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Fixed point theorem satisfying cyclical conditions in b -Menger spaces

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ABSTRACT. In this work, we prove a fixed point theorem for mapping with cyclical conditions using comparison function in b -Menger spaces. We support our results by an example.

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1. Introduction

The notion of statistical metric spaces, as a generalization of metric spaces, with non-deterministic distance, was defined by Menger [6] in 1942. An extension of Banach's contraction [1] in probabilistic metric spaces was showed by Sehgal and Bharucha-Reid [8] in 1972 and their fixed point theorem is further generalized by many authors, for example see [2,5].

Recently, Mbarki et al. [4] introduced the probabilistic b -metris spaces (b -Menger spaces) as a generalization of probabilistic metric spaces (Menger spaces) and they studied topological structures and properties and showed the fixed point property for nonlinear contractions in these spaces.

In 2003, Kirk et al. [3] gave the cyclic contraction mappings as follows:

Let A and B be nonempty subsets of a metric space (X, d) . A mapping $T : A \cup B \rightarrow A \cup B$ is said a cyclic contraction mapping if we have

$$(1) \quad T(A) \subseteq B \text{ and } T(B) \subseteq A.$$

$$(2) \quad d(Tx, Ty) \leq kd(x, y), \quad \forall x \in A, \forall y \in B, \text{ where } k \in [0, 1).$$

And they proved that T has a unique fixed point in $A \cap B$.

In this paper, we prove the existence and uniqueness of the fixed point for mapping with cyclical conditions using comparison function in a b -Menger spaces and we give an example which support the main result.

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2. Preliminaries

Throughout this work, we adopt the usual terminology, notation and conventions of the theory of probabilistic b -metric spaces, as in [4]

Definition 2.1. A distance distribution function (briefly, a d.d.f.) is a nondecreasing function F defined on $\mathbb{R}^+ \cup \{\infty\}$ that satisfies $F(0) = 0$ and $F(\infty) = 1$, and is left continuous on $(0, \infty)$. The set of all d.d.f.'s will be noted by Δ^+ ; and the set of all F in Δ^+ for which $\lim_{t \rightarrow \infty} F(t) = 1$ by D^+ .

A simple example of distribution function is Heavyside function in D^+

$$H(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases}$$

A triangular norm (briefly, a t -norm) is a binary operation T on $[0, 1]$, which satisfies the following conditions:

- (1) $T(x, y) = T(y, x)$, for all $x, y \in [0, 1]$.
- (2) $T(x, y) \leq T(z, w)$, whenever $x \leq z, y \leq w$, for each $x, y, z, w \in [0, 1]$.
- (3) $T(x, 1) = x$, for all $x \in [0, 1]$.
- (4) $T(T(x, y), z) = T(x, T(y, z))$, for all $x, y, z \in [0, 1]$.

Example 2.1. The following t -norms are continuous:

- (1) The t -norm minimum, $T_M(x, y) = \min(x, y)$.
- (2) The t -norm product, $T_p(x, y) = xy$.
- (3) The t -norm T_L , $T_L(x, y) = \max(x + y - 1, 0)$.

Definition 2.2. [4] A b -Menger space is a quadruple (M, F, T, s) where M is a nonempty set, F is a function from $M \times M$ into Δ^+ , T is a continuous t -norm, $s \geq 1$ is a real number, and the following conditions are satisfied: for all $p, r, q \in M$ and $x, y > 0$,

$$(1) F_{pp} = H \quad (H \text{ is the Heavyside function}),$$

$$(2) F_{pr} = H \Rightarrow p = r,$$

$$(3) F_{pr} = F_{rp},$$

$$(4) F_{pr}(s(x + y)) \geq T(F_{pq}(x), F_{qr}(y)).$$

It should be noted that a Menger space is a b -Menger with $s = 1$.

