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Two-Point Quadrature Rules for Riemann–Stieltjes Integrals with L^p –error estimates

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ABSTRACT. In this work, we construct a new general two-point quadrature rules for the Riemann–Stieltjes integral $\int_a^b f(t) du(t)$, where the integrand f is assumed to be satisfied with the Hölder condition on $[a, b]$ and the integrator u is of bounded variation on $[a, b]$. The dual formulas under the same assumption are proved. Some sharp error L^p –Error estimates for the proposed quadrature rules are also obtained. **2010 Mathematics Subject Classification.** Primary: 41A55, 65D30, 65D32.

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1. Introduction

The number of proposed quadrature rules that provides approximation for the Riemann–Stieltjes integral ($\mathcal{RS}$ –integral) $\int_a^b f(t) du(t)$ using derivatives or without using derivatives are very rare in comparison with the large number of methods available to approximate the classical Riemann integral $\int_a^b f(t) dt$.

The problem of introducing quadrature rules for $\mathcal{RS}$ –integral $\int_a^b f dg$ was studied via theory of inequalities by many authors. Two famous real inequalities were used in this approach, which are the well known Ostrowski and Hermite–Hadamard inequalities and their modifications. For this purpose and in order to approximate the $\mathcal{RS}$ –integral $\int_a^b f(t) du(t)$, a generalization of closed Newton–Cotes quadrature rules of $\mathcal{RS}$ –integrals without using derivatives provides a simple and robust solution to a significant problem in the evaluation of certain applied probability models was presented by Tortorella in [32].

In 2000, Dragomir [16] introduced the Ostrowski’s approximation formula (which is of One-point type formula) as follows:

$$\int_a^b f(t) du(t) \cong f(x) [u(b) - u(a)] \quad \forall x \in [a, b].$$

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Several error estimations for this approximation had been done in the works [15] and [16].

From different point of view, the authors of [17] (see also [11, 12]) considered the problem of approximating the Stieltjes integral $\int_a^b f(t) du(t)$ via the generalized trapezoid formula:

$$\int_a^b f(t) du(t) \cong [u(x) - u(a)] f(a) + [u(b) - u(x)] f(b).$$

Many authors have studied this quadrature rule under various assumptions of integrands and integrators. For full history of these two quadratures see [6] and the references therein.

Another trapezoid type formula was considered in [20], which reads:

$$\int_a^b f(t) du(t) \cong \frac{f(a) + f(b)}{2} [u(b) - u(a)] \quad \forall x \in [a, b].$$

Some related results had been presented by the same author in [18] and [19]. For other connected results see [13] and [14].

In 2008, Mercer [27] introduced the following trapezoid type formula for the $\mathcal{RS}$ -integral

$$\int_a^b f dg \cong [G - g(a)] f(a) + [g(b) - G] f(b), \quad (1.1)$$

where $G = \frac{1}{b-a} \int_a^b g(t) dt$.

Recently, Alomari and Dragomir [4], proved several new error bounds for the Mercer-Trapezoid quadrature rule (1.1) for the $\mathcal{RS}$ -integral under various assumptions involved the integrand f and the integrator g .

Follows Mercer approach in [27], Alomari and Dragomir [10] introduced the following three-point quadrature formula:

$$\begin{aligned} \int_a^b f(t) dg(t) &\cong [G(a, x) - g(a)] f(a) + [G(x, b) - G(a, x)] f(x) \\ &\quad + [g(b) - G(x, b)] f(b) \end{aligned} \quad (1.2)$$

for all $a < x < b$, where $G(\alpha, \beta) := \frac{1}{\beta - \alpha} \int_\alpha^\beta g(t) dt$.

Several error estimations of Mercer's type quadrature rules for $\mathcal{RS}$ -integral under various assumptions about the function involved have been considered in [4] and [7].

Motivated by Guessab-Schmeisser inequality (see [22]) which is of Ostrowski's type, Alomari in [5] and [9] presented the following approximation formula for $\mathcal{RS}$ -integrals:

$$\int_a^b f(t) du(t) \cong \left[u\left(\frac{a+b}{2}\right) - u(a) \right] f(x) + \left[u(b) - u\left(\frac{a+b}{2}\right) \right] f(a+b-x), \quad (1.3)$$

for all $x \in \left[a, \frac{a+b}{2}\right]$. For other related results see [6]. For different approaches variant quadrature formulae the reader may refer to [1], [8], [21] and [28].

Among others the L^∞ -norm gives the highest possible degree of precision; so that it is recommended to be 'almost' the norm of choice. However, in some cases we cannot access the L^∞ -norm, so that L^p -norm ($1 \leq p < \infty$) is considered to be a variant norm in error estimations.

In this work, several L^p -error estimates ($1 \leq p < \infty$) of general two and three points quadrature rules for Riemann-Stieltjes integrals are presented. The presented proofs depend on new triangle type inequalities for $\mathcal{RS}$ -integrals.

