta(|x(q) - u(q)|)] dq \\
\leq & \int_a^b K_1(p, q) [\max\{\lambda, \mu\} \theta(|x(q) - u(q)|) + \max\{\lambda, \mu\} \theta(|v(q) - y(q)|)] dq \\
& - \int_a^b K_2(p, q) [\max\{\lambda, \mu\} \theta(|v(q) - y(q)|) + \max\{\lambda, \mu\} \theta(|x(q) - u(q)|)] dq
\end{aligned}$$

as all the quantities on the right hand side of (4.6) are non-negative. Now by using (4.5), we get

$$\begin{aligned}
& |T(x, y) - T(u, v)| \tag{4.8} \\
\leq & \max\{\lambda, \mu\} \int_a^b [K_1(p, q) - K_2(p, q)] dq. [\theta(|x(q) - u(q)|) + \theta(|v(q) - y(q)|)] \\
\leq & \max\{\lambda, \mu\} \sup_{p \in I} \int_a^b [K_1(p, q) - K_2(p, q)] dq. [\theta(|x(q) - u(q)|) + \theta(|v(q) - y(q)|)] \\
\leq & \frac{\theta(|x - u|) + \theta(|v - y|)}{4},
\end{aligned}$$

Thus

$$2|T(x, y) - T(u, v)| \leq \frac{\theta(|x - u|) + \theta(|v - y|)}{2} \tag{4.9}$$

Now, since θ is nondecreasing, we have

$$\begin{aligned}
\theta(|x - u|) & \leq \theta(|x - u|) + \theta(|y - v|), \\
\theta(|y - v|) & \leq \theta(|x - u|) + \theta(|y - v|),
\end{aligned} \tag{4.10}$$

which, by using (ii_θ) , this implies

$$\begin{aligned} \frac{\theta(|x-u|) + \theta(|y-v|)}{2} &\leq \theta(|x-u| + |y-v|) \\ &\leq |x-u| + |y-v| \\ &\leq 2 \max\{|x-u|, |y-v|\}, \end{aligned} \quad (4.11)$$

and so

$$\frac{\theta(|x-u|) + \theta(|y-v|)}{2} \leq \max\{|x-u|, |y-v|\} \quad (4.12)$$

Thus, by (4.9) and (4.12), we get

$$2|T(x, y) - T(u, v)| \leq \max\{|x-u|, |y-v|\} \quad (4.13)$$

Now, it follows that

$$\begin{aligned} F_{T(x,y), T(u,v)}(kt) &= F_{T(x,y), T(u,v)}\left(\frac{t}{2}\right) \\ &= \frac{\frac{t}{2}}{\frac{t}{2} + |T(x, y) - T(u, v)|} \\ &= \frac{t}{t + 2|T(x, y) - T(u, v)|} \\ &\geq \frac{t}{t + \max\{|x-u|, |y-v|\}} \\ &\geq \min\left\{\frac{t}{t + |x-u|}, \frac{t}{t + |y-v|}\right\} \\ &\geq \min\{F_{x,u}(t), F_{y,v}(t)\}, \end{aligned}$$

Thus

$$F_{T(x,y), T(u,v)}(kt) \geq F_{x,u}(t) * F_{y,v}(t)$$

show that all hypotheses of Corollary 3.2 are satisfied.

This proves that T has a unique fixed point $a \in X$, that is, $a = T(a, a)$ and therefore $a \in C(I, \mathbb{R})$ is the unique solution of the integral equation (4.1).

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