Example 2.2. [4] Let $M = \{1, 2, 3, 4\}$. Define $F : M \times M \rightarrow \Delta^+$ as follow:

$$F_{pq}(t) = \begin{cases} H(t) & \text{if } p = q, \\ H(t - 3) & \text{if } [p = 2 \text{ and } q = 3] \text{ or } [p = 3 \text{ and } q = 2], \\ H(t - 1) & \text{otherwise.} \end{cases}$$

It is easy to check that $(M, F, T_M, 2)$ is a b -Menger space, and since:

$$F_{32}(\frac{5}{2}) = 0 < 1 = H(\frac{1}{4}) = \min(F_{31}(\frac{5}{4}), F_{12}(\frac{5}{4}))$$

So (M, F, T_M) is not a standard Menger space.

Example 2.3. [4] Let $M = [0, \infty)$. Define $F : M \times M \rightarrow \Delta^+$ as follow:

$$F_{pq}(t) = H(t - |p - q|^2)$$

It easy to check that $(M, F, T_M, 2)$ is a b -Menger space, but (M, F, T_M) is not a standard Menger space since we have:

$$F_{32}(\frac{2}{3}) = 0 < 1 = H(\frac{1}{12}) = \min(F_{3\frac{5}{2}}(\frac{1}{3}), F_{\frac{5}{2}2}(\frac{1}{3}))$$

Definition 2.3. Let $\{x_n\}$ be a sequence in a b -Menger space (M, F, T, s) and $x \in M$.

(1) A sequence $\{x_n\}$ is convergent to x , if

$$\lim_{n \rightarrow \infty} F_{x_n x}(t) = 1 \text{ for all } t > 0.$$

(2) A sequence $\{x_n\}$ is a G -Cauchy sequence if

$$\lim_{n \rightarrow \infty} F_{x_n + r x_n}(t) = 1 \text{ for any } r > 0 \text{ and } t > 0.$$

(3) The b -Menger space (M, F, T, s) is said to be G -complete if every G -Cauchy sequence is convergent.

Definition 2.4. Let (M, F) be a probabilistic semimetric space (i.e., (i), (ii) and (iii) of Definition 2.2 are satisfied). For p in M and $t > 0$, the strong t -neighborhood of p is the set

$$N_p(t) = \{q \in M : F_{pq}(t) > 1 - t\}.$$

The strong neighborhood system at p is the collection

$$\wp_p = \{N_p(t) : t > 0\},$$

and the strong neighborhood system for M is the union

$$\wp = \bigcup_{p \in M} \wp_p.$$

Remark 2.1. Let (M, F) be a probabilistic semimetric space.

(1) $\forall p \in M, N_p(t_1) \subset N_p(t_2)$ whenever $t_1 \leq t_2$.

(2) Since $\wp_p$ and the collection of neighborhoods $\{N_p(\frac{1}{n}) : n = 1, 2, \dots\}$ are equivalent at every point in M , $\wp$ is first-countable.

Mbarki et al.[4] proved that if (M, F, T, s) is a b -Menger space with T is continuous, then the family $\wp$ consisting of $\wp$ and all unions of elements of this strong neighborhood system for M determines a topology for M . Moreover, we have the following assertions

(1) (M, F, T, s) endowed with the topology $\wp$ is a Hausdorff space provided that T is continuous.

(2) The function F is in general not continuous.

We confirm the last assertion by the following example

Example 2.4. [4] Let $M = \mathbb{N} \cup \{\infty\}$, $0 < a \leq 1$. Define $F^a : M \times M \rightarrow \Delta^+$ as follow:

$$F_{pq}^a(t) = \begin{cases} H(t) & \text{if } p = q, \\ H(t-7) & \text{if } p \text{ and } q \text{ are odd and } p \neq q, \\ H(t - |\frac{a}{p} - \frac{a}{q}|) & \text{if } p \text{ and } q \text{ are even or } pq = \infty, \\ H(t-3) & \text{otherwise.} \end{cases}$$

It easy to show that $(M, F^a, T_M, 4)$ is a b -Menger space with T_M is continuous. In the sequel, we take $a = 1$. Consider the sequence $x_n = 2n$, $n \in \mathbb{N}$. Then $F_{2n\infty}(t) = H(t - \frac{1}{2n})$. Therefore $x_n \rightarrow \infty$, but $F_{2n1}(t) = H(t-3) \neq H(t-1) = F_{1\infty}(t)$. Hence F is not continuous at ∞ .

