Moroccan J. of Pure and Appl. Anal. (MJPAA)

Volume 5(2), 2019, Pages 179–185

ISSN: Online 2351-8227 - Print 2605-6364

DOI 10.2478/mjpaa-2019-0013

Some Identities and inequalities related to the Riemann zeta function

GREGORY ABE-I-KPENG^{1,a}, MOHAMMED MUNIRU IDDRISU^{1,b} AND KWARA NANTOMAH^{1,c}

ABSTRACT. A new proof of Euler's formula for calculating $\zeta(2k)$ is given. Some new inequalities and identities for $\zeta(2k+1)$ have also been given. The Riemann's functional equation together with trigonometric identities were used to establish the results.

2010 Mathematics Subject Classification. 11S40, 11M41, 26D20.

Key words and phrases. Riemann zeta function, Inequalities, Bernoulli numbers.

1. Introduction

The zeta function was introduced by Leonhard Euler as a function of a real variable without using complex analysis which was not well established at that time. He established some properties and computational values of $\zeta(2n)$ where n is a natural number.

Bernhard Riemann extended Euler's work to a complex variable and established its meromorphic continuation and functional equations and defined a connection between the distribution of prime numbers and its zeros. He showed that this function has zeros for negative even numbers which he termed trivial zeros. He also defined a connection between the distribution of primes and its nontrivial zeros and gave a hypothesis that the non trivial zeros are found in the critical strip.

Identities and inequalities related to the Riemann zeta function have been studied in the works [2] and [6] and there still remains a lot to be studied.

Laforgia and Natalini [6] used the Schwarz inequality to give new Turan-type inequalities for the Riemann zeta

Received September 13, 2019 - Accepted November 19, 2019.

©The Author(s) 2017. This article is published with open access by Sidi Mohamed Ben Abdallah University.

¹Department of Mathematics, Faculty of Mathematical Sciences, University for Development studies, Navrongo Campus, Navrongo, Ghana.

^a e-mail: gabeikpeng@gmail.com; gregory.kpeng@uds.edu.gh

^b e-mail: mmuniru@uds.edu.gh

^c e-mail: knantomah@uds.edu.gh .

function. Ahsan et al [2] established an inequality for the Riemann zeta function which implies that in [6]. The properties of the Riemann zeta function have been studied extensively in [3], [5] and [8].

2. Materials and Methods

In his paper, Riemann [9] developed novel method of analyzing functions of a complex variable and used them to derive a new representation of the zeta function as

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^{s-1}}{e^x - 1} dx, \quad (2.1)$$

where $s = \sigma + it$, $i = \sqrt{-1}$ and $\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx$.

Riemann defined the zeta function in connection with the product of primes as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left[1 - \frac{1}{p^s} \right]^{-1}, \quad (2.2)$$

where $s = \sigma + it$ and $\text{Re}(s) > 1$.

Riemann also introduced other functional equations for all s except $s = 1$ and $s = 0$ as

$$\zeta(s) = 2^s \pi^{s-1} \sin(\pi s/2) \Gamma(1-s) \zeta(1-s) \quad (2.3)$$

and

$$\zeta(1-s) = 2^{1-s} \pi^{-s} \cos(\pi(1-s)/2) \Gamma(s) \zeta(s), \quad (2.4)$$

where $\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx$.

Euler in 1734 found values of $\zeta(2k)$ for $k = 1, 2, \dots$ using the formular

$$\zeta(2k) = \frac{(-1)^{k-1} (2\pi)^{2k} B_{2k}}{2(2k)!}, \quad (2.5)$$

where B_{2k} is a Bernuolli number.

Equation (2.5) can be re-written as

$$\zeta(2k) = -\frac{(2\pi)^{2k} B_{2k}}{2(-1)^k (2k)!}. \quad (2.6)$$

Euler's original proof of $\zeta(2k)$ was established from two different representations of $z \cot z$, a power series expansion obtainable from (2.10).

$z \cot z$ can be expressed as

$$z \cot z = 1 - 2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{\pi^{2k}} z^{2k} \quad (2.7)$$

and

$$z \cot z = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k 2^{2k} B_{2k}}{(2k)!} z^{2k}. \quad (2.8)$$

Comparing (2.7) and (2.8) yields (3.1).

