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Fixed Point Theorems in Complex Valued Extended *b*-Metric Spaces

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ABSTRACT. In this article, inspired by the concepts of extended b-metric spaces, we introduce the notion of complex valued extended b-metric spaces. Using this new idea, some fixed point theorems involving rational contractive inequalities are proved. The established results herein augment several significant work in the comparable literature.
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1. Introduction and Preliminary

Fixed point theory is a well known and vast field of research in mathematical sciences. This field is known as the mixture of analysis which includes topology, geometry and algebra. In particular, fixed point technique is a central tool in the study of non-linear analysis. In this area, a huge involvement has been made by Banach [10], who gave the notion of contraction mapping due to a complete metric space to locate fixed point of the specified function. The classical fixed point theorem due to Banach [10] has been generalized by many researchers in various ways (see, for example, [7, 5, 13, 17]) and the references therein. One may also consult Rhoades [24] for different definitions of contractive type mappings.

In 1969, Nadler [23] initiated the study of fixed point theorems for multi-valued mappings. Nadler's contraction principle encouraged many researchers and as a result, the idea has been developed in different directions

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(see, for instance, [8, 7, 12, 22]). Moreover, all the generalizations of Banach fixed point theorem is further classified in two directions-either the contractive condition is replaced with a more generalized one or the axioms characterizing the ground set is enlarged or weakened. In the second case, some of these metric-like spaces are called semimetric, quasimetric, pseudodometric, b-metric, K-metric, to mention but a few. Along this line, by replacing the set of real numbers as the usual co-domain of a metric, Huang and Zhang [18] established the concept of cone metric as a generalization of metric spaces, thereby, establishing some fixed point theorems for contractive mappings on cone metric spaces. Starting from the year 2007, many authors have come up with various significant fixed point results in the frame of cone metric spaces (see, for example, [19, 26]). The interested researcher may also want to go deeply into a comprehensive new survey on cone metric spaces by Aleksić et al. [4]

It is well-known that fixed point results involving rational contractive conditions cannot be extended or even meaningless in cone metric spaces. To tame this restriction, Azam et al. [5] initiated the concept of complex valued metric spaces and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying contractive type conditions involving rational expressions. Thereafter, the study of fixed point theorems concerning rational inequalities in complex valued metric spaces have been growing vigorously (see, for example, [1, 2, 6, 7, 14, 15, 21]). Along the line, the idea of b-metric space was presented by Bakhtin [9] in 1989. Rao [25] introduced the notion of fixed point results on complex valued b-metric spaces, which is broader than complex valued metric spaces. However, every complex valued b-metric space is a cone b-metric space over Banach algebra $\mathbb{C}$ in which the cone is normal with the coefficient of normality $K = 1$, and where the cone has non-empty interior (that is, solid cone). Following [25], various authors have demonstrated fixed point results for different mappings fulfilling rational inequalities with regards to complex valued b-metric spaces (see, for instance, [3, 8]).

Our contribution in this paper is to generalize the idea of extended b-metric spaces to complex valued extended b-metric spaces. In what follows hereafter, we recall some basic concepts that are necessary in the establishment of our main results. Most of these preliminaries are recorded from [3, 5, 11, 20].

Definition 1.1. *Let $\mathbb{C}$ be the set of all complex numbers and $l_1, l_2 \in \mathbb{C}$. The partial order on $\mathbb{C}$ is defined as:*
$l_1 \preceq l_2$, if and only if $Re(l_1) \leq Re(l_2)$ and $Im(l_1) \leq Im(l_2)$. This implies that $l_1 \preceq l_2$ if one of the below conditions is fulfilled:

(i) $Re(l_1) = Re(l_2),\ Im(l_1) < Im(l_2)$,
(ii) $Re(l_1) < Re(l_2),\ Im(l_1) = Im(l_2)$,
(iii) $Re(l_1) < Re(l_2),\ Im(l_1) < Im(l_2)$,
(iv) $Re(l_1) = Re(l_2),\ Im(l_1) = Im(l_2)$.

Definition 1.2. *Let $\mathbb{G}$ be a non-empty set. If the mapping $d : \mathbb{G} \times \mathbb{G} \to \mathbb{C}$ satisfies the following conditions :*
(i) $0 \preceq d(l,m)$ *and* $d(l,m) = 0 \iff l = m$;
(ii) $d(l,m) = d(m,l)$;
(iii) $d(l,m) \preceq d(l,n) + d(n,m),\ \forall\, l,\, n,\, m \in \mathbb{G}$,
then d is known as a complex valued (C.V.) metric on $\mathbb{G}$, and the pair (G,d) is said to be a C.V metric space.

Example 1.1. *Let $X = [0,1]$ and $l, m \in \mathbb{G}$. Define $d : \mathbb{G} \times \mathbb{G} \to \mathbb{C}$ by*

$$d(l,m) = \begin{cases} 0, & \text{if} \quad l = m \\ \frac{i}{2}, & \text{if} \quad l \neq m. \end{cases}$$

Then d is a C.V metric and hence $(\mathbb{G}, d)$ is a complex valued metric space.

Definition 1.3. *Let (G,d) be a C.V metric space.*
(i) *We say that a point $l \in \mathbb{G}$ is an interior point of a set $M \subseteq \mathbb{G}$, whenever there exists $0 \prec r \in \mathbb{C}$ such that*

$$B(l,r) = \{m \in \mathbb{G} : d(m,l) \prec r\},$$

where $B(l,r)$ is an open ball with radius r, centered at l.
(ii) *We say that a point $l \in \mathbb{G}$ is the limit point of a set $M \subseteq \mathbb{G}$ whenever for every $0 \prec r \in \mathbb{C}$, we have*

$$B(l,r) \bigcap M - \{l\} \neq \emptyset.$$

Lemma 1.1. [3] *Let (G,d) be a C.V metric space and $\{l_p\}$ be a sequence in $\mathbb{G}$. Then $\{l_p\}$ is a convergent sequence if and only if $|d(l_p,l)| \to 0$ as $p \to \infty$.*

Lemma 1.2. [3] *Let (G,d) be a C.V metric space and $\{l_p\}$ be a sequence in $\mathbb{G}$. Then $\{l_p\}$ is a Cauchy sequence if and only if $|d(l_p,l_q)| \to 0$ as $p,q \to \infty$.*

Definition 1.4. *Let $(\mathbb{G},d_c)$ be a C.V metric space. We denote*

$$s(u) = \{z \in \mathbb{C} : u \preceq z\},$$

for $l \in \mathbb{G}$ and $\mathbb{N} \in CB(\mathbb{G})$

$$s(l,\mathbb{N}) = \bigcup_{n\in\mathbb{N}} s(d_c(l,n)) = \bigcup_{n\in\mathbb{N}} \{z \in \mathbb{C} : d_c(l,n) \preceq z\},$$

for $\mathbb{M},\mathbb{N} \in CB(\mathbb{G})$, we have

$$s(\mathbb{M},\mathbb{N}) = \Big(\bigcap_{n\in\mathbb{N}} s(n,\mathbb{M})\Big) \cap \Big(\bigcap_{m\in\mathbb{M}} s(m,\mathbb{N})\Big).$$

Definition 1.5. *Let (G,d) be a C.V metric space.*
(i) Let $\mathbb{T} : \mathbb{G} \to CB(\mathbb{G})$ be a multi-valued mapping. For $l \in \mathbb{G}$ and $M \in CB(G)$, define

$$W_l(M) = \{d(l,a) : a \in M\},$$

and for $l,m \in \mathbb{G}$, and $\mathbb{T}m \in CB(\mathbb{G})$, we have

$$W_l(\mathbb{T}m) = \{d(l,u) : u \in \mathbb{T}m\}.$$

(ii) A mapping $F : \mathbb{G} \to 2^{\mathbb{C}}$ is said to bounded below if for each $l \in \mathbb{G}$ there exists $w \in \mathbb{C}$ such that $z_l \preceq w$ for all $w \in F_l$.

(iii) For multi-valued mapping $J : \mathbb{G} \to CB(\mathbb{G})$, we say that it has lower bound property on $(\mathbb{G},d_c)$ if for any $l \in \mathbb{G}$ the mapping $F_l : \mathbb{G} \to 2^{\mathbb{C}}$ defined by $F_l(Jv) = W_l(Fv)$ is bounded below. This means that for $l,v \in \mathbb{G}$ there is an element $l_l(Jv) \in \mathbb{C}$ such that $l_l(Jv) \preceq a$ for all $a \in W_l(Jv)$, where $l_l(Jv)$ is said to be a lower bound of J corresponding to (l,v).

(iv) For multi-valued mapping $J : \mathbb{G} \to CB(\mathbb{G})$, we say that it has greatest lower bound property (g.l.b property) on $(\mathbb{G},d_c)$ if the g.l.b of $W_l(Jv)$ exists in $\mathbb{C}$ for all $l,v \in \mathbb{G}$. We denote the g.l.b of $W_l(Jv)$ by $d_c(l,Jv)$ and define it as:

$$d_c(l,Jv) = inf\{d_c(l,a) : a \in Jv\}.$$

Definition 1.6. *Let $(\mathbb{G},d_c)$ be a C.V metric space and $\mathbb{S},\mathbb{T} : \mathbb{G} \to CB(\mathbb{G})$ be multi- valued mappings.*
(i) A point $l \in \mathbb{G}$ is called a fixed point of $\mathbb{T}$ if $l \in \mathbb{T}l$.
(ii) A point $l \in \mathbb{G}$ is called a common fixed point of $\mathbb{S}$ and $\mathbb{T}$ if $l \in \mathbb{S}l$ and $l \in \mathbb{T}l$.

Definition 1.7. [9] *Let $\mathbb{G}$ be a non-empty set and $\tau \geq 1$ be a real number. A function $d_c : \mathbb{G} \times \mathbb{G} \to \mathbb{C}$ is called C.V b-metric space, if for all $l,m,n \in \mathbb{G}$, the following conditions hold.*
(i) $0 \preceq d_c(l,m)$ and $d_c(l,m) = 0$ if and only if $l = m$;
(ii) $d_c(l,m) = d_c(m,l)$;
(iii) $d_c(l,m) \preceq \tau[d_c(l,n) + d_c(n,m)]$.
Then d_c is called a C.V b-metric on $\mathbb{G}$ and the pair $(\mathbb{G},d_c)$ is called a C.V b-metric space.

Example 1.2. [30] *Let $\mathbb{G} = [0,1]$. Define a mapping $d_c : \mathbb{G} \times \mathbb{G} \to \mathbb{C}$ by*

$$d_c(l,m) = |l-m|^2 + i|l-m|^2$$

for all $l,m \in \mathbb{G}$. Then $(\mathbb{G},d_c)$ is a C.V b-metric space with $\tau = 2$.

Definition 1.8. *Let $\mathbb{G}$ be a non-empty set and $\theta : \mathbb{G} \times \mathbb{G} \to [1,\infty)$ be a function. Then $d_\theta : \mathbb{G} \times \mathbb{G} \to \mathbb{C}$ is known as a C.V extended b-metric space if the following are satisfied for all l, m, n$\in \mathbb{G}$:*
(i) $0 \preceq d_\theta(l,m)$ and $d_\theta(l,m) = 0$ if and only if $l = m$;
(ii) $d_\theta(l,m) = d_\theta(m,l)$;
(iii) $d_\theta(l,n) \preceq \theta(l,n)[d_\theta(l,m) + d_\theta(m,n)]$.
then the pair $(\mathbb{G},d_\theta)$ is known as a C.V extended b-matric space.

Example 1.3. [10] *Let $\mathbb{G}$ be a non-empty set and $\theta : \mathbb{G} \times \mathbb{G} \to [1, \infty)$ be defined as:*

$$\theta(l, m) = \frac{1 + l + m}{l + m}$$

.

Further, Let

(i) $d_\theta(l, m) = \frac{i}{lm}$, *for all* $l, m \in (0, 1]$;

(ii) $d_\theta(l, m) = 0 \Leftrightarrow l = m$ *for all* $l, m \in [0, 1]$;

(iii) $d_\theta(l, 0) = d_\theta(0, l) = \frac{i}{l}$ *for all* $l \in (0, 1]$.

Then the pair $(\mathbb{G}, d_\theta)$ is known as a C.V extended b-metric space.

Example 1.4. *Let $\mathbb{G} = [0, \infty)$. $\theta : \mathbb{G} \times \mathbb{G} \to [1, \infty)$ be a function defined by $\theta(l, m) = 1 + l + m$ and $d_\theta : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{C}$ be given as*

$$d_\theta(l, m) = \begin{cases} 0, & \text{if} \quad l = m \\ i, & \text{if} \quad l \neq m. \end{cases}$$

Then $(\mathbb{G}, d_\theta)$ is known as a C.V extended b-metric space.

In what follows hereafter, we present our main results.

2. Fixed point results for almost contraction in complex valued extended b-metric spaces

We start this section with the following definition.

Definition 2.1. *Let $(\mathbb{G}, d_\theta)$ be a complex valued extended b-metric space. A mapping $\mathbb{T} : \mathbb{G} \to \mathbb{G}$ is said to be an almost contraction if there exists $r \in (0, 1)$ and $\mathbb{L} \geq 0$ such that for all $l, m \in \mathbb{G}$,*

$$d_\theta(\mathbb{T}l, \mathbb{T}m) \leq r d_\theta(l, m) + \mathbb{L} d_\theta(m, \mathbb{T}l).$$

Theorem 2.1. *Let $(\mathbb{G}, d_\theta)$ be a complete complex valued extended b-metric space; let $\theta : \mathbb{G} \times \mathbb{G} \to [1, \infty)$ and $\mathbb{S}, \mathbb{T} : \mathbb{G} \times \mathbb{G} \to CB(\mathbb{G})$ be a pair of multi-valued mappings with g.l.b property such that*

$$\begin{aligned} &\lambda_1 d_\theta(l, m) + \lambda_2[d_\theta(l, \mathbb{S}l) + d_\theta(m, \mathbb{T}m)] + \lambda_3[d_\theta(m, \mathbb{S}l) + d_\theta(l, \mathbb{T}m)] \\ &+ \lambda_4 \frac{d_\theta(m, \mathbb{T}m)[1 + d_\theta(l, \mathbb{S}l)]}{1 + d_\theta(l, m)} + \lambda_5 \frac{d_\theta(m, \mathbb{S}l)[1 + d_\theta(l, \mathbb{T}m)]}{1 + d_\theta(l, m)} \\ &+ \lambda_6 \frac{d_\theta(l, m)[1 + d_\theta(l, \mathbb{S}l) + d_\theta(m, \mathbb{S}l)]}{1 + d_\theta(l, m)} + \mathbb{L} d_\theta(m, \mathbb{S}l) \in s(\mathbb{S}l, \mathbb{T}m) \end{aligned} \tag{2.1}$$

for all $l, m \in \mathbb{G}$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6$, $\mathbb{L}$ are nonnegative real numbers with $\lambda_1 + 2\lambda_2 + 2\lambda_3\theta(l_o, l_2) + \lambda_4 + \lambda_6 < 1$, $k(1 - \lambda_2 - \lambda_3\theta(l_o, l_2) - \lambda_4) = (\lambda_1 + \lambda_2 + \lambda_3\theta(l_o, l_2) + \lambda_6)$ where $k \in [0, \infty)$ be such that for each $l_o \in \mathbb{G}$, $\lim_{p,q\to\infty} \theta(l_p, l_q) < \frac{1}{k}$. Then $\mathbb{S}$ and $\mathbb{T}$ have a common fixed point.