Let f be defined on $[a, b]$. If $P := \{x_0, x_1, \dots, x_n\}$ is a partition of $[a, b]$, write

$$\Delta f_i = f(x_i) - f(x_{i-1}),$$

for $i = 1, 2, \dots, n$. A function f is said to be of bounded p -variation if there exists a positive number M such that

$$\left(\sum_{i=1}^n |\Delta f_i|^p \right)^{\frac{1}{p}} \leq M, \quad (1 \leq p < \infty) \text{ for all partition of } [a, b], \text{ (see [26]).}$$

Let f be of bounded p -variation on $[a, b]$, and let $\sum(P)$ denote the sum $\left(\sum_{i=1}^n |\Delta f_i|^p\right)^{\frac{1}{p}}$ corresponding to the partition P of $[a, b]$. The number

$$\bigvee_a^b(f; p) = \sup \{ \sum(P) : P \in \mathcal{P}([a, b]) \}, \quad 1 \leq p < \infty$$

is called the total p -variation of f on the interval $[a, b]$, where $\mathcal{P}([a, b])$ denotes the set of all partitions of $[a, b]$. For $p = 1$ it is the usual variation of $f(x)$ that was introduced by Jordan (see [24], [25]). For very constructive systematic study of Jordan variation we recommend the interested reader to refer to [29].

In special case, we define the variation of order ∞ of f along $[a, b]$ in the classical sense, i.e., if there exists a positive number M such that

$$\sum_{i=1}^n \text{Osc} \left(f; [x_{i-1}^{(n)}, x_i^{(n)}] \right) = \sum_{i=1}^n (\sup - \inf) f(t_i) \leq M, \quad t_i \in [x_{i-1}^{(n)}, x_i^{(n)}],$$

for all partition of $[a, b]$, then f is said to be of bounded ∞ -variation on $[a, b]$. The number

$$\bigvee_a^b(f; \infty) = \sup \{ \sum(P) : P \in \mathcal{P}[a, b] \} := \text{Osc}(f; [a, b]),$$

is called the oscillation of f on $[a, b]$. Equivalently, we may define the oscillation of f as, (see [23]):

$$\begin{aligned} \bigvee_a^b(f; \infty) &= \lim_{p \rightarrow \infty} \bigvee_a^b(f; p) = \sup_{x \in [a, b]} \{f(x)\} - \inf_{x \in [a, b]} \{f(x)\} \\ &= \text{Osc}(f; [a, b]). \end{aligned}$$

Let $\mathcal{W}_p$ denotes the class of all functions of bounded p -variation ($1 \leq p \leq \infty$). For an arbitrary $p \geq 1$ the class $\mathcal{W}_p$ was firstly introduced by Wiener in [30], where he had shown that $\mathcal{W}_p$ can only have discontinuities of the first kind. More generally, if f is a real function of bounded p -variation on an interval $[a, b]$, then:

- f is bounded, and

$$\text{Osc}(f; [a, b]) \leq \bigvee_a^b(f; p) \leq \bigvee_a^b(f; 1).$$

This fact follows by Jensen's inequality applied for $h(p) = \bigvee_a^b(f; p)$ which is log-convex and decreasing for all $p > 1$. Moreover, the inclusions

$$\mathcal{W}_\infty(f) \subset \mathcal{W}_q(f) \subset \mathcal{W}_p(f) \subset \mathcal{W}_1(f)$$

are valid for all $1 < p < q < \infty$, (see [31]).

- f is continuous except at most on a countable set.
- f has one-sided limits everywhere (limits from the left everywhere in $(a, b]$, and from the right everywhere in $[a, b)$;
- The derivative $f'(x)$ exists almost everywhere (i.e. except for a set of measure zero).
- If $f(x)$ is differentiable on $[a, b]$, then

$$\bigvee_a^b(f; p) = \left(\int_a^b |f'(t)|^p dt \right)^{\frac{1}{p}} = \|f'\|_p, \quad 1 \leq p < \infty.$$

Lemma 1.1. [2] Fix $1 \leq p < \infty$. Let $f, g : [a, b] \rightarrow \mathbb{R}$ be such that f is continuous on $[a, b]$ and g is of bounded p -variation on $[a, b]$. Then the Riemann–Stieltjes integral $\int_a^b f(t) dg(t)$ exists and the inequality:

$$\left| \int_a^b w(t) dv(t) \right| \leq \|w\|_\infty \cdot \text{Osc}(v; [a, b]) \leq \|w\|_\infty \cdot \bigvee_a^b(v; p), \quad (1.4)$$

holds. The constant '1' in the both inequalities is the best possible.