Enrique et al [4] used the Taylor series expansion for the tangent function to prove Euler's formula as

$$\zeta(2k) = \left(1 - \frac{1}{4^k}\right)^{-1} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}}, \quad (2.9)$$

where $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^{2k}} = \frac{(-1)^k B_{2k}(1-4^k)}{(2k)!} \frac{\pi^{2k}}{2}$.

Definition 2.1. The Bernuolli numbers $B_0, B_1, \dots$ are defined as the coefficients in the Taylor expansion of $f(z)$ in $B(0, 2\pi)$:

$$\begin{aligned} f(z) &= \frac{z}{e^z - 1} = 1 - \frac{1}{2}z + \frac{1}{2!} \frac{1}{6}z^2 + \frac{1}{4!} \left(-\frac{1}{30}\right)z^4 \\ &= \sum_{k=0}^{\infty} B_k \frac{z^k}{k!}, z \in B(0, 2\pi). \end{aligned} \quad (2.10)$$

Thus, $B_0 = 1$ and

$$\sum_{k=0}^{\infty} \frac{1}{k!(n-k)!} B_k = 0 \quad (2.11)$$

implies

$$\sum_{k=0}^{\infty} (nk) B_k = 0,$$

for $n \geq 2$.

Definition 2.2. The Bernuolli polynomials $B_0(t), B_1(t), \dots$ are defined as $B_0(t)$ and for $n \geq 1$ by the recursive formular

$$B_n(t) = nB_{n-1}(t) \quad (2.12)$$

or

$$B_n(t) = \sum_{j=0}^n \binom{n}{j} B_j t^{n-j} \quad (2.13)$$

and

$$\int_0^1 B_n(t) dt = 0. \quad (2.14)$$

Thus by (2.13) we have

$$B_{2k+1}(t) = \sum_{j=0}^{2k+1} \binom{2k+1}{j} B_j t^{(2k+1)-j}. \quad (2.15)$$

Minac [7] connected the values of the Riemann zeta function for all non positive integers, s , with the partial sums $T_s(x) = \sum_{n=1}^{x-1} n^s$ as

$$\int_0^1 T_s(x) dx = \zeta(-s). \quad (2.16)$$

By [7], $T_s(x)$ is expressible as a polynomials in x , of degree $s+1$ with rational coefficients. For example, $S_1 = \frac{x(x-1)}{2}$ and $S_2 = \frac{x(x-1)(2x-1)}{6}$.

In [7] we can also find

$$\zeta(-s) = (-1)^s \frac{B_{s+1}}{s+1}. \quad (2.17)$$

An inequality involving the Riemann zeta function was presented in [6] as

$$(s+1) \frac{\zeta(s)}{\zeta(s+1)} \geq s \frac{\zeta(s+1)}{\zeta(s+2)}, \quad s > 1. \quad (2.18)$$

In [2], we also find

$$\frac{\zeta(s)}{\zeta(s+1)} > \frac{\zeta(s+1)}{\zeta(s+2)}, \quad s > 1, \quad (2.19)$$

which is sharper than (2.18).

Also in [2] we have

$$\zeta(2k+1) < \frac{(2\pi)^{2k+1}}{2^{\frac{3}{2}}(2k)!} \left[-\frac{B_{2k}B_{2k+2}}{(k+1)(2k+1)} \right]^{\frac{1}{2}} \quad (2.20)$$

and

$$(2\pi)^{2k+1} \left[\frac{B_{4k+2}}{2(4k+2)!} \right]^{\frac{1}{2}} \leq \zeta(2k+1), \quad (2.21)$$

for $k = 1, 2, \dots$

The Riemann Zeta function at odd integers was also established in [1] as

$$\zeta(2k+1) = \frac{(-1)^{k+1}(2\pi)^{2k+1}}{2(2k+1)!} \int_0^1 B_{2k+1}(x) \cot(\pi x) dx, \quad (2.22)$$

for $k = 1, 2, \dots$

3. Results and Discussion

We begin with a new proof of Euler's formula for calculating $\zeta(2k)$. For completeness sake, we restate the result in the following theorem.

Theorem 3.1. For $k \in \mathbb{N}$, the identity

$$\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k}B_{2k}}{2(2k)!} \quad (3.1)$$

holds, where B_{2k} is a Bernoulli number.