Proof. Let $l_o \in \mathbb{G}$, then $\mathbb{T}l_o$ is non empty, so we take $l_1 \in \mathbb{S}l_o$. Thus, from (2.1), we have

$$\begin{aligned} &\lambda_1 d_\theta(l_o, l_1) + \lambda_2[d_\theta(l_o, \mathbb{S}l_o) + d_\theta(l_1, \mathbb{T}l_1)] + \lambda_3[d_\theta(l_1, \mathbb{S}l_o) + d_\theta(l_o, \mathbb{T}l_1)] \\ &+ \lambda_4 \frac{d_\theta(l_1, \mathbb{T}l_1)[1 + d_\theta(l_o, \mathbb{S}l_o)]}{1 + d_\theta(l_o, l_1)} + \lambda_5 \frac{d_\theta(l_1, \mathbb{S}l_o)[1 + d_\theta(l_o, \mathbb{T}l_1)]}{1 + d_\theta(l_o, l_1)} \\ &+ \lambda_6 \frac{d_\theta(l_o, l_1)[1 + d_\theta(l_o, \mathbb{S}l_o) + d_\theta(l_1, \mathbb{S}l_o)]}{1 + d_\theta(l_o, l_1)} + \mathbb{L} d_\theta(l_1, \mathbb{S}l_o) \in s(\mathbb{S}l_o, \mathbb{T}l_1) \end{aligned}$$

$$\begin{aligned} &\lambda_1 d_\theta(l_o, l_1) + \lambda_2[d_\theta(l_o, \mathbb{S}l_o) + d_\theta(l_1, Tl_1)] + \lambda_3[d_\theta(l_1, \mathbb{S}l_o) + d_\theta(l_o, \mathbb{T}l_1)] \\ &+ \lambda_4 \frac{d_\theta(l_1, \mathbb{T}l_1)[1 + d_\theta(l_o, \mathbb{S}l_o)]}{1 + d_\theta(l_o, l_1)} + \lambda_5 \frac{d_\theta(l_1, \mathbb{S}l_o)[1 + d_\theta(l_o, \mathbb{T}l_1)]}{1 + d_\theta(l_o, l_1)} \end{aligned}$$

$$+\lambda_6\frac{d_\theta(l_o,l_1)[1+d_\theta(l_o,\mathbb{S}l_o)+d_\theta(l_1,\mathbb{S}l_o)]}{1+d_\theta(l_o,l_1)}+\mathbb{L}d_\theta(l_1,\mathbb{S}l_o)\in\bigcap_{a\in\mathbb{S}l_o}s(a,\mathbb{T}l_1)$$

This gives

$$\begin{aligned}&\lambda_1 d_\theta(l_o,l_1)+\lambda_2[d_\theta(l_o,\mathbb{S}l_o)+d_\theta(l_1,\mathbb{T}l_1)]+\lambda_3[d_\theta(l_1,\mathbb{S}l_o)+d_\theta(l_o,\mathbb{T}l_1)]\\&+\lambda_4\frac{d_\theta(l_1,\mathbb{T}l_1)[1+d_\theta(l_o,\mathbb{S}l_o)]}{1+d_\theta(l_o,l_1)}+\lambda_5\frac{d_\theta(l_1,\mathbb{S}l_o)[1+d_\theta(l_o,\mathbb{T}l_1)]}{1+d_\theta(l_o,l_1)}\\&+\lambda_6\frac{d_\theta(l_o,l_1)[1+d_\theta(l_o,\mathbb{S}l_o)+d_\theta(l_1,\mathbb{S}l_o)]}{1+d_\theta(l_o,l_1)}+\mathbb{L}d_\theta(l_1,\mathbb{S}l_o)\in s(a,\mathbb{T}l_1),\qquad\forall a\in\mathbb{S}l_o.\end{aligned}$$

Since $l_1\in\mathbb{S}l_o$, then

$$\begin{aligned}&\lambda_1 d_\theta(l_o,l_1)+\lambda_2[d_\theta(l_o,\mathbb{S}l_o)+d_\theta(l_1,\mathbb{T}l_1)]+\lambda_3[d_\theta(l_1,\mathbb{S}l_o)+d_\theta(l_o,\mathbb{T}l_1)]\\&+\lambda_4\frac{d_\theta(l_1,\mathbb{T}l_1)[1+d_\theta(l_o,\mathbb{S}l_o)]}{1+d_\theta(l_o,l_1)}+\lambda_5\frac{d_\theta(l_1,\mathbb{S}l_o)[1+d_\theta(l_o,\mathbb{T}l_1)]}{1+d_\theta(l_o,l_1)}\\&+\lambda_6\frac{d_\theta(l_o,l_1)[1+d_\theta(l_o,\mathbb{S}l_o)+d_\theta(l_1,\mathbb{S}l_o)]}{1+d_\theta(l_o,l_1)}+\mathbb{L}d_\theta(l_1,\mathbb{S}l_o)\in s(l_1,\mathbb{T}l_1)\end{aligned}$$

$$\begin{aligned}&\lambda_1 d_\theta(l_o,l_1)+\lambda_2[d_\theta(l_o,\mathbb{S}l_o)+d_\theta(l_1,\mathbb{T}l_1)]+\lambda_3[d_\theta(l_1,\mathbb{S}l_o)+d_\theta(l_o,\mathbb{T}l_1)]\\&+\lambda_4\frac{d_\theta(l_1,\mathbb{T}l_1)[1+d_\theta(l_o,\mathbb{S}l_o)]}{1+d_\theta(l_o,l_1)}+\lambda_5\frac{d_\theta(l_1,\mathbb{S}l_o)[1+d_\theta(l_o,\mathbb{T}l_1)]}{1+d_\theta(l_o,l_1)}\\&+\lambda_6\frac{d_\theta(l_o,l_1)[1+d_\theta(l_o,\mathbb{S}l_o)+d_\theta(l_1,\mathbb{S}l_o)]}{1+d_\theta(l_o,l_1)}+\mathbb{L}d_\theta(l_1,\mathbb{S}l_o)\in\bigcup_{b\in\mathbb{T}l_1}s(l_1,b).\end{aligned}$$

Therefore, there exists $l_2\in\mathbb{T}l_1$ such that

$$\begin{aligned}&\lambda_1 d_\theta(l_o,l_1)+\lambda_2[d_\theta(l_o,\mathbb{S}l_o)+d_\theta(l_1,\mathbb{T}l_1)]+\lambda_3[d_\theta(l_1,\mathbb{S}l_o)+d_\theta(l_o,\mathbb{T}l_1)]\\&+\lambda_4\frac{d_\theta(l_1,\mathbb{T}l_1)[1+d_\theta(l_o,\mathbb{S}l_o)]}{1+d_\theta(l_o,l_1)}+\lambda_5\frac{d_\theta(l_1,\mathbb{S}l_o)[1+d_\theta(l_o,\mathbb{T}l_1)]}{1+d_\theta(l_o,l_1)}\\&+\lambda_6\frac{d_\theta(l_o,l_1)[1+d_\theta(l_o,\mathbb{S}l_o)+d_\theta(l_1,\mathbb{S}l_o)]}{1+d_\theta(l_o,l_1)}+\mathbb{L}d_\theta(l_1,\mathbb{S}l_o)\in s(d_\theta(l_1,l_2)).\end{aligned}$$

By definition and the g.l.b property of $\mathbb{S}$ and $\mathbb{T}$, we have

$$\begin{aligned}d_\theta(l_1,l_2)\preceq{}&\lambda_1 d_\theta(l_o,l_1)+\lambda_2[d_\theta(l_o,\mathbb{S}l_o)+d_\theta(l_1,\mathbb{T}l_1)]+\lambda_3[d_\theta(l_1,\mathbb{S}l_o)+d_\theta(l_o,\mathbb{T}l_1)]\\&+\lambda_4\frac{d_\theta(l_1,\mathbb{T}l_1)[1+d_\theta(l_o,\mathbb{S}l_o)]}{1+d_\theta(l_o,l_1)}+\lambda_5\frac{d_\theta(l_1,\mathbb{S}l_o)[1+d_\theta(l_o,\mathbb{T}l_1)]}{1+d_\theta(l_o,l_1)}\\&+\lambda_6\frac{d_\theta(l_o,l_1)[1+d_\theta(l_o,\mathbb{S}l_o)+d_\theta(l_1,\mathbb{S}l_o)]}{1+d_\theta(l_o,l_1)}+\mathbb{L},d_\theta(l_1,\mathbb{S}l_o)\end{aligned}$$

from which we have

$$\begin{aligned}d_\theta(l_1,l_2)\preceq{}&\lambda_1 d_\theta(l_o,l_1)+\lambda_2[d_\theta(l_o,l_1)+d_\theta(l_1,l_2)]+\lambda_3[d_\theta(l_1,l_1)+d_\theta(l_o,l_2)]\\&+\lambda_4\frac{d_\theta(l_1,l_2)[1+d_\theta(l_o,l_1)]}{1+d_\theta(l_o,l_1)}+\lambda_5\frac{d_\theta(l_1,l_1)[1+d_\theta(l_o,l_2)]}{1+d_\theta(l_o,l_1)}\\&+\lambda_6\frac{d_\theta(l_o,l_1)[1+d_\theta(l_o,l_1)+d_\theta(l_1,l_1)]}{1+d_\theta(l_o,l_1)}+\mathbb{L}d_\theta(l_1,l_1).\end{aligned}$$

This implies

$$\begin{aligned}|d_\theta(l_1,l_2)|\leq{}&\lambda_1|d_\theta(l_o,l_1)|+\lambda_2[|d_\theta(l_o,l_1)|+|d_\theta(l_1,l_2)|]\\&+\lambda_3|d_\theta(l_o,l_2)|+\lambda_4|d_\theta(l_1,l_2)|+\lambda_6|d_\theta|(l_o,l_1)|.\end{aligned}$$

Since $|d_\theta(l_1, l_2)| < 1 + |d_\theta(l_1, l_2)|$,

$$|d_\theta(l_1, l_2)| \leq \frac{(\lambda_1 + \lambda_2 + \lambda_3\theta(l_o, l_2) + \lambda_6)}{(1 - \lambda_2 - \lambda_3\theta(l_o, l_2) - \lambda_4)} |d_\theta(l_o, l_1)|.$$

Then, we get

$$|d_\theta(l_1, l_2)| \leq k|d_\theta(l_o, l_1)|.$$

Inductively, we can find a sequence $\{l_p\}$ in $\mathbb{G}$ such that

$$\begin{aligned}
|d_\theta(l_1, l_2)| &\leq \kappa |d_\theta(l_o, l_1)| \\
|d_\theta(l_2, l_3)| &\leq \kappa^2 |d_\theta(l_o, l_1)| \\
&. \\
&. \\
&. \\
|d_\theta(l_p, l_{p+1})| &\leq \kappa^p |d_\theta(l_o, l_1)|.
\end{aligned}$$

Now, by triangular inequality, for $q > p$, we have

$$\begin{aligned}
d_\theta(l_p, l_q) &\preceq \theta(l_p, l_q)k^p d_\theta(l_o, l_1) + \theta(l_p, l_q)\theta(l_{p+1}, l_q)k^{p+1} d_\theta(l_o, l_1) + \cdots \\
&\quad + \theta(l_p, l_q)\theta(l_{p+1}, l_q)\cdots\theta(l_{q-2}, l_q)\theta(l_{q-1}, l_q)k^{q-1} d_\theta(l_o, l_1) \\
d_\theta(l_p, l_q) &\preceq d_\theta(l_0, l_1)[\theta(l_p, l_q)k^p + \theta(l_p, l_q)\theta(l_{p+1}, l_q)k^{p+1} + \cdots \\
&\quad + \theta(l_p, l_q)\theta(l_{p+1}, l_q)...\theta(l_{q-2}, l_q)\theta(l_{q-1}, l_p)k^{q-1}]
\end{aligned}$$

Since $\lim_{p,q\to\infty} \theta(l_p, l_q)k < 1$, so the series $\sum_{p=1}^{\infty} k^p \prod_{i=1}^{p} \theta(l_i, l_q)$ converges by ratio test for each $q \in \mathbb{N}$. Let

$$S = \sum_{p=1}^{\infty} k^p \prod_{i=1}^{p} \theta(l_i, l_q), \; S_p = \sum_{j=1}^{p} k^j \prod_{i=1}^{p} \theta(l_i, l_q).$$

Thus, for $q > p$, the above can be written as

$$d_\theta(l_p, l_q) \preceq d_\theta(l_o, l_1)[S_{q-1} - S_P]$$

and

$$|d_\theta(l_p, l_q)| \leq |d_\theta(l_o, l_1)|[S_{q-1} - S_p]$$

.