Lemma 1.2. [2] Let $1 \leq p < \infty$. Let $w, v : [a, b] \rightarrow \mathbb{R}$ be such that $w \in L^p[a, b]$ and v has a Lipschitz property on $[a, b]$. Then the inequality

$$\left| \int_a^b w(t) dv(t) \right| \leq L(b-a)^{1-\frac{1}{p}} \cdot \|w\|_p, \quad (1.5)$$

holds and the constant '1' in the right hand side is the best possible, where

$$\|w\|_p = \left(\int_a^b |w(t)|^p dt \right)^{1/p}, \quad (1 \leq p < \infty).$$

In this paper, we establish two-point of Ostrowski's integral inequality for the Riemann-Stieltjes integral $\int_a^b f(t) du(t)$, where f is assumed to be of r -H-Hölder type on $[a, b]$ and u is of bounded variation on $[a, b]$, are given. The dual formulas under the same assumption are proved. Some sharp error L^p -Error estimates for the proposed quadrature rules are also obtained.

2. The Results

Consider the quadrature rule

$$\int_a^b f(s) du(s) = \mathcal{Q}^{[a,b]}(f, u; t_0, x, t_1) + \mathcal{R}^{[a,b]}(f, u; t_0, x, t_1) \quad (2.1)$$

where $\mathcal{Q}^{[a,b]}(f, u; t_0, x, t_1)$ is the quadrature formula

$$\mathcal{Q}^{[a,b]}(f, u; t_0, x, t_1) = [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) \quad (2.2)$$

for all $a \leq t_0 \leq x \leq t_1 \leq b$.

Hence, the remainder term $\mathcal{R}^{[a,b]}(f, u; t_0, x, t_1)$ is given by

$$\mathcal{R}^{[a,b]}(f, u; t_0, x, t_1) := \int_a^b f(s) du(s) - [u(x) - u(a)] f(t_0) - [u(b) - u(x)] f(t_1) \quad (2.3)$$

The following Two-point Ostrowski's inequality for Riemann-Stieltjes integral holds.

Theorem 2.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be Hölder continuous of order r , ($0 < r \leq 1$), and $u : [a, b] \rightarrow \mathbb{R}$ is a mapping of bounded p -variation ($1 \leq p \leq \infty$) on $[a, b]$. Then we have the inequality

$$\begin{aligned} & \left| \mathcal{R}^{[a,b]}(f, u; t_0, x, t_1) \right| \\ & \leq H \max \left\{ \left[\frac{x-a}{2} + \left| t_0 - \frac{a+x}{2} \right| \right], \left[\frac{b-x}{2} + \left| t_1 - \frac{x+b}{2} \right| \right] \right\}^r \cdot \bigvee_a^b(u; p) \end{aligned} \quad (2.4)$$

for all $a \leq t_0 \leq x \leq t_1 \leq b$. Furthermore, the first half of each max-term is the best possible in the sense that it cannot be replaced by a smaller one, for all $r \in (0, 1]$.

Proof. Using the integration by parts formula for Riemann-Stieltjes integral, we have

$$\begin{aligned} & \int_a^x [f(t_0) - f(s)] du(s) + \int_x^b [f(t_1) - f(s)] du(s) \\ & = \int_a^x f(t_0) du(s) + \int_x^b f(t_1) du(s) - \int_a^b f(s) du(s) \\ & = [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \\ & = -\mathcal{R}^{[a,b]}(f, u; t_0, x, t_1), \end{aligned}$$

It is well known that if $p : [c, d] \rightarrow \mathbb{R}$ is continuous and $v : [c, d] \rightarrow \mathbb{R}$ is of p -bounded variation ($1 \leq p < \infty$), then the Riemann-Stieltjes integral $\int_c^d p(t)dv(t)$ exists and the following inequality holds:

$$\left| \int_c^d p(t) dv(t) \right| \leq \sup_{t \in [c, d]} |p(t)| \bigvee_c^d(v). \quad (2.5)$$

Applying the inequality (2.5) for $v(t) = u(t)$, $p(t) = f(t_0) - f(s)$, for all $s \in [a, x]$; and then for $p(t) = f(t_1) - f(s)$, $v(t) = u(t)$ for all $t \in (x, b]$, we get

$$\begin{aligned} & \left| [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \right| \\ &= \left| \int_a^x [f(t_0) - f(s)] du(s) + \int_x^b [f(t_1) - f(s)] du(s) \right| \\ &\leq \left| \int_a^x [f(t_0) - f(s)] du(s) \right| + \left| \int_x^b [f(t_1) - f(s)] du(s) \right| \\ &\leq \sup_{s \in [a, x]} |f(t_0) - f(s)| \cdot \bigvee_a^x(u; p) + \sup_{s \in [x, b]} |f(t_1) - f(s)| \cdot \bigvee_x^b(u; p). \end{aligned} \quad (2.6)$$