Let f be a self map on M . Power of f at $p \in M$ are defined by $f^0 p = p$ and $f^{n+1} p = f(f^n p)$, $n \geq 0$. We will use the notation $p_n = f^n p$, in particular $p_0 = p$, $p_1 = fp$.

Let T be a t -norm, define the operator T^n by

$T^0(x_1) = x_1$ and $T^n(x_1, x_2, \dots, x_{n+1}) = T(T^{n-1}(x_1, x_2, \dots, x_n), x_{n+1})$ for all $n \geq 1$ and $x_i \in [0, 1]$ for $i = 1, 2, \dots, n+1$.

3. Main result

Definition 3.1. [9] A function $\psi : [0, 1] \rightarrow [0, 1]$ is called a comparison function if it satisfies

- (1) ψ is nondecreasing and left continuous.
- (2) $\psi(t) > t$ for all $t \in (0, 1)$.

Lemma 3.1. [9] Let ψ be a comparison function, then

- (1) $\psi(1) = 1$.
- (2) $\lim_{n \rightarrow \infty} \psi^n(t) = 1$ for all $t \in (0, 1)$, where $\psi^n(t)$ denotes the composition of $\psi(t)$ with itself n times.

Now, we can state and prove the main fixed point of this paper.

Theorem 3.1. Let (M, F, T, s) be a G -complete b -Menger space under a continuous t -norm T such that $F_{pq}(t) > 0$ for each $p, q \in M$ and $t > 0$. Let A and B be nonempty closed subsets of M and let $f : A \cup B \rightarrow A \cup B$ be a mapping and satisfies

- (1) $f(A) \subset B$ and $f(B) \subset A$.
- (2) There exists a comparison function $\psi : [0, 1] \rightarrow [0, 1]$ such that

$$F_{f p f q}(t) \geq \psi(F_{pq}(t)),$$

$$\forall p \in A, \forall q \in B \text{ and } t > 0.$$

Then f has a unique fixed point in $A \cap B$.

Proof. Let $p \in A \cup B$ and $t > 0$, we have

$$\begin{aligned} F_{p_n p_{n+1}}(t) &= F_{f p_{n-1} f p_n}(t) \\ &\geq \psi(F_{p_{n-1} p_n}(t)) \\ &\vdots \\ &\geq \psi^n(F_{p_0 p_1}(t)). \end{aligned}$$

Thus, for any $r > 0$, we have

$$\begin{aligned} F_{p_n p_{n+r}}(t) &\geq T(F_{p_n p_{n+1}}(\frac{t}{2s}), F_{p_{n+1} p_{n+r}}(\frac{t}{2s})) \\ &\geq T(\psi^n(F_{p_0 p_1}(\frac{t}{2s})), T(F_{p_{n+1} p_{n+2}}(\frac{t}{(2s)^2}), F_{p_{n+2} p_{n+r}}(\frac{t}{(2s)^2}))) \\ &\geq T(T(\psi^n(F_{p_0 p_1}(\frac{t}{2s})), \psi^{n+1}(F_{p_0 p_1}(\frac{t}{(2s)^2}))), F_{p_{n+2} p_{n+r}}(\frac{t}{(2s)^2})) \\ &\vdots \\ &\geq T^{r-1}(\psi^n(F_{p_0 p_1}(\frac{t}{2s})), \psi^{n+1}(F_{p_0 p_1}(\frac{t}{(2s)^2})), \psi^{n+2}(F_{p_0 p_1}(\frac{t}{(2s)^3})), \dots, \psi^{n+r-1}(F_{p_0 p_1}(\frac{t}{(2s)^r}))). \end{aligned}$$