Now, by taking $p \to \infty$, we get

$$|d_\theta(l_p, l_q)| \to 0.$$

By lemma (2.4), $\{l_p\}$ is a Cauchy sequence in $\mathbb{G}$. By completeness of $\mathbb{G}$, there exists some $r \in \mathbb{G}$ such that

$$\lim_{p\to\infty} l_p = r.$$

Now, we show that $r \in \mathbb{S}r$ and $r \in \mathbb{T}r$. From (2.1), we have

$$\begin{aligned}
&\lambda_1 d_\theta(l_{2p}, r) + \lambda_2[d_\theta(l_{2p}, \mathbb{S}l_{2p}) + d_\theta(r, \mathbb{T}r)] + \lambda_3[d_\theta(r, \mathbb{S}l_{2p}) + d_\theta(l_{2p}, \mathbb{T}r)] \\
&+ \lambda_4 \frac{d_\theta(r, \mathbb{T}r)[1 + d_\theta(l_{2p}, \mathbb{S}l_{2p})]}{1 + d_\theta(l_{2p}, r)} + \lambda_5 \frac{d_\theta(r, \mathbb{S}l_{2p})[1 + d_\theta(l_{2p}, \mathbb{T}r)]}{1 + d_\theta(l_{2p}, r)} \\
&+ \lambda_6 \frac{d_\theta(l_{2p}, r)[1 + d_\theta(l_{2p}, \mathbb{S}l_{2p}) + d_\theta(r, \mathbb{S}l_{2p})]}{1 + d_\theta(l_{2p}, r)} + \mathbb{L} d_\theta(r, \mathbb{S}l_{2p}) \in s(\mathbb{S}l_{2p}, \mathbb{T}r).
\end{aligned}$$

This implies

$$\begin{aligned}&\lambda_1 d_\theta(l_{2p},r)+\lambda_2[d_\theta(l_{2p},\mathbb{S}l_{2p})+d_\theta(r,\mathbb{T}r)]+\lambda_3[d_\theta(r,\mathbb{S}l_{2p})+d_\theta(l_{2p},\mathbb{T}r)]\\&+\lambda_4\frac{d_\theta(r,\mathbb{T}r)[1+d_\theta(l_{2p},\mathbb{S}l_{2p})]}{1+d_\theta(l_{2p},r)}+\lambda_5\frac{d_\theta(r,\mathbb{S}l_{2p})[1+d_\theta(l_{2p},\mathbb{T}r)]}{1+d_\theta(l_{2p},r)}\\&+\lambda_6\frac{d_\theta(l_{2p},r)[1+d_\theta(l_{2p},\mathbb{S}l_{2p})+d_\theta(r,\mathbb{S}l_{2p})]}{1+d_\theta(l_{2p},r)}+\mathbb{L}d_\theta(r,\mathbb{S}l_{2p})\in\bigcap_{a\in\ \mathbb{S}l_{2p}}s(a,\mathbb{T}r).\end{aligned}$$

$$\begin{aligned}&\lambda_1 d_\theta(l_{2p},r)+\lambda_2[d_\theta(l_{2p},\mathbb{S}l_{2p})+d_\theta(r,\mathbb{T}r)]+\lambda_3[d_\theta(r,\mathbb{S}l_{2p})+d_\theta(l_{2p},\mathbb{T}r)]\\&+\lambda_4\frac{d_\theta(r,\mathbb{T}r)[1+d_\theta(l_{2p},\mathbb{S}l_{2p})]}{1+d_\theta(l_{2p},r)}+\lambda_5\frac{d_\theta(r,\mathbb{S}l_{2p})[1+d_\theta(l_{2p},\mathbb{T}r)]}{1+d_\theta(l_{2p},r)}\\&+\lambda_6\frac{d_\theta(l_{2p},r)[1+d_\theta(l_{2p},\mathbb{S}l_{2p})+d_\theta(r,\mathbb{S}l_{2p})]}{1+d_\theta(l_{2p},r)}+\mathbb{L}d_\theta(r,\mathbb{S}l_{2p})\in s(a,\mathbb{T}r),\quad \forall a\in\mathbb{S}l_{2n}.\end{aligned}$$

Since $l_{2p+1}\in\mathbb{S}l_{2p}$, therefore

$$\begin{aligned}&\lambda_1 d_\theta(l_{2p},r)+\lambda_2[d_\theta(l_{2p},\mathbb{S}l_{2p})+d_\theta(r,\mathbb{T}r)]+\lambda_3[d_\theta(r,\mathbb{S}l_{2p})+d_\theta(l_{2p},\mathbb{T}r)]\\&+\lambda_4\frac{d_\theta(r,\mathbb{T}r)[1+d_\theta(l_{2p},\mathbb{S}l_{2p})]}{1+d_\theta(l_{2p},r)}+\lambda_5\frac{d_\theta(r,\mathbb{S}l_{2p})[1+d_\theta(l_{2p},\mathbb{T}r)]}{1+d_\theta(l_{2p},r)}\\&+\lambda_6\frac{d_\theta(l_{2p},r)[1+d_\theta(l_{2p},\mathbb{S}l_2p)+d_\theta(r,\mathbb{S}l_{2p})]}{1+d_\theta(l_{2p},r)}+\mathbb{L}d_\theta(r,\mathbb{S}l_{2p})\in s(l_{2p+1},\mathbb{T}r),\end{aligned}$$

$$\begin{aligned}&\lambda_1 d_\theta(l_{2p},r)+\lambda_2[d_\theta(l_{2p},\mathbb{S}l_{2p})+d_\theta(r,\mathbb{T}r)]+\lambda_3[d_\theta(r,\mathbb{S}l_{2p})+d_\theta(l_{2p},\mathbb{T}r)]\\&+\lambda_4\frac{d_\theta(r,\mathbb{T}r)[1+d_\theta(l_{2p},\mathbb{S}l_{2p})]}{1+d_\theta(l_{2p},r)}+\lambda_5\frac{d_\theta(r,\mathbb{S}l_{2p})[1+d_\theta(l_{2p},\mathbb{T}r)]}{1+d_\theta(l_{2p},r)}\\&+\lambda_6\frac{d_\theta(l_{2p},r)[1+d_\theta(l_{2p},\mathbb{S}l_2p)+d_\theta(r,\mathbb{S}l_{2p})]}{1+d_\theta(l_{2p},r)}+\mathbb{L}d_\theta(r,\mathbb{S}l_{2p})\in\bigcup_{b\in\mathbb{T}r}s(d_\theta(l_{2p+1},b)).\end{aligned}$$

This implies that there exists some $r_p\in\mathbb{T}r$ such that

$$\begin{aligned}&\lambda_1 d_\theta(l_{2p},r)+\lambda_2[d_\theta(l_{2p},\mathbb{S}l_{2p})+d_\theta(r,\mathbb{T}r)]+\lambda_3[d_\theta(r,\mathbb{S}l_{2p})+d_\theta(l_{2p},\mathbb{T}r)]\\&+\lambda_4\frac{d_\theta(r,\mathbb{T}r)[1+d_\theta(l_{2p},\mathbb{S}l_{2p})]}{1+d_\theta(l_{2p},r)}+\lambda_5\frac{d_\theta(r,\mathbb{S}l_{2p})[1+d_\theta(l_{2p},\mathbb{T}r)]}{1+d_\theta(l_{2p},r)}\\&+\lambda_6\frac{d_\theta(l_{2p},r)[1+d_\theta(l_{2p},\mathbb{S}l_{2p})+d_\theta(r,\mathbb{S}l_{2p})]}{1+d_\theta(l_{2p},r)}+\mathbb{L}d_\theta(r,\mathbb{S}l_{2p})\in s(d_(l_{2p+1},r_p)),\end{aligned}$$

$$\begin{aligned}d_\theta(l_{2p+1},r_p)\preceq{}&\lambda_1 d_\theta(l_{2p},r)+\lambda_2[d_\theta(l_{2p},\mathbb{S}l_{2p})+d_\theta(r,\mathbb{T}r)]+\lambda_3[d_\theta(r,\mathbb{S}l_{2p})+d_\theta(l_{2p},\mathbb{T}r)]\\&+\lambda_4\frac{d_\theta(r,\mathbb{T}r)[1+d_\theta(l_{2p},\mathbb{S}l_{2p})]}{1+d_\theta(l_{2p},r)}+\lambda_5\frac{d_\theta(r,\mathbb{S}l_{2p})[1+d_\theta(l_{2p},\mathbb{T}r)]}{1+d_\theta(l_{2p},r)}\\&+\lambda_6\frac{d_\theta(l_{2p},r)[1+d_\theta(l_{2p},\mathbb{S}l_{2p})+d_\theta(r,\mathbb{S}l_{2p})]}{1+d_\theta(l_{2p},r)}+\mathbb{L}d_\theta(r,\mathbb{S}l_{2p}),\end{aligned}$$

$$\begin{aligned}d_\theta(l_{2p+1},r_p)\preceq{}&\lambda_1 d_\theta(l_{2p},r)+\lambda_2[d_\theta(l_{2p},l_{2p+1})+d_\theta(r,r_p)]+\lambda_3[d_\theta(r,l_{2p+1})+d_\theta(l_{2p},r_p)]\\&+\lambda_4\frac{d_\theta(r,r_p)[1+d_\theta(l_{2p},l_{2p+1})]}{1+d_\theta(l_{2p},r)}+\lambda_5\frac{d_\theta(r,l_{2p+1})[1+d_\theta(l_{2p},r_p)]}{1+d_\theta(l_{2p},r)}\end{aligned}$$

$$+\lambda_6 \frac{d_\theta(l_{2p},r)[1+d_\theta(l_{2p},l_{2p+1})+d_\theta(r,l_{2p+1})]}{1+d_\theta(l_{2p},r)} + \mathbb{L} d_\theta(r,l_{2p+1}).$$

We know that

$$d_\theta(r,r_p) \preceq \theta(r,r_p)[d_\theta(r,l_{2p+1})+d(l_{2p+1},r_p)]$$

, hence,

$$\begin{aligned} d_\theta(r,r_p) &\preceq \theta(r,r_p)d_\theta(l_{2p+1},r)+\theta(r,r_p)\lambda_1 d_\theta(l_{2p},r)+\theta(r,r_p)\lambda_2[d_\theta(l_{2p},l_{2p+1})+d_\theta(r,r_p)] \\ &+\theta(r,r_p)\lambda_3[d_\theta(r,l_{2p+1})+d_\theta(l_{2p},r_p)]+\theta(r,r_p)\lambda_4\frac{d_\theta(r,r_p)[1+d_\theta(l_{2p},l_{2p+1})]}{1+d_\theta(l_{2p},r)} \\ &+\theta(r,r_p)\lambda_5\frac{d_\theta(r,l_{2p+1})[1+d_\theta(l_{2p},r_p)]}{1+d_\theta(l_{2p},r)}+\theta(r,r_p)\lambda_6 \\ &\frac{d_\theta(l_{2p},r)[1+d_\theta(l_{2p},l_{2p+1})+\theta(r,r_p)d_\theta(r,l_{2p+1})]}{1+d_\theta(l_{2p},r)}+\theta(r,r_p)\mathbb{L}d_\theta(r,l_{2p+1}) \end{aligned}$$

$$\begin{aligned} |d_\theta(r,r_p)| &\le |d_\theta(l_{2p+1},r)|+\lambda_1|d_\theta(l_{2p},r)|+\lambda_2[|d_\theta(l_{2p},l_{2p+1})|+|d_\theta(r,r_p)|]+\lambda_3[|d_\theta(r,l_{2p+1})| \\ &+|d_\theta(l_{2p},r_p)|]+\lambda_4\left|\frac{d_\theta(r,r_p)[1+|d_\theta(l_{2p},l_{2p+1})]}{1+d_\theta(l_{2p},r)}\right|+\lambda_5\left|\frac{d_\theta(r,l_{2p+1})[1+d_\theta(l_{2p},r_p)]}{1+d_\theta(l_{2p},r)}\right| \\ &+\lambda_6\left|\frac{d_\theta(l_{2p},r)[1+d_\theta(l_{2p},l_{2p+1})+d_\theta(r,l_{2p+1})]}{1+d_\theta(l_{2p},r)}\right|+\mathbb{L}|d_\theta(r,l_{2p+1})|. \end{aligned}$$

By letting $p \to \infty$ in the above expression, we get

$$|d_\theta(r,r_p)| \to 0 \; as \; p \to \infty.$$

By Lemma (2.4), $r_p \to r$ as $p \to \infty$; also since $\mathbb{T}r$ is closed, thus $r \in \mathbb{T}r$. Similarly, we have $r \in \mathbb{S}r$. Hence, $\mathbb{S}$ and $\mathbb{T}$ have a common fixed point.

In what follows, we deduce some consequences of Theorem 2.1. For $\lambda_6 = 0$, we have

Corollary 2.1. *Let $(\mathbb{G}, d_\theta)$ be a complete complex valued extended b-metric space, $\theta : \mathbb{G}\times\mathbb{G} \to [1,\infty)$ and $\mathbb{S},\mathbb{T} : \mathbb{G}\times\mathbb{G} \to CB(\mathbb{G})$ be a pair of multi-valued mappings with g.l.b property such that*

$$\begin{aligned} &\lambda_1 d_\theta(l,m)+\lambda_2[d_\theta(l,\mathbb{S}l)+d_\theta(m,\mathbb{T}m)]+\lambda_3[d_\theta(m,\mathbb{S}l)+d_\theta(l,\mathbb{T}m)] \\ &+\lambda_4\frac{d_\theta(m,\mathbb{T}m)[1+d_\theta(l,\mathbb{S}l)]}{1+d_\theta(l,m)}+\lambda_5\frac{d_\theta(m,\mathbb{S}l)[1+d_\theta(l,\mathbb{T}m)]}{1+d_\theta(l,m)}+\mathbb{L}d_\theta(m,\mathbb{T}l) \in s(\mathbb{S}l,\mathbb{T}m), \end{aligned}$$

for all $l,m \in \mathbb{G}$ and $\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\mathbb{L}$ are nonnegative real numbers with $\lambda_1+2\lambda_2+2\lambda_3\theta(l_o,l_2)+\lambda_4<1$, $k(1-\lambda_2-\lambda_3\theta(l_o,l_2)-\lambda_4) = (\lambda_1+\lambda_2+\lambda_3\theta(l_o,l_2))$. Then $\mathbb{S}$ and $\mathbb{T}$ have a common fixed point.

For $\lambda_i = 0$, where $i = 5,6$, we have the next corollary.

Corollary 2.2. *Let $(\mathbb{G}, d_\theta)$ be a complete complex valued extended b-metric space, $\theta : \mathbb{G}\times\mathbb{G} \to [1,\infty)$ and $\mathbb{S},\mathbb{T} : \mathbb{G}\times\mathbb{G} \to CB(\mathbb{G})$ be a pair of multi-valued mappings with g.l.b property such that*

$$\begin{aligned} &\lambda_1 d_\theta(l,m)+\lambda_2[d_\theta(l,\mathbb{S}l)+d_\theta(m,\mathbb{T}m)]+\lambda_3[d_\theta(m,\mathbb{S}l)+d_\theta(l,\mathbb{T}m)] \\ &+\lambda_4\frac{d_\theta(m,\mathbb{T}m)[1+d_\theta(l,\mathbb{S}l)]}{1+d_\theta(l,m)}+\mathbb{L}d_\theta(m,\mathbb{T}l) \in s(\mathbb{S}l,\mathbb{T}m), \end{aligned}$$

for all $l,m \in \mathbb{G}$ and $\lambda_1,\lambda_2,\lambda_3,\lambda_4,\mathbb{L}$ are nonnegative real numbers with $\lambda_1+2\lambda_2+2\lambda_3\theta(l_o,l_2)+\lambda_4<1$, $k(1-\lambda_2-\lambda_3\theta(l_o,l_2)-\lambda_4) = (\lambda_1+\lambda_2+\lambda_3\theta(l_o,l_2))$. Then $\mathbb{S}$ and $\mathbb{T}$ have a common fixed point.