As f is of r -H-Hölder type, we have

$$\begin{aligned} \sup_{s \in [a, x]} |f(t_0) - f(s)| &\leq \sup_{s \in [a, x]} [H |t_0 - s|^r] \\ &= H \max \{(x - t_0)^r, (t_0 - a)^r\} \\ &= H [\max \{(x - t_0), (t_0 - a)\}]^r \\ &= H \left[\frac{x - a}{2} + \left| t_0 - \frac{a + x}{2} \right| \right]^r, \end{aligned}$$

and

$$\begin{aligned} \sup_{s \in [x, b]} |f(t_1) - f(s)| &\leq \sup_{s \in [x, b]} [H |t_1 - s|^r] \\ &= H \max \{(t_1 - x)^r, (b - t_1)^r\} \\ &= H [\max \{(t_1 - x), (b - t_1)\}]^r \\ &= H \left[\frac{b - x}{2} + \left| t_1 - \frac{x + b}{2} \right| \right]^r. \end{aligned}$$

Therefore, by (2.6), we have

$$\begin{aligned} & \left| [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \right| \\ &\leq H \left[\frac{x - a}{2} + \left| t_0 - \frac{a + x}{2} \right| \right]^r \cdot \bigvee_a^x(u; p) + H \left[\frac{b - x}{2} + \left| t_1 - \frac{x + b}{2} \right| \right]^r \cdot \bigvee_x^b(u; p) \\ &\leq H \max \left\{ \left[\frac{x - a}{2} + \left| t_0 - \frac{a + x}{2} \right| \right]^r, \left[\frac{b - x}{2} + \left| t_1 - \frac{x + b}{2} \right| \right]^r \right\} \cdot \bigvee_a^b(u; p) \\ &= H \max \left\{ \left[\frac{x - a}{2} + \left| t_0 - \frac{a + x}{2} \right| \right]^r, \left[\frac{b - x}{2} + \left| t_1 - \frac{x + b}{2} \right| \right]^r \right\} \cdot \bigvee_a^b(u; p) \end{aligned}$$

To prove the sharpness of the constant $\frac{1}{2^r}$ for any $r \in (0, 1]$, assume that (2.4) holds with a constant $C > 0$, that is,

$$\begin{aligned} & \left| [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \right| \\ & \leq H \max \left\{ \left[C(x-a) + \left| t_0 - \frac{a+x}{2} \right| \right], \left[C(b-x) + \left| t_1 - \frac{x+b}{2} \right| \right] \right\}^r \cdot \bigvee_a^b(u; p). \end{aligned} \quad (2.7)$$

Choose $f(t) = t^r$, $r \in (0, 1]$, $t \in [0, 1]$ and $u : [0, 1] \rightarrow [0, \infty)$ given by

$$u(t) = \begin{cases} 0, & t \in (0, 1] \\ -1, & t = 0 \end{cases}$$

As

$$|f(x) - f(y)| = |x^r - y^r| \leq |x - y|^r, \quad \forall x \in [0, 1], \quad r \in (0, 1],$$

it follows that f is r -H-Hölder type with the constant $H = 1$.

By using the integration by parts formula for Riemann-Stieltjes integrals, we have:

$$\int_0^1 f(t) du(t) = f(1)u(1) - f(0)u(0) - \int_0^1 u(t) df(t) = 0,$$

and $\bigvee_0^1(u; p) = 1$. Consequently, by (2.7), we get

$$|t_0^r| \leq \max \left\{ \left[Cx + \left| t_0 - \frac{x}{2} \right| \right], \left[C(1-x) + \left| t_1 - \frac{x+1}{2} \right| \right] \right\}^r, \quad \forall t_0 \in [0, 1].$$

For $t_0 = \frac{x}{2}$ and $t_1 = x = 1$ we get $\frac{1}{2^r} \leq C^r$, which implies that $C \geq \frac{1}{2}$.

It remains to prove the second part, so we consider

$$u(t) = \begin{cases} 0, & t \in [0, 1) \\ 1, & t = 1 \end{cases}$$

therefore as we have obtained previously

$$\int_0^1 f(t) du(t) = 0 \quad \text{and} \quad \bigvee_0^1(u; p) = 1.$$

Consequently, by (2.4), we get

$$|t_1^r| \leq \max \left\{ \left[Cx + \left| t_0 - \frac{x}{2} \right| \right], \left[C(1-x) + \left| t_1 - \frac{x+1}{2} \right| \right] \right\}^r, \quad \forall t_0 \in [0, 1].$$

For $t_0 = x = 0$ and $t_1 = \frac{1}{2}$ we get $\frac{1}{2^r} \leq C^r$, which implies that $C \geq \frac{1}{2}$, and the theorem is completely proved.