Since $F_{p_0 p_1}(t) > 0$ for all $t > 0$, by Lemma 3.1, for every $i \in \{0, 1, 2, \dots, r-1\}$, we obtain that

$$\lim_{n \rightarrow \infty} \psi^{n+i}(t) = 1.$$

By continuity of t -norm T , we get

$$\lim_{n \rightarrow \infty} F_{p_{n+r} p_n}(t) = 1.$$

It shows that $\{p_n\}$ is a G -Cauchy sequence in the G -complete subset $A \cup B$. So there exists $q \in M$ such that

$$\lim_{n \rightarrow \infty} p_n = q.$$

The subsequence $\{p_{2n}\}$ is in either the set A or in the set B and a subsequence $\{p_{2n+1}\}$ is in other set, and A and B are closed sets then $q \in A \cap B$. For any $t > 0$, we have

$$F_{f q q}(t) \geq T(F_{f q f p_n}(\frac{t}{2s}), F_{f p_n q}(\frac{t}{2s}))$$

$$\geq T(\psi(F_{qp_n}(\frac{t}{2s})), F_{p_{n+1}q}(\frac{t}{2s})) \rightarrow T(1, 1) = 1 \text{ as } n \rightarrow \infty.$$

Then

$$fq = q.$$

To show uniqueness, we assume that $u \in A \cap B$ is another fixed point of f with $u \neq q$, we have

$$\begin{aligned} F_{qu}(t) &= F_{fqfu}(t) \\ &\geq \psi(F_{qu}(t)) \\ &> F_{qu}(t), \end{aligned}$$

which is a contradiction. Thus q is the unique fixed point for f .

The following example support our result.

Example 3.1. Let M be the subset of $\mathbb{R}^2$ defined by

$$M = \{O, P, Q, R, S\}$$

where $O = (0, 0)$, $P = (0, 1)$, $Q = (1, 1)$, $R = (0, 2)$, $S = (3, 3)$.

Define

$$F_{pq}(t) = \exp(-\frac{d(p, q)}{t})$$

where

$$d(p, q) = (p_1 - q_1)^2 + (p_2 - q_2)^2$$

for all $p = (p_1, p_2) \in \mathbb{R}^2$ and $q = (q_1, q_2) \in \mathbb{R}^2$.

We have

$$d(p, q) \leq 2(d(p, r) + d(r, q)) \quad \text{for all } p, q, r \in \mathbb{R}^2.$$

Then

$$d(p, q) \leq 2(\frac{t_1 + t_2}{t_1}d(p, r) + \frac{t_1 + t_2}{t_2}d(r, q)) \quad \text{for all } p, q, r \in \mathbb{R}^2 \text{ and } t_1 > 0, t_2 > 0.$$

Then

$$\frac{-d(p, q)}{t_1 + t_2} \geq \frac{-2}{t_1}d(p, r) - \frac{2}{t_2}d(r, q) \quad \text{for all } p, q, r \in \mathbb{R}^2 \text{ and } t_1 > 0, t_2 > 0.$$

So

$$\exp(\frac{-d(p, q)}{2(t_1 + t_2)}) \geq \exp(\frac{-d(p, r)}{t_1}) \cdot \exp(\frac{-d(r, q)}{t_2}) \quad \text{for all } p, q, r \in \mathbb{R}^2 \text{ and } t_1 > 0, t_2 > 0.$$

So

$$F_{pq}(2(t_1 + t_2)) \geq T_p(F_{pr}(t_1), F_{rq}(t_2)) \quad \text{for all } p, q, r \in \mathbb{R}^2 \text{ and } t_1 > 0, t_2 > 0$$

where T_p is the t -norm product. Therefore, $(M, F, T_p, 2)$ is a b -Menger space.

Let $f : M \rightarrow M$ be given by

$$f(O) = f(P) = f(Q) = f(R) = P, \quad f(S) = O.$$

Set $A = \{O, P, Q, R\}$ and $B = \{P, R, S\}$. $f(A) \subset B$ and $f(B) \subset A$.

In other hand, it can easily be verified that

$$F_{fpfq}(t) \geq \psi(F_{pq}(t)) \quad \text{for every } p \in A \text{ and } q \in B \text{ and } t > 0$$

where $\psi(t) = \sqrt{t}$. In addition, $(M, F, T_p, 2)$ is a G -complete b -Menger space and $F_{pq}(t) > 0$ for all $p, q \in M$ and $t > 0$, then all the conditions of Theorem 3.1 are satisfied and then f has a unique fixed point, that is $P \in A \cap B$.

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