For $\lambda_i = 0$, where $i = 4,5,6$, we have the following result. follow:

Corollary 2.3. *Let* $(\mathbb{G}, d_\theta)$ *be a complete complex valued extended b-metric space,* $\theta : \mathbb{G} \times \mathbb{G} \to [1, \infty)$ *and* $\mathbb{S}, \mathbb{T} : \mathbb{G} \times \mathbb{G} \to CB(\mathbb{G})$ *be a pair of multi-valued mappings with g.l.b property such that*

$$\lambda_1 d_\theta(l, m) + \lambda_2[d_\theta(l, \mathbb{S}l) + d_\theta(m, \mathbb{T}m)] + \lambda_3[d_\theta(m, \mathbb{S}l) + d_\theta(l, \mathbb{T}m)] + \mathbb{L}d_\theta(m, \mathbb{T}l) \in s(\mathbb{S}l, \mathbb{T}m),$$

for all $l, m \in \mathbb{G}$ *and* $\lambda_1, \lambda_2, \lambda_3, \mathbb{L}$ *are nonnegative real numbers with* $\lambda_1 + 2\lambda_2 + 2\lambda_3\theta(l_o, l_2) < 1$. $k(1 - \lambda_2 - \lambda_3\theta(l_o, l_2) = (\lambda_1 + \lambda_2 + \lambda_3\theta(l_o, l_2))$. *Then* $\mathbb{S}$ *and* $\mathbb{T}$ *have a common fixed point.*

For $\mathbb{S} = \mathbb{T}$, we have the following corollary.

Corollary 2.4. *Let* $(\mathbb{G}, d_\theta)$ *be a complete complex valued extended b-metric space,* $\theta : \mathbb{G} \times \mathbb{G} \to [1, \infty)$ *and* $\mathbb{T} : \mathbb{G} \to CB(\mathbb{G})$ *be a multi-valued mapping with g.l.b property such that*

$$\begin{aligned} &\lambda_1 d_\theta(l, m) + \lambda_2[d_\theta(l, \mathbb{T}l) + d_\theta(m, \mathbb{T}m)] + \lambda_3[d_\theta(m, \mathbb{T}l) + d_\theta(l, \mathbb{T}m)] \\ &+ \lambda_4 \frac{d_\theta(m, \mathbb{T}m)[1 + d_\theta(l, \mathbb{T}l)]}{1 + d_\theta(l, m)} + \lambda_5 \frac{d_\theta(m, \mathbb{T}l)[1 + d_\theta(l, \mathbb{T}m)]}{1 + d_\theta(l, m)} \\ &+ \lambda_6 \frac{d_\theta(l, m)[1 + d_\theta(l, \mathbb{T}l) + d_\theta(m, \mathbb{T}l)]}{1 + d_\theta(l, m)} + \mathbb{L}d_\theta(m, \mathbb{T}l) \in s(\mathbb{T}l, \mathbb{T}m) \end{aligned} \tag{2.2}$$

for all $l, m \in \mathbb{G}$ *and* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \mathbb{L}$ *are nonnegative real numbers with* $\lambda_1 + 2\lambda_2 + 2\lambda_3\theta(l, m) + \lambda_4 + \lambda_5 + \lambda_6 < 1$, $k(1 - \lambda_2 - \lambda_3\theta(l, m) - \lambda_4) = (\lambda_1 + \lambda_2 + \lambda_3\theta(l, m) + \lambda_6)$ *where* $k \in [0, \infty)$ *be such that for each* $l_o \in \mathbb{G}$ $\lim_{p,q\to\infty} \theta(l_p, l_q) < \frac{1}{k}$. *Then* $\mathbb{T}$ *has a fixed point.*

Remark 2.1. *From Corollary* 4.6, *we can deduce several corollaries by putting* $\lambda_6 = 0$, $\lambda_6, \lambda_5 = 0$ *and* $\lambda_6, \lambda_5, \lambda_4 = 0$.

Now, we provide an example to validate the hypotheses of Theorem 2.1 as follows.

Example 2.1. *Let* $\mathbb{G} = [0, \infty)$. *Define* $\theta : \mathbb{G} \times \mathbb{G} \longrightarrow [1, \infty)$ *by*

$$\theta(l, m) = \frac{5 + l + m}{1 + l + m}, \text{ for all } l, m \in \mathbb{G},$$

and $d_\theta : \mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{C}$ *by*

$$d_\theta(l, m) = |l - m|^2 + i|l - m|^2, \text{ for all } l, m \in \mathbb{G}.$$

Then $(\mathbb{G}, d_\theta)$ *is a complete complex-valued extended b-metric space. Consider the mappings* $\mathbb{S}, \mathbb{T} : \mathbb{G} \longrightarrow CB(\mathbb{G})$, *defined by*

$$\mathbb{S}l = \begin{cases} \left[0, \frac{l}{7}\right], & \text{if } l \in [0, 1] \\ [3l, 5l], & \text{otherwise.} \end{cases}$$

$$\mathbb{T}l = \begin{cases} \left[0, \frac{l}{14}\right], & \text{if } l \in [0, 1] \\ [5l, 9l], & \text{otherwise.} \end{cases}$$

If $l = m = 0$, *the contractive condition of Theorem* 2.1 *holds trivially. Assume without loss of generality that both* l *and* m *are nonzero with* $l < m$. *Then*

$$d_\theta(l, \mathbb{S}l) = \left|l - \frac{l}{7}\right|^2 + i\left|l - \frac{l}{7}\right|^2, \; d_\theta(m, \mathbb{T}m) = \left|m - \frac{m}{14}\right|^2 + i\left|m - \frac{m}{14}\right|^2,$$

$$d_\theta(m, \mathbb{S}l) = \left|m - \frac{l}{7}\right|^2 + i\left|m - \frac{l}{7}\right|^2, \; d_\theta(l, \mathbb{T}m) = \left|l - \frac{m}{14}\right|^2 + i\left|l - \frac{m}{14}\right|^2,$$

$$s(\mathbb{S}l, \mathbb{T}m) = s\left(\left|\frac{l}{7} - \frac{m}{14}\right|^2 + i\left|\frac{l}{7} - \frac{m}{14}\right|^2\right).$$

By taking $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0$ *and* $\lambda_1 = \frac{1}{2}$, *it can be verified directly that all the conditions of Theorem* 2.1 *are satisfied. In this case, 0 is a common fixed point of* $\mathbb{S}$ *and* $\mathbb{T}$.

3. Banach Type Contractive Mapping

Theorem 3.1. *Let $(\mathbb{G}, d_\theta)$ be a complete C.V extended b-metric space, $\theta : \mathbb{G} \times \mathbb{G} \to [1, \infty)$ be a function and $\mathbb{S}, \mathbb{T} : \mathbb{G} \to CB(\mathbb{G})$ be a pair multi-valued mappings satisfying the g.l.b property such that*

$$\lambda_1 d_\theta(l, m) + \frac{\lambda_2 d_\theta(l, \mathbb{S}l) d_\theta(m, \mathbb{T}m) + \lambda_3 d_\theta(m, \mathbb{S}l) d_\theta(l, \mathbb{T}m)}{1 + d_\theta(l, m)} \in s(\mathbb{S}l, \mathbb{T}m), \tag{3.1}$$

for all $l, m \in \mathbb{G}$ and $\lambda_1, \lambda_2, \lambda_3$ are non negative real numbers with $\lambda_1 + \lambda_2 + \lambda_3 < 1$, and assumed that $k(1 - \lambda_2) = \lambda_1$ where $k \in [0, 1)$ for each $l_o \in \mathbb{G}$, $\lim_{p,q\to\infty} \theta(l_p, m_q)k < 1$. Then $\mathbb{S}$ and $\mathbb{T}$ have a common fixed point.

Proof. Let $l_o \in \mathbb{G}$, then Tl_o is not empty; so we can take $l_1 \in Tl_o$. Thus, from (3.1),

$$\lambda_1 d_\theta(l_o, l_1) + \frac{\lambda_2 d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1) + \lambda_3 d_\theta(l_1, \mathbb{S}l_o) d_\theta(l_o, \mathbb{T}l_1)}{1 + d_\theta(l_0, l_1)} \in s(\mathbb{S}l_o, \mathbb{T}l_1).$$

This implies that

$$\lambda_1 d_\theta(l_o, l_1) + \frac{\lambda_2 d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1) + \lambda_3 d_\theta(l_1, \mathbb{S}l_o) d_\theta(l_o, \mathbb{T}l_1)}{1 + d_\theta(l_0, l_1)} \in \bigcap_{a' \in \mathbb{S}l_o} s(a', \mathbb{T}l_1),$$

$$\lambda_1 d_\theta(l_o, l_1) + \frac{\lambda_2 d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1) + \lambda_3 d_\theta(l_1, \mathbb{S}l_o) d_\theta(l_o, \mathbb{T}l_1)}{1 + d_\theta(l_0, l_1)} \in s(a', \mathbb{T}l_1), \quad \forall a' \in \mathbb{S}l_o.$$

Since $l_1 \in \mathbb{S}l_0$, we have

$$\lambda_1 d_\theta(l_o, l_1) + \frac{\lambda_2 d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1) + \lambda_3 d_\theta(l_1, \mathbb{S}l_o) d_\theta(l_o, \mathbb{T}l_1)}{1 + d_\theta(l_0, l_1)} \in s(l_1, \mathbb{T}l_1)$$

$$\lambda_1 d_\theta(l_o, l_1) + \frac{\lambda_2 d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1) + \lambda_3 d_\theta(l_1, \mathbb{S}l_o) d_\theta(l_o, \mathbb{T}l_1)}{1 + d_\theta(l_0, l_1)} \in \bigcup_{b' \in \mathbb{T}l_1} s(d_\theta(l_1, b')).$$

Therefore, there exist $l_2 \in \mathbb{T}l_2$ such that

$$\lambda_1 d_\theta(l_o, l_1) + \frac{\lambda_2 d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1) + \lambda_3 d_\theta(l_1, \mathbb{S}l_o) d_\theta(l_o, \mathbb{T}l_1)}{1 + d_\theta(l_0, l_1)} \in s(d_\theta(l_1, l_2)).$$

By definition and the g.l.b property of $\mathbb{S}$ and $\mathbb{T}$, we get

$$d_\theta(l_1, l_2) \preceq \lambda_1 d_\theta(l_o, l_1) + \frac{\lambda_2 d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1) + \lambda_3 d_\theta(l_1, \mathbb{S}l_o) d_\theta(l_o, \mathbb{T}l_1)}{1 + d_\theta(l_0, l_1)},$$

$$d_\theta(l_1, l_2) \preceq \lambda_1 d_\theta(l_o, l_1) + \frac{\lambda_2 d_\theta(l_o, l_1) d_\theta(l_1, l_2) + \lambda_3 d_\theta(l_1, l_1) d_\theta(l_o, l_2)}{1 + d_\theta(l_0, l_1)}.$$

This implies

$$\begin{aligned} |d_\theta(l_1, l_2)| &\leq \lambda_1 |d_\theta(l_o, l_1)| + \frac{\lambda_2 |d_\theta(l_o, l_1)||d_\theta(l_1, l_2)|}{1 + |d_\theta(l_0, l_1)|} \\ &= \lambda_1 |d_\theta(l_o, l_1)| + \lambda_2 |d_\theta(l_1, l_2)| \frac{|d_\theta(l_o, l_1)|}{|1 + d_\theta(l_0, l_1)|}. \end{aligned}$$

Or

$$\begin{aligned} |d_\theta(l_1, l_2)| &\leq \lambda_1 |d_\theta(l_o, l_1)| + \lambda_2 |d_\theta(l_1, l_2)| \\ (1 - \lambda_2)|d_\theta(l_1, l_2)| &\leq \lambda_1 |d_\theta(l_o, l_1)| \\ |d_\theta(l_1, l_2)| &\leq \frac{\lambda_1}{(1 - \lambda_2)} |d_\theta(l_o, l_1)|. \end{aligned}$$

Inductively, we develop a sequence $\{l_p\}$ in $\mathbb{G}$ such that

$$|d_\theta(l_1, l_2)| \leq k|d_\theta(l_o, l_1)|.$$
$$|d_\theta(l_2, l_3)| \leq k^2|d_\theta(l_o, l_1)|.$$
$$\vdots$$
$$|d_\theta(l_p, l_{p+1})| \leq k^p|d_\theta(l_o, l_1)|.$$

Now, by triangular inequality, for $q > p$, we have

$$\begin{aligned} d_\theta(l_p, l_q) &\preceq \theta(l_p, l_q)k^p d_\theta(l_o, l_1) + \theta(l_p, l_q)\theta(l_{p+1}, l_q)k^{p+1}d_\theta(l_o, l_1) + \cdots \\ &+ \theta(l_p, l_q)\theta(l_{p+1}, l_q) \cdots \theta(l_{q-2}, l_q)\theta(l_{q-1}, l_q)k^{q-1}d_\theta(l_o, l_1) \\ d_\theta(l_p, l_q) &\preceq d_\theta(l_0, l_1)[\theta(l_p, l_q)k^p + \theta(l_p, l_q)\theta(l_{p+1}, l_q)k^{p+1} + \cdots \\ &+ \theta(l_p, l_q)\theta(l_{p+1}, l_q)...\theta(l_{q-2}, l_q)\theta(l_{q-1}, l_p)k^{q-1}]. \end{aligned}$$

Since $\lim_{p,q\to\infty} \theta(l_p, l_q)k < 1$, so the series $\sum_{p=1}^{\infty} k^p \prod_{i=1}^{p} \theta(l_i, l_q)$ converges by ratio test for each $q \in \mathbb{N}$. Let

$$S = \sum_{p=1}^{\infty} k^p \prod_{i=1}^{p} \theta(l_i, l_q), \; S_p = \sum_{j=1}^{p} k^j \prod_{i=1}^{p} \theta(l_i, l_q)$$

.