The following inequalities are hold:

Corollary 2.1. *Let f and u as in Theorem 2.1. In 2.4 choose*

(1) $t_0 = a$ and $t_1 = b$, then we get the following trapezoid type inequality

$$\left| \mathcal{R}^{[a,b]}(f, u; a, x, b) \right| \leq H \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^r \cdot \bigvee_a^b(u; p).$$

or equivalently, we may write using parts formula for Riemann-Stieltjes integral

$$\left| [f(b) - f(a)] u(x) - \int_a^b u(s) df(s) \right| \leq H \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^r \cdot \bigvee_a^b(u; p).$$

The constant $\frac{1}{2}$ is the best possible for all $r \in (0, 1]$.

(2) $x = \frac{a+b}{2}$, then we get the following mid-point type inequality

$$\left| \mathcal{R}^{[a,b]} \left(f, u; t_0, \frac{a+b}{2}, t_1 \right) \right| \leq H \max \left\{ \left[\frac{b-a}{4} + \left| t_0 - \frac{3a+b}{4} \right| \right], \left[\frac{b-a}{4} + \left| t_1 - \frac{a+3b}{4} \right| \right] \right\}^r \cdot \bigvee_a^b(u; p).$$

The constant $\frac{1}{4}$ is the best possible for all $r \in (0, 1]$. For instance, setting $t_0 = y$ and $t_1 = a + b - y$, we get

$$\left| \mathcal{R}^{[a,b]} \left(f, u; y, \frac{a+b}{2}, a+b-y \right) \right| \leq H \left[\frac{b-a}{4} + \left| y - \frac{3a+b}{4} \right| \right]^r \cdot \bigvee_a^b(u; p).$$

for all $y \in \left[a, \frac{a+b}{2} \right]$.

(3) $t_0 = \frac{a+x}{2}$ and $t_1 = \frac{x+b}{2}$, then

$$\left| \mathcal{R}^{[a,b]} \left(f, u; \frac{a+x}{2}, x, \frac{x+b}{2} \right) \right| \leq \frac{H}{2^r} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right]^r \cdot \bigvee_a^b(u; p)$$

Both constants $\frac{1}{2^r}$ and $\frac{1}{2}$ are the best possible for all $r \in (0, 1]$.

Corollary 2.2. Let f be a Hölder continuous function of order r ($0 < r \leq 1$), on $[a, b]$, and $g : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$. Then we have the inequality

$$\left| f(t_0) \int_a^x g(s) ds + f(t_1) \int_x^b g(s) ds - \int_a^b f(s) g(s) ds \right| \leq H \max \left\{ \left[\frac{x-a}{2} + \left| t_0 - \frac{a+x}{2} \right| \right], \left[\frac{b-x}{2} + \left| t_1 - \frac{x+b}{2} \right| \right] \right\}^r \cdot \|g\|_p,$$

for all $a \leq t_0 \leq x \leq t_1 \leq b$, where $\|g\|_p = \left(\int_a^b |g(t)|^p dt \right)^{1/p}$.

Proof. Define the mapping $u : [a, b] \rightarrow \mathbb{R}$, $u(t) = \int_a^t g(s) ds$. Then u is differentiable on (a, b) and $u'(t) = g(t)$. Using the properties of the Riemann-Stieltjes integral, we have

$$\int_a^b f(t) du(t) = \int_a^b f(t) g(t) dt,$$

and

$$\bigvee_a^b(u; p) = \left(\int_a^b |u'(t)|^p dt \right)^{1/p} = \left(\int_a^b |g(t)|^p dt \right)^{1/p},$$

which gives the required result.

Theorem 2.2. Let $1 \leq p < \infty$. Let $f, u : [a, b] \rightarrow \mathbb{R}$ be such that is $f \in L^p[a, b]$ and u has a Lipschitz property on $[a, b]$. If f is r -Hölder continuous, then the inequality

$$\left| \mathcal{R}^{[a,b]}(f, u; t_0, x, t_1) \right| \leq HL \left[(x-a)^{1-\frac{1}{p}} \left(\frac{(t_0-a)^{rp+1} + (x-t_0)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} + (b-x)^{1-\frac{1}{p}} \left(\frac{(t_1-x)^{rp+1} + (b-t_1)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} \right] \quad (2.8)$$

holds for all $p > 1$ and $r \in (0, 1]$.