Proof. Putting $s = -2k + 1$ into (2.3) we have

$$\begin{aligned} \zeta(-2k+1) &= 2^{-2k+1} \pi^{-2k} \sin\left(\pi \frac{(-2k+1)}{2}\right) \Gamma(2k) \zeta(2k) \\ &= 2(2\pi)^{-2k} (-1)^k (2k-1)! \zeta(2k). \end{aligned}$$

Letting $s = 2k - 1$ in (2.17) and making the appropriate substitution yields

$$-\frac{B_{2k}}{2k} = 2(2\pi)^{-2k} (-1)^k (2k-1)! \zeta(2k). \quad (3.2)$$

which is equivalent to (3.1) as required.

Theorem 3.2. The inequality

$$\zeta(2k+1) \leq \frac{(2\pi)^{2k+1}}{2(2k)!} \left[-\frac{B_{2k}B_{2k+2}}{(2k+1)(2k+2)} \right]^{\frac{1}{2}} \quad (3.3)$$

holds for $k \in \mathbb{N}$.

Proof. Putting $s = 2k$ into (2.19) gives

$$\zeta^2(2k+1) < \zeta(2k)\zeta(2k+2). \quad (3.4)$$

Replacing $2k$ with $(2k+2)$ in (2.6) and with the appropriate substitution in (3.4) yields

$$\zeta^2(2k+1) \leq \frac{(2\pi)^{4k+2} B_{2k} B_{2k+2}}{4(-1)^{2k+1} (2k)! (2k+2)!} \quad (3.5)$$

$$\leq \frac{\left((2\pi)^{2k+1}\right)^2 B_{2k} B_{2k+2}}{4(-1)^{2k} (-1) (2k)! (2k+1) (2k+2) (2k)!}. \quad (3.6)$$

$$\leq \frac{\left((2\pi)^{2k+1}\right)^2}{4(-1)^{2k} ((2k)!)^2} \left[-\frac{B_{2k} B_{2k+2}}{(2k+1)(2k+2)} \right].$$

Taking square root on both sides completes the proof.

Theorem 3.3. *The inequality*

$$\zeta(2k+1) \leq \frac{(2\pi)^{2k+1}}{2(2k)!} \left[-\frac{B_{2k} B_{2k+2}}{2k(2k+2)} \right]^{\frac{1}{2}}$$

holds for $k \in \mathbb{N}$.

Proof. Putting $s = 2k$ into (2.18) gives

$$(2k+1)\zeta(2k)\zeta(2k+2) \geq 2k\zeta^2(2k+1).$$

By using (2.6) we obtain

$$(2k+1) \frac{(2\pi)^{4k+2} B_{2k} B_{2k+2}}{4(-1)^{2k+1} (2k)! (2k+2)!} \geq 2k\zeta^2(2k+1),$$

which simplifies to

$$\frac{\left((2\pi)^{2k+1}\right)^2}{4((2k)!)^2} \left[-\frac{B_{2k} B_{2k+2}}{(2k)(2k+2)} \right] \geq \zeta^2(2k+1).$$

Taking square root of both sides completes the proof.

Theorem 3.4. *The identity*

$$\zeta(2k+1) = \frac{2\pi\zeta(2k)}{B_{2k}(2k+1)} \int_0^1 \sum_{j=0}^{2k+1} \binom{2k+1}{j} B_j t^{(2k+1)-j} \cot(\pi t) dt$$

holds for $k \in \mathbb{N}$.

Proof. Substituting $(2k+1)! = (2k+1)(2k)!$ into (2.22) yields

$$\zeta(2k+1) = \frac{(-1)^{k-1} (2\pi)^{2k} 2\pi}{2(2k+1)(2k)!} \int_0^1 B_{2k+1}(t) \cot(\pi t) dt,$$

which implies

$$\zeta(2k+1) = \frac{2\pi}{(2k+1)} \cdot \frac{(-1)^{k-1} (2\pi)^{2k}}{2(2k)!} \int_0^1 B_{2k+1}(t) \cot(\pi t) dt.$$

By (3.1) we obtain

$$\zeta(2k+1) = \frac{2\pi\zeta(2k)}{B_{2k}(2k+1)} \int_0^1 B_{2k+1}(t) \cot(\pi t) dt. \quad (3.7)$$

Using (2.15) completes the proof.