Thus, for $q > p$, the above can be written as

$$d_\theta(l_p, l_q) \preceq d_\theta(l_o, l_1)[S_{q-1} - S_P]$$

In other words,

$$|d_\theta(l_p, l_q)| \leq |d_\theta(l_o, l_1)|[S_{q-1} - S_p].$$

Now, by taking $p \to \infty$, we get

$$|d_\theta(l_p, l_q)| \to 0.$$

By lemma (2.4), we conclude that $\{l_p\}$ is a Cauchy sequence. Since $\mathbb{G}$ is complete, then, there exists an element r such that $l_p \to r \in \mathbb{G}$ as $p \to \infty$.
Now, to show $r \in \mathbb{S}r$ and $r \in \mathbb{T}r$. For this, we have from (5.1)

$$\lambda_1 d_\theta(l_{2p}, r) + \frac{\lambda_2 d_\theta(l_{2p}, \mathbb{S}l_{2p})d_\theta(r, \mathbb{T}r) + \lambda_3 d_\theta(r, \mathbb{S}l_{2p})d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \in s(\mathbb{S}l_{2p}, \mathbb{T}r),$$

$$\lambda_1 d_\theta(l_{2p}, r) + \frac{\lambda_2 d_\theta(l_{2p}, \mathbb{S}l_{2p})d_\theta(r, \mathbb{T}r) + \lambda_3 d_\theta(r, \mathbb{S}l_{2p})d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \in \bigcap_{a' \in \mathbb{S}l_{2p}} s(a', \mathbb{T}r).$$

Since $l_{2p+1} \in \mathbb{S}l_{2p}$, we have

$$\lambda_1 d_\theta(l_{2p}, r) + \frac{\lambda_2 d_\theta(l_{2p}, \mathbb{S}l_{2p})d_\theta(r, \mathbb{T}r) + \lambda_3 d_\theta(r, \mathbb{S}l_{2p})d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \in s(l_{2p+1}, \mathbb{T}r),$$

$$\lambda_1 d_\theta(l_{2p}, r) + \frac{\lambda_2 d_\theta(l_{2p}, \mathbb{S}l_{2p})d_\theta(r, \mathbb{T}r) + \lambda_3 d_\theta(r, \mathbb{S}l_{2p})d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \in \bigcup_{b' \in \mathbb{T}r} s(d_\theta(l_{2p+1}, b')).$$

This implies that there exists $r_p \in \mathbb{T}r$ such that

$$\lambda_1 d_\theta(l_{2p}, r) + \frac{\lambda_2 d_\theta(l_{2p}, \mathbb{S}l_{2p})d_\theta(r, \mathbb{T}r) + \lambda_3 d_\theta(r, \mathbb{S}l_{2p})d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \in s(d_\theta(l_{2p+1}, r_p)).$$

That is,

$$d_\theta(l_{2p+1}, r_p) \preceq \lambda_1 d_\theta(l_{2p}, r) + \frac{\lambda_2 d_\theta(l_{2p}, l_{2p+1}) d_\theta(r, r_p) + \lambda_3 d_\theta(r, \mathbb{S}l_{2p}) d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)}.$$

Now,

$$d_\theta(r, r_p) \preceq \theta(r, r_p)[d_\theta(r, l_{2p+1}) + d_\theta(l_{2p+1}, r_p)]$$

$$d_\theta(r, r_p) \preceq \theta(r, r_p) d_\theta(r, l_{2p+1}) + \theta(r, r_p)\lambda_1 d_\theta(l_{2p}, r) + \frac{\theta(r, r_p)\lambda_2 d_\theta(l_{2p}, l_{2p+1}) d_\theta(r, r_p) + \theta(r, r_p)\lambda_3 d_\theta(r, l_{2p+1}) d_\theta(l_{2p}, r_p)}{1 + d_\theta(l_{2p}, r)}.$$

It follows that

$$|d_\theta(r, r_p)| \leq \theta(r, r_p)|d_\theta(r, l_{2p+1})| + \theta(r, r_p)\lambda_1 |d_\theta(l_{2p}, r)| + \frac{\theta(r, r_p)\lambda_2 |d_\theta(l_{2p}, l_{2p+1})||d_\theta(r, r_p)| + \theta(r, r_p)\lambda_3 |d_\theta(r, l_{2p+1})||d_\theta(l_{2p}, r_p)|}{1 + |d_\theta(l_{2p}, r)|}.$$

Letting $p \to \infty$, we get $|d_\theta(r, r_p)| \to 0$.
By using Lemma 2.4, we have $r_p \to r$. Since $\mathbb{T}r$ is closed, so $r \in \mathbb{T}r$. On similar steps, we can prove that $r \in \mathbb{S}r$. Thus, $\mathbb{T}$ and $\mathbb{S}$ have a common fixed point.

By putting $\lambda_3 = 0$ in Theorem 3.1, we have the next result.

Corollary 3.1. *Let $(\mathbb{G}, d_\theta)$ ba a complete C.V extended b-metric $\theta : \mathbb{G} \times \mathbb{G} \to [1, \infty)$ be a function, and $\mathbb{S}, \mathbb{T} : \mathbb{G} \to CB(\mathbb{G})$ be a pair of multi-valued mappings satisfying the g.l.b property such that*

$$\lambda_1 d_\theta(l, m) + \frac{\lambda_2 d_\theta(l, \mathbb{S}l) d_\theta(m, \mathbb{T}m)}{1 + d_\theta(l, m)} \in s(\mathbb{S}l, \mathbb{T}m), \tag{3.2}$$

for all $l, m \in \mathbb{G}$ and λ_1, λ_2 are non negative reals with $\lambda_1 + \lambda_2 < 1$ and assumed $k(1 - \lambda_2) = \lambda_1$ where $k \in [0, 1)$ such that for each $l_o \in \mathbb{G}$, $\lim_{p,q\to\infty} \theta(l_p, l_q)k < 1$. Then, $\mathbb{S}$ and $\mathbb{T}$ have a common fixed point.

By putting $\mathbb{S} = \mathbb{T}$ in Theorem 3.1, we have the following. corollary.

Corollary 3.2. *Let $(\mathbb{G}, d_\theta)$ be a complete C.V extended b-metric space, $\theta : \mathbb{G} \times \mathbb{G} \to [1, \infty)$ be a function, and $\mathbb{T} : \mathbb{G} \to CB(\mathbb{G})$ be a multi-valued mapping satisfying the g.l.b property such that*

$$\lambda_1 d_\theta(l, m) + \frac{\lambda_2 d_\theta(l, \mathbb{T}l) d_\theta(m, \mathbb{T}m) + \lambda_3 d_\theta(m, \mathbb{T}l) d_\theta(l, \mathbb{T}m)}{1 + d_\theta(l, m)} \in s(\mathbb{T}l, \mathbb{T}m) \tag{3.3}$$

for all $l, m \in \mathbb{G}$ and $\lambda_1, \lambda_2, \lambda_3$ are non negative reals with $\lambda_1 + \lambda_2 + \lambda_3 < 1$ and assumed that $k(1 - \lambda_2) = \lambda_1$ where $k \in [0, 1)$ for each, $l_o \in \mathbb{G}$ $\lim_{p,q\to\infty} \theta(l_p, m_q)k < 1$. Then $\mathbb{T}$ has a fixed point.

4. Kannan Type Contractive Mapping

Theorem 4.1. *Let $(\mathbb{G}, d_\theta)$ be a complete C.V extended b-metric space and $\theta : \mathbb{G} \times \mathbb{G} \to [1, \infty)$ be a function. Let $\mathbb{S}, \mathbb{T} : \mathbb{G} \to CB(\mathbb{G})$ be a pair of multi-valued mappings satisfying the g.l.b property such that*

$$\lambda_1 d_\theta(l, \mathbb{S}l) + \lambda_2 d_\theta(m, \mathbb{T}m) + \frac{\lambda_3 d_\theta(l, \mathbb{S}l) d_\theta(m, \mathbb{T}m)}{1 + d_\theta(l, m)} \in s(\mathbb{S}l, \mathbb{T}m), \tag{4.1}$$

for all $l, m \in \mathbb{G}$ and $\lambda_1, \lambda_2, \lambda_3$ are non negative reals with $\lambda_1 + \lambda_2 + \lambda_3 < 1$ and assumed $k(1 - \lambda_2 - \lambda_3) = \lambda_1$, where $k \in [0, 1)$ is such that for each $l_o \in \mathbb{G}$, $\lim_{p,q\to\infty} \theta(l_p, l_q)k < 1$. Then $\mathbb{S}$ and $\mathbb{T}$ have a common fixed point.

Proof. Let $l_o \in \mathbb{G}$, then Tl_o is non-empty; so we can take $l_1 \in Tl_o$. Thus from (4.1),

$$\lambda_1 d_\theta(l_o, \mathbb{S}l_o) + \lambda_2 d_\theta(l_1, \mathbb{T}l_1) + \frac{\lambda_3 d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1)}{1 + d_\theta(l_o, l_1)} \in s(\mathbb{S}l_o, \mathbb{T}l_1),$$

$$\lambda_1 d_\theta(l_o, \mathbb{S}l_o) + \lambda_2 d_\theta(l_1, \mathbb{T}l_1) + \frac{\lambda_3 d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1)}{1 + d_\theta(l_o, l_1)} \in \bigcap_{a' \in \mathbb{S}l_o} s(a', \mathbb{T}l_1),$$

$$\lambda_1 d_\theta(l_o, \mathbb{S}l_o) + \lambda_2 d_\theta(l_1, \mathbb{T}l_1) + \frac{\lambda_3 d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1)}{1 + d_\theta(l_o, l_1)} \in s(a', \mathbb{T}l_1), \quad \forall a' \in \mathbb{S}l_o.$$

Since $l_1 \in \mathbb{S}l_o$, thus

$$\lambda_1 d_\theta(l_o, \mathbb{S}l_o) + \lambda_2 d_\theta(l_1, \mathbb{T}l_1) + \frac{\lambda_3 d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1)}{1 + d_\theta(l_o, l_1)} \in s(l_1, \mathbb{T}l_1),$$

$$\lambda_1 d_\theta(l_o, \mathbb{S}l_o) + \lambda_2 d_\theta(l_1, \mathbb{T}l_1) + \frac{\lambda_3 d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1)}{1 + d_\theta(l_o, l_1)} \in \bigcup_{b' \in \mathbb{T}l_1} s(d_\theta(l_1, b')).$$

Therefore, there exist $l_2 \in \mathbb{T}l_1$ such that

$$\lambda_1 d_\theta(l_o, \mathbb{S}l_o) + \lambda_2 d_\theta(l_1, \mathbb{T}l_1) + \frac{\lambda_3 d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1)}{1 + d_\theta(l_o, l_1)} \in s(d_\theta(l_1, l_2)).$$

By definition, we get

$$d_\theta(l_1, l_2) \preceq \lambda_1 d_\theta(l_o, \mathbb{S}l_o) + \lambda_2 d_\theta(l_1, \mathbb{T}l_1) + \frac{\lambda_3 d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1)}{1 + d_\theta(l_o, l_1)}.$$

Or

$$d_\theta(l_1, l_2) \preceq \lambda_1 d_\theta(l_o, l_1) + \lambda_2 d_\theta(l_1, l_2) + \frac{\lambda_3 d_\theta(l_o, l_1) d_\theta(l_1, l_2)}{1 + d_\theta(l_o, l_1)},$$

from which we have

$$|d_\theta(l_1, l_2)| \leq \lambda_1 |d_\theta(l_o, l_1)| + \lambda_2 |d_\theta(l_1, l_2)| + \frac{\lambda_3 |d_\theta(l_o, l_1)||d_\theta(l_1, l_2)|}{1 + |d_\theta(l_o, l_1)|}.$$

That is,

$$|d_\theta(l_1, l_2)| \leq \lambda_1 |d_\theta(l_o, l_1)| + \lambda_2 |d_\theta(l_1, l_2)| + \lambda_3 |d_\theta(l_1, l_2)|,$$

or

$$|d_\theta(l_1, l_2)| \leq \frac{\lambda_1}{(1 - \lambda_2 - \lambda_3)} |d_\theta(l_o, l_1)|.$$

Inductively, we develop a sequence $\{l_p\}$ in $\mathbb{G}$ such that

$$\begin{aligned} |d_\theta(l_1, l_2)| &\leq k |d_\theta(l_o, l_1)|. \\ |d_\theta(l_2, l_3)| &\leq k^2 |d_\theta(l_o, l_1)|. \\ &. \\ &. \\ &. \\ |d_\theta(l_p, l_{p+1})| &\leq k^p |d_\theta(l_o, l_1)|. \end{aligned}$$

Now, by triangular inequality and for $q > p$, we have the following

$$\begin{aligned} d_\theta(l_p, l_q) \preceq\ & \theta(l_p, l_q) k^p d_\theta(l_o, l_1) + \theta(l_p, l_q)\theta(l_{p+1}, l_q) k^{p+1} d_\theta(l_o, l_1) + \cdots \\ & + \theta(l_p, l_q)\theta(l_{p+1}, l_q) \cdots \theta(l_{q-2}, l_q)\theta(l_{q-1}, l_q) k^{q-1} d_\theta(l_o, l_1).d_\theta(l_p, l_q) \end{aligned}$$

$$d_\theta(l_p,l_q) \preceq d_\theta(l_0,l_1)[\theta(l_p,l_q)k^p + \theta(l_p,l_q)\theta(l_{p+1},l_q)k^{p+1} + \cdots + \theta(l_p,l_q)\theta(l_{p+1},l_q)...\theta(l_{q-2},l_q)\theta(l_{q-1},m_p)k^{q-1}]$$

Since $\lim_{p,q\to\infty}\theta(l_p,l_q)k < 1$, so the series $\sum_{p=1}^{\infty} k^p \prod_{i=1}^{p}\theta(l_i,l_q)$ converges by ratio test for each $q \in N$. Let

$$S = \sum_{p=1}^{\infty} k^p \prod_{i=1}^{p}\theta(l_i,l_q), \quad S_p = \sum_{j=1}^{p} k^j \prod_{i=1}^{p}\theta(l_i,l_q)$$

.

Thus, for $q > p$, the above can be written as

$$d_\theta(l_p,l_q) \preceq d_\theta(l_o,l_1)[S_{q-1} - S_P],$$

or

$$|d_\theta(l_p,l_q)| \leq |d_\theta(l_o,l_1)|[S_{q-1} - S_p]$$

Now, by taking $p \to \infty$, we get

$$|d_\theta(l_p,l_q)| \to 0.$$

By Lemma 2.4, we conclude that $\{l_p\}$ is a Cauchy sequence. Since $\mathbb{G}$ is complete, then there exists an element r such that $l_p \to r \in \mathbb{G}$ as $p \to \infty$.