Proof. From Lemma 1.2 we have

$$\begin{aligned}
& \left| [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \right| \\
&= \left| \int_a^x [f(t_0) - f(s)] du(s) + \int_x^b [f(t_1) - f(s)] du(s) \right| \\
&\leq \left| \int_a^x [f(t_0) - f(s)] du(s) \right| + \left| \int_x^b [f(t_1) - f(s)] du(s) \right| \\
&\leq L \left[(x-a)^{1-\frac{1}{p}} \left(\int_a^x |f(t_0) - f(s)|^p ds \right)^{\frac{1}{p}} \right. \\
&\quad \left. + (b-x)^{1-\frac{1}{p}} \left(\int_x^b |f(t_1) - f(s)|^p ds \right)^{\frac{1}{p}} \right] \\
&\leq HL \left[(x-a)^{1-\frac{1}{p}} \left(\int_a^x |t_0 - s|^{rp} ds \right)^{\frac{1}{p}} + (b-x)^{1-\frac{1}{p}} \left(\int_x^b |t_1 - s|^{rp} ds \right)^{\frac{1}{p}} \right] \\
&= HL \left[(x-a)^{1-\frac{1}{p}} \left(\frac{(t_0-a)^{rp+1} + (x-t_0)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} \right. \\
&\quad \left. + (b-x)^{1-\frac{1}{p}} \left(\frac{(t_1-x)^{rp+1} + (b-t_1)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} \right].
\end{aligned}$$

which proves the required result.

Corollary 2.3. Let f and u as in Theorem 2.2. In (2.8) choose

(1) $t_0 = a$ and $t_1 = b$, then we get the following trapezoid type inequality

$$\left| \mathcal{R}^{[a,b]}(f, u; a, x, b) \right| \leq HL \left[(x-a)^{1-\frac{1}{p}} \left(\frac{(x-a)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} + (b-x)^{1-\frac{1}{p}} \left(\frac{(b-x)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} \right].$$

or equivalently, we may write using parts formula for Riemann-Stieltjes integral

$$\left| [f(b) - f(a)] u(x) - \int_a^b u(s) df(s) \right| \leq HL \left[(x-a)^{1-\frac{1}{p}} \left(\frac{(x-a)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} + (b-x)^{1-\frac{1}{p}} \left(\frac{(b-x)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} \right].$$

(2) $x = \frac{a+b}{2}$, then we get the following mid-point type inequality

$$\left| \mathcal{R}^{[a,b]} \left(f, u; t_0, \frac{a+b}{2}, t_1 \right) \right| \leq HL \left[\left(\frac{b-a}{2} \right)^{1-\frac{1}{p}} \left(\frac{(t_0-a)^{rp+1} + \left(\frac{a+b}{2} - t_0 \right)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} + \left(\frac{b-a}{2} \right)^{1-\frac{1}{p}} \left(\frac{\left(t_1 - \frac{a+b}{2} \right)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} \right].$$

For instance, setting $t_0 = y$ and $t_1 = a+b-y$, we get

$$\left| \mathcal{R}^{[a,b]} \left(f, u; y, \frac{a+b}{2}, a+b-y \right) \right| \leq 2HL \left[\left(\frac{b-a}{2} \right)^{1-\frac{1}{p}} \left(\frac{(t_0-a)^{rp+1} + \left(\frac{a+b}{2} - t_0 \right)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} \right].$$

for all $y \in \left[a, \frac{a+b}{2}\right]$.

(3) $t_0 = \frac{3a+b}{4}$, $x = \frac{a+b}{2}$ and $t_1 = \frac{a+3b}{4}$, then

$$\left| \mathcal{R}^{[a,b]} \left(f, u; \frac{3a+b}{4}, \frac{a+b}{2}, \frac{a+3b}{4} \right) \right| \leq HL \frac{(b-a)^{1+r}}{2^{2r+\frac{1}{p}} (rp+1)^{\frac{1}{p}}}.$$

Now, let I be a real interval such that $[a, b] \subseteq I^\circ$ the interior of I , $a, b \in \mathbb{R}$ with $a < b$. Consider $\mathfrak{U}^p(I)$ ($p > 1$) be the space of all positive n -th differentiable functions f whose n -th derivatives $f^{(n)}$ is positive locally absolutely continuous on I° with $\int_a^b (f^{(n)}(t))^p dt < \infty$, and $f^{(n)}(a) = f^{(n)}(b) = 0$.

L^p -error estimates for Riemann–Stieltjes $\int_a^b f(t) du(t)$ where f belongs to $\mathfrak{U}^p(I)$ is considered in the following result.

Theorem 2.3. *Let $1 \leq p < \infty$. Let $f, u : [a, b] \rightarrow \mathbb{R}$ be such that is $f \in \mathfrak{U}^p(I)$ and u has a Lipschitz property on $[a, b]$. If f is r -H–Hölder continuous, then the inequality holds for all $p > 1$ and $r \in (0, 1]$.*

$$\begin{aligned} \left| \mathcal{R}^{[a,b]}(f, u; t_0, x, t_1) \right| &\leq L \left(\frac{p \sin\left(\frac{\pi}{p}\right)}{\pi \sqrt[p]{p-1}} \right)^n \left\{ (x-a)^{1-\frac{1}{p}} \left[\frac{x-a}{2} + \left| t_0 - \frac{x+a}{2} \right| \right]^n \right. \\ &\quad \left. + (b-x)^{1-\frac{1}{p}} \left[\frac{b-x}{2} + \left| t_1 - \frac{x+b}{2} \right| \right]^n \right\} \|f^{(n)}\|_{p,[a,b]} \end{aligned} \quad (2.9)$$