Remark 3.1. Theorem 3.1 connects the odd values of the Riemann zeta function with its even values. By making $\zeta(2k)$ the subject from (3.7), one can calculate the even values of the Riemann zeta function, an approach different from that of the Euler formular.

Corollary 3.1. The identity

$$\zeta(2k+1) = \frac{(-1)^{1-k}(2\pi)^{2k+1}}{2(2k+1)!} \int_0^1 B_{2k+1}(t) \cot(\pi t) dt.$$

holds for $k \in \mathbb{N}$.

Proof. Using Theorem 3.1 and (3.7) yields the result.

Theorem 3.5. The identity

$$\zeta(2k+1) = \frac{2^{2k}\pi^{2k+1}\zeta(1-2k)}{B_{2k}(2k+1)(-1)^k(2k-1)!} \int_0^1 B_{2k+1}(t) \cot(\pi t) dt$$

holds for $k \in \mathbb{N}$.

Proof. From (3.7) we have

$$\zeta(2k) = \frac{B_{2k}(2k+1)\zeta(2k+1)}{2\pi} \left(\int_0^1 B_{2k+1}(t) \cot(\pi t) dt \right)^{-1}. \quad (3.8)$$

Putting $s = 1 - 2k$ into (2.3) gives

$$\begin{aligned} \zeta(1-2k) &= 2^{1-2k}\pi^{-2k} \sin\left(\frac{\pi}{2}(1-2k)\right) \Gamma(2k)\zeta(2k), \\ &= 2^{1-2k}\pi^{-2k}(-1)^k(2k-1)!\zeta(2k). \end{aligned}$$

Then by (3.8) we obtain

$$\begin{aligned} \zeta(1-2k) &= 2^{1-2k}\pi^{-2k}(-1)^k(2k-1)! \frac{B_{2k}(2k+1)\zeta(2k+1)}{2\pi} \left(\int_0^1 B_{2k+1}(t) \cot(\pi t) dt \right)^{-1}. \end{aligned}$$

Thus

$$\begin{aligned} \zeta(2k+1) &= \frac{2\pi\zeta(1-2k)}{2^{1-2k}\pi^{-2k}(-1)^k(2k-1)!B_{2k}(2k+1)} \int_0^1 B_{2k+1}(t) \cot(\pi t) dt, \end{aligned}$$

which completes the proof.

4. Conclusion

A new proof of the Euler formular for calculating $\zeta(2k)$ was given. The study also presented some new identities and inequalities for the Riemann zeta function. A connection between $\zeta(2k+1)$ and $\zeta(2k)$ as well as its negative odd values were also established.

References

- [1] M. Abramowitz, I.M. Stegun, *Handbook of Mathematical Functions and Constants*, National Bureau of Standards, Applied Mathematics Series, Dover Publications Inc., Tenth Printing (1972).
- [2] Z. Ahsan, P. Lam-Estrada, J. Lopez-Bonilla, *On the Laforgia-Natalini Inequality for the Riemann zeta function*, Transnational Journal of Mathematical Analysis and Applications, **1** (2013), 1–4.
- [3] R.J. Dwilewicz, J. Minac, *Values of the Riemann zeta function at integer values*. Materials Mathematics, **6** (2009), 1–26.
- [4] D.D. Enrigue, M.D. Carrilo, and J. Fernandez-Sanchez, *Another proof of Euler's formular for $\zeta(2k)$* ., Proceedings of the American Mathematical Soceity, **189** (2012), 1–4.
- [5] A. Ivic, *The Riemann Zeta Function: Theory and Applications*, John Wiley and Sons, New York (Dover Publications),(1985).
- [6] A. Laforgia, P. Natalini, *Turan-type inequalities for some special functions*, Journal of inequalities in Pure and Applied Mathematics, **7** (2006), 1–3.
- [7] J. Minac, *A Remark on the values of the Riemann zeta function*, Expo.Math., **12** (1994), 459–462.
- [8] E. Titchmarsh, *The Theory of the Riemann Zeta Function*, Oxford University Press, London, (1986).
- [9] B. Riemann, *Ueber die Anzahl der Primzahlen unter einer gegebenen Groe*, Ges. Math. Werke und Wissenschaftlicher Nachla , **2**, Aufl, (1859), 145–155.