To show that $r \in \mathbb{S}r$ and $r \in \mathbb{T}r$, from (6.1), we have

$$\lambda_1 d_\theta(l_{2p},\mathbb{S}l_{2p}) + \lambda_2 d_\theta(r,\mathbb{T}r) + \frac{\lambda_3 d_\theta(l_{2p},\mathbb{S}l_{2p})d_\theta(r,\mathbb{T}r)}{1+d_\theta(l_{2p},r)} \in s(\mathbb{S}l_{2p},\mathbb{T}r),$$

$$\lambda_1 d_\theta(l_{2p},\mathbb{S}l_{2p}) + \lambda_2 d_\theta(r,\mathbb{T}r) + \frac{\lambda_3 d_\theta(l_{2p},\mathbb{S}l_{2p})d_\theta(r,\mathbb{T}r)}{1+d_\theta(l_{2p},r)} \in \bigcap_{a'\in Sl_{2p}} s(a',\mathbb{T}r).$$

Since $l_{2p+1} \in \mathbb{S}l_{2p}$, we have

$$\lambda_1 d_\theta(l_{2p},\mathbb{S}l_{2p}) + \lambda_2 d_\theta(r,\mathbb{T}r) + \frac{\lambda_3 d_\theta(l_{2p},\mathbb{S}l_{2p})d_\theta(r,\mathbb{T}r)}{1+d_\theta(l_{2p},r)} \in s(l_{2p+1},\mathbb{T}r)$$

$$\lambda_1 d_\theta(l_{2p},\mathbb{S}l_{2p}) + \lambda_2 d_\theta(r,\mathbb{T}r) + \frac{\lambda_3 d_\theta(l_{2p},\mathbb{S}l_{2p})d_\theta(r,\mathbb{T}r)}{1+d_\theta(l_{2p},r)} \in \bigcup_{b'\in\mathbb{T}r} s(d_\theta(l_{2p+1},b').$$

So, there exist some $r_p \in \mathbb{T}r$ such that

$$\lambda_1 d_\theta(l_{2p},\mathbb{S}l_{2p}) + \lambda_2 d_\theta(r,\mathbb{T}r) + \frac{\lambda_3 d_\theta(l_{2p},\mathbb{S}l_{2p})d_\theta(r,\mathbb{T}r)}{1+d_\theta(l_{2p},r)} \in s(d_\theta(l_{2p+1},r_p).$$

Using definition 2.5, we have

$$d_\theta(l_{2p+1},r_p) \preceq \lambda_1 d_\theta(l_{2p},\mathbb{S}l_{2p}) + \lambda_2 d_\theta(r,\mathbb{T}r) + \frac{\lambda_3 d_\theta(l_{2p},\mathbb{S}l_{2p})d_\theta(r,\mathbb{T}r)}{1+d_\theta(l_{2p},r)}.$$

The g.l.b property of $\mathbb{T}$ yields

$$d_\theta(l_{2p+1},r_p) \preceq \lambda_1 d_\theta(l_{2p},l_{2p+1}) + \lambda_2 d_\theta(r,r_p) + \frac{\lambda_3 d_\theta(l_{2p},l_{2p+1})d_\theta(r,r_p)}{1+d_\theta(l_{2p},r)}.$$

Now,

$$d_\theta(r,r_p) \preceq \theta(r,r_p)[d_\theta(r,l_{2p+1}) + d_\theta(l_{2p+1},r_p)]$$

$$d_\theta(r,r_p) \preceq \theta(r,r_p)d_\theta(r,l_{2p+1}) + \theta(r,r_p)\lambda_1 d_\theta(l_{2p},l_{2p+1})$$

$$+\theta(r,r_p)\lambda_2 d_\theta(r,r_p)+\frac{\theta(r,r_p)\lambda_3 d_\theta(l_{2p},l_{2p+1})d_\theta(r,r_p)}{1+d_\theta(l_{2p},r)}$$

This implies

$$|d_\theta(r,r_p)|\leq \theta(r,r_p)|d_\theta(r,l_{2p+1})|+\theta(r,r_p)\lambda_1|d_\theta(l_{2p},l_{2p+1})|$$
$$+\theta(r,r_p)\lambda_2|d_\theta(r,r_p)|+\frac{\theta(r,r_p)\lambda_3|d_\theta(l_{2p},l_{2p+1})||d_\theta(r,r_p)|}{1+|d_\theta(l_{2p},r)|}.$$

Taking the limit as $p\to\infty$, we get $|d_\theta(r,r_p)|\to 0$.
By Lemma 2.4, we have $r_p\to r$. Since $\mathbb{T}$ is closed, then $r\in\mathbb{T}r$. By similar arguments, we can show that $r\in\mathbb{S}r$. Thus, $\mathbb{S}$ and $\mathbb{T}$ have a common fixed point.

By setting $\mathbb{S}=\mathbb{T}$ in Theorem 4.1, we get

Corollary 4.1. *Let $(\mathbb{G},d_\theta)$ be a complete C.V extended b-metric space, $\theta:\mathbb{G}\times\mathbb{G}\to[1,\infty)$ be a function and $T:\mathbb{G}\to CB(\mathbb{G})$ be a multi-valued mapping satisfying the g.l.b property such that*

$$\lambda_1 d_\theta(l,\mathbb{T}l)+\lambda_2 d_\theta(m,\mathbb{T}m)+\frac{\lambda_3 d_\theta(l,\mathbb{T}r)d_\theta(m,\mathbb{T}m)}{1+d_\theta(l,m)}\in s(\mathbb{T}l,\mathbb{T}m), \tag{4.2}$$

for all $l,m\in\mathbb{G}$ and $\lambda_1,\lambda_2,\lambda_3$ are non negative reals with $\lambda_1+\lambda_2+\lambda_3<1$, and assumed that $k(1-\lambda_2-\lambda_3)=\lambda_1$, where $k\in[0,1)$ is such that for each $l_o\in\mathbb{G}$, $\lim_{p,q\to\infty}\theta(l_p,l_q)k<1$. Then $\mathbb{T}$ has a fixed point.

Theorem 4.2. *Let $(\mathbb{G},d_\theta)$ be a complete C.V extended b-metric space, $\theta:\mathbb{G}\times\mathbb{G}\to[1,\infty)$ be a function and $\mathbb{T}:\mathbb{G}\to CB(\mathbb{G})$ be a multi-valued mapping fulfilling the g.l.b property such that*

$$\lambda_1 d_\theta(l,m)+\lambda_2\frac{d_\theta(l,\mathbb{T}l)d_\theta(m,\mathbb{T}m)}{1+d(l,m)}+\lambda_3\frac{d_\theta(m,\mathbb{T}l)d_(l,\mathbb{T}m)}{1+d_\theta(l,m)}$$
$$+\lambda_4\frac{d_\theta(l,\mathbb{T}l)d_\theta(l,\mathbb{T}m)}{1+d_(l,m)}+\lambda_5\frac{d_\theta(m,\mathbb{T}l)d_\theta(m,\mathbb{T}m)}{1+d_\theta(l,m)}\in s(\mathbb{T}l,\mathbb{T}m), \tag{4.3}$$

for all $l,m\in\mathbb{G}$ and $\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5$ are nonnegative real numbers with $\lambda_1+\lambda_2+\lambda_3+2\lambda_4\theta(l_0,l_2)+\lambda_5<1$ and assumed that $k(1-\lambda_2-\lambda_4\theta(l_0,l_2))=\lambda_1+\lambda_4\theta(l_0,l_2)$ where $k\in[0,1)$ is such that for each $l_o\in\mathbb{G}$, $\lim_{p,q\to\infty}\theta(l_p,l_q)k<1$. Then $\mathbb{T}$ has a fixed point.

Proof. Let $l_o\in\mathbb{G}$, then Tl_o is non-empty, so we can take $l_1\in Tl_o$; thus from (4.3),

$$\lambda_1 d_\theta(l_o,l_1)+\lambda_2\frac{d_\theta(l_o,\mathbb{T}l_o)d_\theta(l_1,\mathbb{T}l_1)}{1+d(l_o,l_1)}+\lambda_3\frac{d_\theta(l_1,\mathbb{T}l_o)d_(l_o,\mathbb{T}l_1)}{1+d_\theta(l_0,l_1)}$$
$$+\lambda_4\frac{d_\theta(l_o,\mathbb{T}l_o)d_\theta(l_o,\mathbb{T}l_1)}{1+d_(l_o,l_1)}+\lambda_5\frac{d_\theta(l_1,\mathbb{T}l_o)d_(l_1,\mathbb{T}l_1)}{1+d_(l_o,l_1)}\in s(\mathbb{T}l_o,\mathbb{T}l_1),$$

$$\lambda_1 d_\theta(l_o,l_1)+\lambda_2\frac{d_\theta(l_o,\mathbb{T}l_o)d_\theta(l_1,\mathbb{T}l_1)}{1+d(l_o,l_1)}+\lambda_3\frac{d_\theta(l_1,\mathbb{T}l_o)d_(l_o,\mathbb{T}l_1)}{1+d_\theta(l_0,l_1)}$$
$$+\lambda_4\frac{d_\theta(l_o,\mathbb{T}l_o)d_\theta(l_o,\mathbb{T}l_1)}{1+d_(l_o,l_1)}+\lambda_5\frac{d_\theta(l_1,\mathbb{T}l_o)d_(l_1,\mathbb{T}l_1)}{1+d_(l_o,l_1)}\in\bigcap_{a\in\mathbb{T}l_o}s(a,\mathbb{T}l_1),$$

$$\lambda_1 d_\theta(l_o,l_1)+\lambda_2\frac{d_\theta(l_o,\mathbb{T}l_o)d_\theta(l_1,\mathbb{T}l_1)}{1+d(l_o,l_1)}+\lambda_3\frac{d_\theta(l_1,\mathbb{T}l_o)d_(l_o,\mathbb{T}l_1)}{1+d_\theta(l_0,l_1)}$$
$$+\lambda_4\frac{d_\theta(l_o,\mathbb{T}l_o)d_\theta(l_o,\mathbb{T}l_1)}{1+d_(l_o,l_1)}+\lambda_5\frac{d_\theta(l_1,\mathbb{T}l_o)d_(l_1,\mathbb{T}l_1)}{1+d_(l_o,l_1)}\in s(a,\mathbb{T}l_1)\quad\forall a\in\mathbb{T}l_o.$$

Since $l_1 \in \mathbb{T}l_o$, we get

$$\lambda_1 d_\theta(l_o,l_1) + \lambda_2 \frac{d_\theta(l_o,\mathbb{T}l_o)d_\theta(l_1,\mathbb{T}l_1)}{1+d(l_o,l_1)} + \lambda_3 \frac{d_\theta(l_1,\mathbb{T}l_o)d_(l_o,\mathbb{T}l_1)}{1+d_\theta(l_0,l_1)} + \lambda_4 \frac{d_\theta(l_o,\mathbb{T}l_o)d_\theta(l_o,\mathbb{T}l_1)}{1+d_(l_o,l_1)} + \lambda_5 \frac{d_\theta(l_1,\mathbb{T}l_o)d_(l_1,\mathbb{T}l_1)}{1+d_(l_o,l_1)} \in s(l_1,\mathbb{T}l_1),$$

$$\lambda_1 d_\theta(l_o,l_1) + \lambda_2 \frac{d_\theta(l_o,\mathbb{T}l_o)d_\theta(l_1,\mathbb{T}l_1)}{1+d(l_o,l_1)} + \lambda_3 \frac{d_\theta(l_1,\mathbb{T}l_o)d_(l_o,\mathbb{T}l_1)}{1+d_\theta(l_0,l_1)} + \lambda_4 \frac{d_\theta(l_o,\mathbb{T}l_o)d_\theta(l_o,\mathbb{T}l_1)}{1+d_(l_o,l_1)} + \lambda_5 \frac{d_\theta(l_1,\mathbb{T}l_o)d_(l_1,\mathbb{T}l_1)}{1+d_(l_o,l_1)} \in \bigcup_{b\in\mathbb{T}x_1} s(d_\theta(l_1,b)).$$

Thus, there exists some $l_2 \in \mathbb{T}l_1$ such that

$$\lambda_1 d_\theta(l_o,l_1) + \lambda_2 \frac{d_\theta(l_o,\mathbb{T}l_o)d_\theta(l_1,\mathbb{T}l_1)}{1+d(l_o,l_1)} + \lambda_3 \frac{d_\theta(l_1,\mathbb{T}l_o)d_(l_o,\mathbb{T}l_1)}{1+d_\theta(l_0,l_1)} + \lambda_4 \frac{d_\theta(l_o,\mathbb{T}l_o)d_\theta(l_o,\mathbb{T}l_1)}{1+d_(l_o,l_1)} + \lambda_5 \frac{d_\theta(l_1,\mathbb{T}l_o)d_(l_1,\mathbb{T}l_1)}{1+d_(l_o,l_1)} \in s(d_\theta(l_1,l_2)).$$

By definition 2.5

$$d_\theta(l_1,l_2) \preceq \lambda_1 d_\theta(l_o,l_1) + \lambda_2 \frac{d_\theta(l_o,\mathbb{T}l_o)d_\theta(l_1,\mathbb{T}l_1)}{1+d(l_o,l_1)} + \lambda_3 \frac{d_\theta(l_1,\mathbb{T}l_o)d_(l_o,\mathbb{T}l_1)}{1+d_\theta(l_0,l_1)} + \lambda_4 \frac{d_\theta(l_o,\mathbb{T}l_o)d_\theta(l_o,\mathbb{T}l_1)}{1+d_(l_o,l_1)} + \lambda_5 \frac{d_\theta(l_1,\mathbb{T}l_o)d_(l_1,\mathbb{T}l_1)}{1+d_(l_o,l_1)}.$$

Using the g.l.b property of $\mathbb{T}$, we get

$$d_\theta(l_1,l_2) \preceq \lambda_1 d_\theta(l_o,l_1) + \lambda_2 \frac{d_\theta(l_o,l_1)d_\theta(l_1,l_2)}{1+d(l_o,l_1)} + \lambda_3 \frac{d_\theta(l_1,l_1)d_(l_o,l_2)}{1+d_\theta(l_0,l_1)} + \lambda_4 \frac{d_\theta(l_o,l_1)d_\theta(l_o,l_2)}{1+d_(l_o,l_1)} + \lambda_5 \frac{d_\theta(l_1,l_1)d_(l_1,l_2)}{1+d_(l_o,l_1)}$$

The above inequality gives

$$|d_\theta(l_1,l_2)| \leq \lambda_1|d_\theta(l_o,l_1)| + \lambda_2|d_\theta(l_1,l_2)| + \lambda_4|d_\theta(l_o,l_2)|.$$

By using the triangular inequality, we get the following

$$|d_\theta(l_1,l_2)| \leq \lambda_1|d_\theta(l_o,l_1)| + \lambda_2|d_\theta(l_1,l_2)| + \theta(l_0,l_2)\lambda_4|d_\theta(l_o,l_1)| + \theta(l_0,l_2)\lambda_4|d_\theta(l_1,l_2)|.$$

Thus,

$$|d_\theta(l_1,l_2)| \leq \frac{\lambda_1 + \theta(l_0,l_2)\lambda_4}{1-\lambda_2-\theta(l_0,l_2)\lambda_4}|d_\theta(l_o,l_1)|.$$