Proof. As in the proof of Theorem 2.2, we have by Lemma 1.2

$$\begin{aligned} &\left| [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \right| \\ &\leq L \left[(x-a)^{1-\frac{1}{p}} \left(\int_a^x |f(t_0) - f(s)|^p ds \right)^{\frac{1}{p}} \right. \\ &\quad \left. + (b-x)^{1-\frac{1}{p}} \left(\int_x^b |f(t_1) - f(s)|^p ds \right)^{\frac{1}{p}} \right] \\ &\leq L \left[(x-a)^{1-\frac{1}{p}} \left(\frac{p \sin\left(\frac{\pi}{p}\right)}{\pi \sqrt[p]{p-1}} \right)^n \left[\frac{x-a}{2} + \left| t_0 - \frac{x+a}{2} \right| \right]^n \|f^{(n)}\|_{p,[a,x]} \right. \\ &\quad \left. + (b-x)^{1-\frac{1}{p}} \left(\frac{p \sin\left(\frac{\pi}{p}\right)}{\pi \sqrt[p]{p-1}} \right)^n \left[\frac{b-x}{2} + \left| t_1 - \frac{x+b}{2} \right| \right]^n \|f^{(n)}\|_{p,[x,b]} \right] \\ &\leq L \left(\frac{p \sin\left(\frac{\pi}{p}\right)}{\pi \sqrt[p]{p-1}} \right)^n \left\{ (x-a)^{1-\frac{1}{p}} \left[\frac{x-a}{2} + \left| t_0 - \frac{x+a}{2} \right| \right]^n \right. \\ &\quad \left. + (b-x)^{1-\frac{1}{p}} \left[\frac{b-x}{2} + \left| t_1 - \frac{x+b}{2} \right| \right]^n \right\} \|f^{(n)}\|_{p,[a,b]} \end{aligned}$$

which proves the required result, where we have used that fact that if $h \in \mathfrak{U}^p(I)$ then for all $\xi \in (a, b)$ we have

$$\int_a^b |h(t) - h(\xi)|^p dt \leq \left(\frac{p \sin^p\left(\frac{\pi}{p}\right)}{\pi^p (p-1)} \right)^n \left[\frac{b-a}{2} + \left| \xi - \frac{a+b}{2} \right| \right]^{np} \cdot \int_a^b (h^{(n)}(x))^p dx. \quad (2.10)$$

In case $n = 1$, the inequality (2.10) is sharp, see [3].

Remark 2.1. If $f \in \mathcal{L}^p(I)$ and $f^{(n)}$ is bounded on I , so that as $p \rightarrow \infty$ in (2.9), then since $\lim_{p \rightarrow \infty} \frac{p \sin\left(\frac{\pi}{p}\right)}{\sqrt[p]{p-1}} = \pi$, therefore we have

$$\left| \mathcal{R}^{[a,b]}(f, u; t_0, x, t_1) \right| \leq L \left\{ (x-a) \left[\frac{x-a}{2} + \left| t_0 - \frac{x+a}{2} \right| \right]^n + (b-x) \left[\frac{b-x}{2} + \left| t_1 - \frac{x+b}{2} \right| \right]^n \right\} \|f^{(n)}\|_{\infty, [a,b]} \quad (2.11)$$

In what follows we observe several general quadrature rules for the Riemann–Stieltjes integral $\int_a^b f(t) du(t)$ where f is n -times differentiable whose derivatives belongs ton $L^p([a, b])$. To the best of our knowledge, this is the first time of such result concerning Riemann–Stieltjes integral without using interpolation.

Corollary 2.4. Let f and u as in Theorem 2.3. In (2.9) choose

(1) $t_0 = a$ and $t_1 = b$, then we get the following trapezoid type inequality

$$\left| \mathcal{R}^{[a,b]}(f, u; a, x, b) \right| \leq L \left(\frac{p \sin\left(\frac{\pi}{p}\right)}{\pi \sqrt[p]{p-1}} \right)^n \left\{ (x-a)^{n+1-\frac{1}{p}} + (b-x)^{n+1-\frac{1}{p}} \right\} \|f^{(n)}\|_{p, [a,b]}.$$

or equivalently, we may write using parts formula for Riemann–Stieltjes integral

$$\left| [f(b) - f(a)] u(x) - \int_a^b u(s) df(s) \right| \leq L \left(\frac{p \sin\left(\frac{\pi}{p}\right)}{\pi \sqrt[p]{p-1}} \right)^n \left\{ (x-a)^{n+1-\frac{1}{p}} + (b-x)^{n+1-\frac{1}{p}} \right\} \|f^{(n)}\|_{p, [a,b]}.$$