Inductively, we develop a sequence $\{l_p\}$ in $\mathbb{G}$ such that

$$|d_\theta(l_1,l_2)| \leq k|d_\theta(l_o,l_1)|$$
$$|d_\theta(l_2,l_3)| \leq k^2|d_\theta(l_o,l_1)|$$

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$$|d_\theta(l_p, l_{p+1})| \leq k^p |d_\theta(l_o, l_1)|.$$

Now, by triangular inequality and for $q > p$, we have the following

$$\begin{aligned} d_\theta(l_p, l_q) &\preceq \theta(l_p, l_q) k^p d_\theta(l_o, l_1) + \theta(l_p, l_q)\theta(l_{p+1}, l_q) k^{p+1} d_\theta(l_o, l_1) + \cdots \\ &\quad + \theta(l_p, l_q)\theta(l_{p+1}, l_q) \cdots \theta(l_{q-2}, l_q)\theta(l_{q-1}, l_q) k^{q-1} d_\theta(l_o, l_1).d_\theta(l_p, l_q) \\ d_\theta(l_p, l_q) &\preceq d_\theta(l_0, l_1)[\theta(l_p, l_q) k^p + \theta(l_p, l_q)\theta(l_{p+1}, l_q) k^{p+1} + \cdots \\ &\quad + \theta(l_p, l_q)\theta(l_{p+1}, l_q) ... \theta(l_{q-2}, l_q)\theta(l_{q-1}, m_p) k^{q-1}]. \end{aligned}$$

Since $\lim_{p,q\to\infty} \theta(l_p, l_q)k < 1$, so the series $\sum_{p=1}^{\infty} k^p \prod_{i=1}^{p} \theta(l_i, l_q)$ converges by ratio test for each $q \in N$. Let

$$S = \sum_{p=1}^{\infty} k^p \prod_{i=1}^{p} \theta(l_i, l_q), \quad S_p = \sum_{j=1}^{p} k^j \prod_{i=1}^{p} \theta(l_i, l_q)$$

.
Thus, for $q > p$, the above can be written as

$$d_\theta(l_p, l_q) \preceq d_\theta(l_o, l_1)[S_{q-1} - S_P]$$

$$|d_\theta(l_p, l_q)| \leq |d_\theta(l_o, l_1)|[S_{q-1} - S_p].$$

Now, by taking $p \to \infty$, we get

$$|d_\theta(l_p, l_q)| \to 0.$$

By Lemma 2.4, we conclude that $\{l_p\}$ is a Cauchy sequence. By the completeness of $\mathbb{G}$, there exists an element r such that $l_p \to r \in \mathbb{G}$ as $p \to \infty$.
We show $r \in \mathbb{T}r$. From (6.3), we have

$$\begin{aligned} &\lambda_1 d_\theta(l_{2p}, r) + \lambda_2 \frac{d_\theta(l_{2p}, \mathbb{T}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d(l_{2p}, r)} + \lambda_3 \frac{d_\theta(r, \mathbb{T}l_{2p}) d_(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \\ &+ \lambda_4 \frac{d_\theta(l_{2p}, \mathbb{T}l_{2p}) d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_(l_{2p}, r)} + \lambda_5 \frac{d_\theta(r, \mathbb{T}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \in s(\mathbb{T}l_{2p}, \mathbb{T}r), \end{aligned}$$

$$\begin{aligned} &\lambda_1 d_\theta(l_{2p}, r) + \lambda_2 \frac{d_\theta(l_{2p}, \mathbb{T}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d(l_{2p}, r)} + \lambda_3 \frac{d_\theta(r, \mathbb{T}l_{2p}) d_(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \\ &+ \lambda_4 \frac{d_\theta(l_{2p}, \mathbb{T}l_{2p}) d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_(l_{2p}, r)} + \lambda_5 \frac{d_\theta(r, \mathbb{T}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \in \bigcap_{a \in \mathbb{T}l_{2p}} s(a, \mathbb{T}r). \end{aligned}$$

Since $l_{2p+1} \in \mathbb{T}l_{2p}$, we get

$$\begin{aligned} &\lambda_1 d_\theta(l_{2p}, r) + \lambda_2 \frac{d_\theta(l_{2p}, \mathbb{T}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d(l_{2p}, r)} + \lambda_3 \frac{d_\theta(r, \mathbb{T}l_{2p}) d_(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \\ &+ \lambda_4 \frac{d_\theta(l_{2p}, \mathbb{T}l_{2p}) d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_(l_{2p}, r)} + \lambda_5 \frac{d_\theta(r, \mathbb{T}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \in s(l_{2p+1}, \mathbb{T}r), \end{aligned}$$

$$\begin{aligned} &\lambda_1 d_\theta(l_{2p}, r) + \lambda_2 \frac{d_\theta(l_{2p}, \mathbb{T}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d(l_{2p}, r)} + \lambda_3 \frac{d_\theta(r, \mathbb{T}l_{2p}) d_(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \\ &+ \lambda_4 \frac{d_\theta(l_{2p}, \mathbb{T}l_{2p}) d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_(l_{2p}, r)} + \lambda_5 \frac{d_\theta(r, \mathbb{T}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \in \bigcup_{b \in \mathbb{T}r} s(d_\theta(l_{2p+1}, b)). \end{aligned}$$

So, there exists some $r_p \in \mathbb{T}r$ such that

$$\lambda_1 d_\theta(l_{2p}, r) + \lambda_2 \frac{d_\theta(l_{2p}, \mathbb{T}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d(l_{2p}, r)} + \lambda_3 \frac{d_\theta(r, \mathbb{T}l_{2p}) d_(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)}$$
$$+ \lambda_4 \frac{d_\theta(l_{2p}, \mathbb{T}l_{2p}) d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_(l_{2p}, r)} + \lambda_5 \frac{d_\theta(r, \mathbb{T}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \in s(d_\theta(l_{2p+1}, r_p)).$$

Therefore, by definition (2.5), we have

$$d_\theta(l_{2p+1}, r_p) \preceq \lambda_1 d_\theta(l_{2p}, r) + \lambda_2 \frac{d_\theta(l_{2p}, \mathbb{T}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d(l_{2p}, r)} + \lambda_3 \frac{d_\theta(r, \mathbb{T}l_{2p}) d_(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)}$$
$$+ \lambda_4 \frac{d_\theta(l_{2p}, \mathbb{T}l_{2p}) d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_(l_{2p}, r)} + \lambda_5 \frac{d_\theta(r, \mathbb{T}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)}.$$

Using the g.l.b property of $\mathbb{T}$, gives

$$d_\theta(l_{2p+1}, r_p) \preceq \lambda_1 d_\theta(l_{2p}, r) + \lambda_2 \frac{d_\theta(l_{2p}, l_{2p+1}) d_\theta(r, r_p)}{1 + d(l_{2p}, r)} + \lambda_3 \frac{d_\theta(r, l_{2p+1}) d_(l_{2p}, r_P)}{1 + d_\theta(l_{2p}, r)}$$
$$+ \lambda_4 \frac{d_\theta(l_{2p}, l_{2p+1}) d_\theta(l_{2p}, r_p)}{1 + d_(l_{2p}, r)} + \lambda_5 \frac{d_\theta(r, l_{2p+1}) d_\theta(r, r_p)}{1 + d_\theta(l_{2p}, r)}.$$

Now, we have

$$d_\theta(r, r_p) \preceq \theta(r, r_p)[d_\theta(r, l_{2p+1}) + d_\theta(l_{2p+1}, r_p)].$$

Thus,

$$d_\theta(r, r_p) \preceq \theta(r, r_p) d_\theta(r, l_{2p+1}) + \theta(r, r_p)\lambda_1 d_\theta(l_{2p}, r) + \theta(r, r_p)\lambda_2 \frac{d_\theta(l_{2p}, l_{2p+1}) d_\theta(r, r_p)}{1 + d(l_{2p}, r)}$$
$$+ \theta(r, r_p)\lambda_3 \frac{d_\theta(r, l_{2p+1}) d_(l_{2p}, r_P)}{1 + d_\theta(l_{2p}, r)} + \theta(r, r_p)\lambda_4 \frac{d_\theta(l_{2p}, l_{2p+1}) d_\theta(l_{2p}, r_p)}{1 + d_(l_{2p}, r)}$$
$$+ \theta(r, r_p)\lambda_5 \frac{d_\theta(r, l_{2p+1}) d_\theta(r, r_p)}{1 + d_\theta(l_{2p}, r)}.$$

Which implies

$$|d_\theta(r, r_p)| \leq \theta(r, r_p)|d_\theta(r, l_{2p+1})| + \theta(r, r_p)\lambda_1 |d_\theta(l_{2p}, r)| + \theta(r, r_p)\lambda_2 \left| \frac{d_\theta(l_{2p}, l_{2p+1}) d_\theta(r, r_p)}{1 + d(l_{2p}, r)} \right|$$
$$+ \theta(r, r_p)\lambda_3 \left| \frac{d_\theta(r, l_{2p+1}) d_(l_{2p}, r_P)}{1 + d_\theta(l_{2p}, r)} \right| + \theta(r, r_p)\lambda_4 \left| \frac{d_\theta(l_{2p}, l_{2p+1}) d_\theta(l_{2p}, r_p)}{1 + d_(l_{2p}, r)} \right|$$
$$+ \theta(r, r_p)\lambda_5 \left| \frac{d_\theta(r, l_{2p+1}) d_\theta(r, r_p)}{1 + d_\theta(l_{2p}, r)} \right|.$$

Taking limit as $p \to \infty$, we get

$$|d_\theta(r, r_p)| \to 0.$$

By Lemma 2.4, $r_p \to r$ as $p \to \infty$, Also, since $\mathbb{T}r$ is closed ,then $r \in \mathbb{T}r$. Thus $\mathbb{T}$ has a fixed point.

Theorem 4.3. *Let $(\mathbb{G}, d_\theta)$ be a complete C.V extended b-metric space, $\theta : \mathbb{G} \times \mathbb{G} \to [1, \infty)$ be a function and let $\mathbb{S}, \mathbb{T} : \mathbb{G} \to CB(\mathbb{G})$ be a pair of multi-valued mappings fulfilling the g.l.b property such that*

$$\begin{aligned} &\lambda_1 d_\theta(l,m) + \lambda_2 \frac{d_\theta(l, \mathbb{S}l) d_\theta(m, \mathbb{T}m)}{1 + d(l,m)} + \lambda_3 \frac{d_\theta(m, \mathbb{S}l) d_(l, \mathbb{T}m)}{1 + d_\theta(l,m)} \\ &+ \lambda_4 \frac{d_\theta(l, \mathbb{S}l) d_\theta(l, \mathbb{T}m)}{1 + d_(l,m)} + \lambda_5 \frac{d_\theta(m, \mathbb{S}l) d_\theta(m, \mathbb{T}m)}{1 + d_\theta(l,m)} \in s(\mathbb{S}l, \mathbb{T}m), \end{aligned} \tag{4.4}$$

for all $l, m \in \mathbb{G}$ and $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$, are nonnegative real numbers with $\lambda_1 + \lambda_2 + \lambda_3$
$+ 2\theta\lambda_4 + \lambda_5 < 1$ and assume $k(1 - \lambda_2 - \theta\lambda_4) = 1 + \theta\lambda_4$
where $k \in [0, 1)$ is such that for each $l_o \in \mathbb{G}$, $\lim_{p,q\to\infty} \theta(l_p, l_q)k < 1$. Then $\mathbb{S}$ and $\mathbb{T}$ have common fixed point.

Proof. Let $l_o \in \mathbb{G}$, then Tl_o is non-empty, so we can take $l_1 \in Tl_o$. Thus, from (4.4),

$$\begin{aligned} &\lambda_1 d_\theta(l_o, l_1) + \lambda_2 \frac{d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1)}{1 + d(l_o, l_1)} + \lambda_3 \frac{d_\theta(l_1, \mathbb{S}l_o) d_(l_o, \mathbb{T}l_1)}{1 + d_\theta(l_0, l_1)} \\ &+ \lambda_4 \frac{d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_o, \mathbb{T}l_1)}{1 + d_(l_o, l_1)} + \lambda_5 \frac{d_\theta(l_1, \mathbb{S}l_o) d_(l_1, \mathbb{T}l_1)}{1 + d_(l_o, l_1)} \in s(\mathbb{S}l_o, \mathbb{T}l_1). \end{aligned}$$

This implies

$$\begin{aligned} &\lambda_1 d_\theta(l_o, l_1) + \lambda_2 \frac{d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1)}{1 + d(l_o, l_1)} + \lambda_3 \frac{d_\theta(l_1, \mathbb{S}l_o) d_(l_o, \mathbb{T}l_1)}{1 + d_\theta(l_0, l_1)} \\ &+ \lambda_4 \frac{d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_o, \mathbb{T}l_1)}{1 + d_(l_o, l_1)} + \lambda_5 \frac{d_\theta(l_1, \mathbb{S}l_o) d_(l_1, \mathbb{T}l_1)}{1 + d_(l_o, l_1)} \in \bigcap_{a \in \mathbb{S}l_o} s(a, \mathbb{T}l_1), \end{aligned}$$

$$\begin{aligned} &\lambda_1 d_\theta(l_o, l_1) + \lambda_2 \frac{d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1)}{1 + d(l_o, l_1)} + \lambda_3 \frac{d_\theta(l_1, \mathbb{S}l_o) d_(l_o, \mathbb{T}l_1)}{1 + d_\theta(l_0, l_1)} \\ &+ \lambda_4 \frac{d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_o, \mathbb{T}l_1)}{1 + d_(l_o, l_1)} + \lambda_5 \frac{d_\theta(l_1, \mathbb{S}l_o) d_(l_1, \mathbb{T}l_1)}{1 + d_(l_o, l_1)} \in s(a, \mathbb{S}l_1), \quad \forall a \in \mathbb{S}l_o. \end{aligned}$$

Since $l_1 \in \mathbb{S}l_o$, we get

$$\begin{aligned} &\lambda_1 d_\theta(l_o, l_1) + \lambda_2 \frac{d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1)}{1 + d(l_o, l_1)} + \lambda_3 \frac{d_\theta(l_1, \mathbb{S}l_o) d_(l_o, \mathbb{T}l_1)}{1 + d_\theta(l_0, l_1)} \\ &+ \lambda_4 \frac{d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_o, \mathbb{T}l_1)}{1 + d_(l_o, l_1)} + \lambda_5 \frac{d_\theta(l_1, \mathbb{S}l_o) d_(l_1, \mathbb{S}l_1)}{1 + d_(l_o, l_1)} \in s(l_1, \mathbb{T}l_1), \end{aligned}$$

$$\begin{aligned} &\lambda_1 d_\theta(l_o, l_1) + \lambda_2 \frac{d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1)}{1 + d(l_o, l_1)} + \lambda_3 \frac{d_\theta(l_1, \mathbb{S}l_o) d_(l_o, \mathbb{T}l_1)}{1 + d_\theta(l_0, l_1)} \\ &+ \lambda_4 \frac{d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_o, \mathbb{T}l_1)}{1 + d_(l_o, l_1)} + \lambda_5 \frac{d_\theta(l_1, \mathbb{S}l_o) d_(l_1, \mathbb{T}l_1)}{1 + d_(l_o, l_1)} \in \bigcup_{b \in \mathbb{T}x_1} s(d_\theta(l_1, b)). \end{aligned}$$

Thus, there exists some $l_2 \in \mathbb{T}l_1$ such that

$$\begin{aligned} &\lambda_1 d_\theta(l_o, l_1) + \lambda_2 \frac{d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_1, \mathbb{T}l_1)}{1 + d(l_o, l_1)} + \lambda_3 \frac{d_\theta(l_1, \mathbb{S}l_o) d_(l_o, \mathbb{T}l_1)}{1 + d_\theta(l_0, l_1)} \\ &+ \lambda_4 \frac{d_\theta(l_o, \mathbb{S}l_o) d_\theta(l_o, \mathbb{T}l_1)}{1 + d_(l_o, l_1)} + \lambda_5 \frac{d_\theta(l_1, \mathbb{S}l_o) d_(l_1, \mathbb{T}l_1)}{1 + d_(l_o, l_1)} \in s(d_\theta(l_1, l_2)). \end{aligned}$$

By definition 2.5, we have

$$d_\theta(l_1,l_2) \preceq \lambda_1 d_\theta(l_o,l_1) + \lambda_2 \frac{d_\theta(l_o,\mathbb{S}l_o)d_\theta(l_1,\mathbb{T}l_1)}{1+d(l_o,l_1)} + \lambda_3 \frac{d_\theta(l_1,\mathbb{S}l_o)d_(l_o,\mathbb{T}l_1)}{1+d_\theta(l_0,l_1)}$$
$$+ \lambda_4 \frac{d_\theta(l_o,\mathbb{S}l_o)d_\theta(l_o,\mathbb{T}l_1)}{1+d_(l_o,l_1)} + \lambda_5 \frac{d_\theta(l_1,\mathbb{S}l_o)d_(l_1,\mathbb{T}l_1)}{1+d_(l_o,l_1)}.$$

Using the g.l.b property of $\mathbb{S}$ and $\mathbb{T}$, gives

$$d_\theta(l_1,l_2) \preceq \lambda_1 d_\theta(l_o,l_1) + \lambda_2 \frac{d_\theta(l_o,l_1)d_\theta(l_1,l_2)}{1+d(l_o,l_1)} + \lambda_3 \frac{d_\theta(l_1,l_1)d_(l_o,l_2)}{1+d_\theta(l_0,l_1)}$$
$$+ \lambda_4 \frac{d_\theta(l_o,l_1)d_\theta(l_o,l_2)}{1+d_(l_o,l_1)} + \lambda_5 \frac{d_\theta(l_1,l_1)d_(l_1,l_2)}{1+d_(l_o,l_1)},$$

from which we have

$$|d_\theta(l_1,l_2)| \leq \lambda_1 |d_\theta(l_o,l_1)| + \lambda_2 |d_\theta(l_1,l_2)| + \lambda_4 |d_\theta(l_o,l_2)|$$

Using the triangular inequality, we get

$$|d_\theta(l_1,l_2)| \leq \lambda_1 |d_\theta(l_o,l_1)| + \lambda_2 |d_\theta(l_1,l_2)|$$
$$+ \theta(l_o,l_2)\lambda_4 |d_\theta(l_o,l_1)| + \theta(l_o,l_2)\lambda_4 |d_\theta(l_1,l_2)|.$$

That is,

$$|d_\theta(l_1,l_2)| \leq \frac{(\lambda_1 + \theta(l_o,l_2)\lambda_4)}{(1-\lambda_2-\theta(l_o,l_2)\lambda_4)} |d_\theta(l_o,l_1)|.$$

Thus,

$$|d_\theta(l_1,l_2)| \leq k|d_\theta(l_o,l_1)|.$$

Inductively, we develop a sequence l_p in $\mathbb{G}$ such that

$$|d_\theta(l_1,l_2)| \leq k|d_\theta(l_o,l_1)|$$
$$|d_\theta(l_2,l_3)| \leq k^2|d_\theta(l_o,l_1)|$$
$$.$$
$$.$$
$$.$$
$$|d_\theta(l_p,l_{p+1})| \leq k^p|d_\theta(l_o,l_1)|.$$

By triangular inequality and for $q > p$, we have

$$d_\theta(l_p,l_q) \preceq \theta(l_p,l_q)k^p d_\theta(l_o,l_1) + \theta(l_p,l_q)\theta(l_{p+1},l_q)k^{p+1}d_\theta(l_o,l_1) + \cdots$$
$$+ \theta(l_p,l_q)\theta(l_{p+1},l_q)\cdots\theta(l_{q-2},l_q)\theta(l_{q-1},l_q)k^{q-1}d_\theta(l_o,l_1).d_\theta(l_p,l_q)$$
$$d_\theta(l_p,l_q) \preceq d_\theta(l_0,l_1)[\theta(l_p,l_q)k^p + \theta(l_p,l_q)\theta(l_{p+1},l_q)k^{p+1} + \cdots$$
$$+ \theta(l_p,l_q)\theta(l_{p+1},l_q)...\theta(l_{q-2},l_q)\theta(l_{q-1},m_p)k^{q-1}]$$

Since $\lim_{p,q\to\infty}\theta(l_p,l_q)k < 1$, so the series $\sum_{p=1}^{\infty} k^p \prod_{i=1}^{p}\theta(l_i,l_q)$ converges by ratio test for each $q \in N$. Take

$$S = \sum_{p=1}^{\infty} k^p \prod_{i=1}^{p}\theta(l_i,l_q), \quad S_p = \sum_{j=1}^{p} k^j \prod_{i=1}^{p}\theta(l_i,l_q)$$

Then, for $q > p$, the above can be written as

$$d_\theta(l_p,l_q) \preceq d_\theta(l_o,l_1)[S_{q-1} - S_P]$$

$$|d_\theta(l_p, l_q)| \leq |d_\theta(l_o, l_1)|[S_{q-1} - S_p].$$

Now, by taking $p \to \infty$ in the above expression, we get

$$|d_\theta(l_p, l_q)| \to 0.$$

By Lemma 2.4, $\{l_p\}$ is a Cauchy sequence in $\mathbb{G}$, which is complete, so there exists some $r \in \mathbb{G}$ such that $limit_{p\to\infty} l_p = r$. We now show that $r \in \mathbb{S}r$ and $r \in \mathbb{T}r$. From (6.4), we have

$$\lambda_1 d_\theta(l_{2p}, r) + \lambda_2 \frac{d_\theta(l_{2p}, \mathbb{S}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d(l_{2p}, r)} + \lambda_3 \frac{d_\theta(r, \mathbb{S}l_{2p}) d_(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)}$$
$$+ \lambda_4 \frac{d_\theta(l_{2p}, \mathbb{S}l_{2p}) d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_(l_{2p}, r)} + \lambda_5 \frac{d_\theta(r, \mathbb{S}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \in s(\mathbb{S}l_{2p}, \mathbb{T}r),$$

$$\lambda_1 d_\theta(l_{2p}, r) + \lambda_2 \frac{d_\theta(l_{2p}, \mathbb{S}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d(l_{2p}, r)} + \lambda_3 \frac{d_\theta(r, \mathbb{S}l_{2p}) d_(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)}$$
$$+ \lambda_4 \frac{d_\theta(l_{2p}, \mathbb{S}l_{2p}) d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_(l_{2p}, r)} + \lambda_5 \frac{d_\theta(r, \mathbb{S}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \in \bigcap_{a \in \mathbb{S}l_{2p}} s(a, \mathbb{T}r).$$

Since $l_{2p+1} \in \mathbb{S}l_{2p}$, we get

$$\lambda_1 d_\theta(l_{2p}, r) + \lambda_2 \frac{d_\theta(l_{2p}, \mathbb{S}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d(l_{2p}, r)} + \lambda_3 \frac{d_\theta(r, \mathbb{S}l_{2p}) d_(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)}$$
$$+ \lambda_4 \frac{d_\theta(l_{2p}, \mathbb{S}l_{2p}) d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_(l_{2p}, r)} + \lambda_5 \frac{d_\theta(r, \mathbb{S}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \in s(l_{2p+1}, \mathbb{T}r),$$

$$\lambda_1 d_\theta(l_{2p}, r) + \lambda_2 \frac{d_\theta(l_{2p}, \mathbb{S}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d(l_{2p}, r)} + \lambda_3 \frac{d_\theta(r, \mathbb{S}l_{2p}) d_(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)}$$
$$+ \lambda_4 \frac{d_\theta(l_{2p}, \mathbb{S}l_{2p}) d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_(l_{2p}, r)} + \lambda_5 \frac{d_\theta(r, \mathbb{S}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \in \bigcup_{b \in \mathbb{T}r} s(d_\theta(l_{2p+1}, b))$$

So, there exists some $r_p \in \mathbb{T}r$ such that

$$\lambda_1 d_\theta(l_{2p}, r) + \lambda_2 \frac{d_\theta(l_{2p}, \mathbb{S}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d(l_{2p}, r)} + \lambda_3 \frac{d_\theta(r, \mathbb{S}l_{2p}) d_(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)}$$
$$+ \lambda_4 \frac{d_\theta(l_{2p}, \mathbb{S}l_{2p}) d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_(l_{2p}, r)} + \lambda_5 \frac{d_\theta(r, \mathbb{S}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)} \in s(d_\theta(l_{2p+1}, r_p)).$$

Therefore, by definition, we have

$$d_\theta(l_{2p+1}, r_p) \preceq \lambda_1 d_\theta(l_{2p}, r) + \lambda_2 \frac{d_\theta(l_{2p}, \mathbb{S}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d(l_{2p}, r)} + \lambda_3 \frac{d_\theta(r, \mathbb{S}l_{2p}) d_(l_{2p}, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)}$$
$$+ \lambda_4 \frac{d_\theta(l_{2p}, \mathbb{S}l_{2p}) d_\theta(l_{2p}, \mathbb{T}r)}{1 + d_(l_{2p}, r)} + \lambda_5 \frac{d_\theta(r, \mathbb{S}l_{2p}) d_\theta(r, \mathbb{T}r)}{1 + d_\theta(l_{2p}, r)}.$$

Using the g.l.b property of $\mathbb{T}$, we obtain

$$d_\theta(l_{2p+1}, r_p) \preceq \lambda_1 d_\theta(l_{2p}, r) + \lambda_2 \frac{d_\theta(l_{2p}, l_{2p+1}) d_\theta(r, r_p)}{1 + d(l_{2p}, r)} + \lambda_3 \frac{d_\theta(r, l_{2p+1}) d_(l_{2p}, r_P)}{1 + d_\theta(l_{2p}, r)}$$

$$+\lambda_4 \frac{d_\theta(l_{2p}, l_{2p+1}) d_\theta(l_{2p}, r_p)}{1 + d_(l_{2p}, r)} + \lambda_5 \frac{d_\theta(r, l_{2p+1}) d_\theta(r, r_p)}{1 + d_\theta(l_{2p}, r)}.$$

By definition 6.4, we get

$$d_\theta(r, r_p) \preceq \theta(r, r_p)[d_\theta(r, l_{2p+1}) + d_\theta(l_{2p+1}, r_p)].$$

Thus,

$$\begin{aligned} d_\theta(l_{2p+1}, r_p) \preceq{} & \theta(r, r_p) d_\theta(r, l_{2p+1}) + \theta(r, r_p)\lambda_1 d_\theta(l_{2p}, r) + \theta(r, r_p)\lambda_2 \frac{d_\theta(l_{2p}, l_{2p+1}) d_\theta(r, r_p)}{1 + d(l_{2p}, r)} \\ & + \theta(r, r_p)\lambda_3 \frac{d_\theta(r, l_{2p+1}) d_(l_{2p}, r_P)}{1 + d_\theta(l_{2p}, r)} + \theta(r, r_p)\lambda_4 \frac{d_\theta(l_{2p}, l_{2p+1}) d_\theta(l_{2p}, r_p)}{1 + d_(l_{2p}, r)} \\ & + \theta(r, r_p)\lambda_5 \frac{d_\theta(r, l_{2p+1}) d_\theta(r, r_p)}{1 + d_\theta(l_{2p}, r)} \end{aligned}$$

$$\begin{aligned} |d_\theta(l_{2p+1}, r_p)| \leq{} & \theta(r, r_p)|d_\theta(r, l_{2p+1})| + \theta(r, r_p)\lambda_1 |d_\theta(l_{2p}, r)| + \theta(r, r_p)\lambda_2 \left| \frac{d_\theta(l_{2p}, l_{2p+1}) d_\theta(r, r_p)}{1 + d(l_{2p}, r)} \right| \\ & + \theta(r, r_p)\lambda_3 \left| \frac{d_\theta(r, l_{2p+1}) d_(l_{2p}, r_P)}{1 + d_\theta(l_{2p}, r)} \right| + \theta(r, r_p)\lambda_4 \left| \frac{d_\theta(l_{2p}, l_{2p+1}) d_\theta(l_{2p}, r_p)}{1 + d_(l_{2p}, r)} \right| \\ & + \theta(r, r_p)\lambda_5 \left| \frac{d_\theta(r, l_{2p+1}) d_\theta(r, r_p)}{1 + d_\theta(l_{2p}, r)} \right|. \end{aligned}$$

Taking limit as $p \to \infty$ in the above inequality, we get

$$|d_\theta(l_{2p+1}, r_p)| \to 0.$$

By Lemma 2.4, $r_p \to r$ as $p \to \infty$. Further, since $\mathbb{T}r$ is closed, then $r \in \mathbb{T}r$. Similarly, $r \in \mathbb{S}r$. Thus, $\mathbb{S}$ and $\mathbb{T}$ have a common fixed point.

Conclusion

In this paper, we studied common fixed point results for almost contraction, Banach and Kannan type contractions in the setting of complex-valued extended b-metric spaces. Starting from the notion of complex-valued metric spaces, our results complement several significant fixed point theorems of b-metric and extended b-metric spaces in the frame of crisp mappings. We hope that our presented idea herein will be a source of motivation for other researchers to extend and improve these results suitable for their applications.

Competing Interests

The authors declare that they have no competing interests.

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