(2) $x = \frac{a+b}{2}$, then we get the following mid-point type inequality

$$\left| \mathcal{R}^{[a,b]} \left(f, u; t_0, \frac{a+b}{2}, t_1 \right) \right| \leq L \left(\frac{b-a}{2} \right)^{1-\frac{1}{p}} \left(\frac{p \sin\left(\frac{\pi}{p}\right)}{\pi \sqrt[p]{p-1}} \right)^n \left\{ \left[\frac{b-a}{4} + \left| t_0 - \frac{3a+b}{4} \right| \right]^n + \left[\frac{b-a}{4} + \left| t_1 - \frac{a+3b}{4} \right| \right]^n \right\} \|f^{(n)}\|_{p, [a,b]}.$$

For instance, setting $t_0 = y$ and $t_1 = a+b-y$, we get

$$\left| \mathcal{R}^{[a,b]} \left(f, u; y, \frac{a+b}{2}, a+b-y \right) \right| \leq L \left(\frac{b-a}{2} \right)^{1-\frac{1}{p}} \left(\frac{p \sin\left(\frac{\pi}{p}\right)}{\pi \sqrt[p]{p-1}} \right)^n \left\{ \left[\frac{b-a}{4} + \left| y - \frac{3a+b}{4} \right| \right]^n + \left[\frac{b-a}{4} + \left| y - \frac{a+3b}{4} \right| \right]^n \right\} \|f^{(n)}\|_{p, [a,b]}.$$

for all $y \in \left[a, \frac{a+b}{2} \right]$.

(3) $t_0 = \frac{3a+b}{4}$, $x = \frac{a+b}{2}$ and $t_1 = \frac{a+3b}{4}$, then

$$\left| \mathcal{R}^{[a,b]} \left(f, u; \frac{3a+b}{4}, \frac{a+b}{2}, \frac{a+3b}{4} \right) \right| \leq \frac{L}{2^{n-\frac{1}{p}}} (b-a)^{n+1-\frac{1}{p}} \left(\frac{p \sin\left(\frac{\pi}{p}\right)}{\pi \sqrt[p]{p-1}} \right)^n \|f^{(n)}\|_{p, [a,b]}$$

3. The dual assumptions

In this section, L^p -error estimates of Two-point quadrature rules for the Riemann–Stieltjes integral $\int_a^b f(t) du(t)$, where the integrand f is of bounded variation on $[a, b]$ and the integrator u is assumed to be satisfied the Hölder condition on $[a, b]$.

Theorem 3.1. Let $u : [a, b] \rightarrow \mathbb{R}$ be a Hölder continuous of order r , ($0 < r \leq 1$), and $f : [a, b] \rightarrow \mathbb{R}$ is a mapping of bounded p -variation ($1 \leq p \leq \infty$) on $[a, b]$. Then we have the inequality

$$\left| \mathcal{R}^{[a,b]}(f, u; t_0, x, t_1) \right| \leq H \max \left\{ (t_0 - a), \left[\frac{t_1 - t_0}{2} + \left| x - \frac{t_0 + t_1}{2} \right| \right], (b - t_1) \right\}^r \cdot \bigvee_a^b(f; p) \quad (3.1)$$

for all $a \leq t_0 \leq x \leq t_1 \leq b$. Furthermore, the constant 1 is the best possible in the sense that it cannot be replaced by a smaller one, for all $r \in (0, 1]$.

Proof. Using the integration by parts formula for Riemann–Stieltjes integral, we have

$$\begin{aligned} \int_a^{t_0} [u(s) - u(a)] df(s) &= [u(t_0) - u(a)] f(t_0) - \int_a^{t_0} f(s) du(s) \\ \int_{t_0}^{t_1} [u(s) - u(x)] df(s) &= [u(t_1) - u(x)] f(t_1) - [u(t_0) - u(x)] f(t_0) - \int_{t_0}^{t_1} f(s) du(s) \\ \int_{t_1}^b [u(s) - u(b)] df(s) &= [u(b) - u(t_1)] f(t_1) - \int_{t_1}^b f(s) du(s), \end{aligned}$$

Adding these identities, we get

$$\begin{aligned} \int_a^{t_0} [u(s) - u(a)] df(s) + \int_{t_0}^{t_1} [u(s) - u(x)] df(s) + \int_{t_1}^b [u(s) - u(b)] df(s) \\ = [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \end{aligned} \quad (3.2)$$

Applying the triangle inequality on the above identity and then use Lemma 1.1, for each term separately, we get

$$\begin{aligned} & \left| [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \right| \\ & \left| \int_a^{t_0} [u(s) - u(a)] df(s) + \int_{t_0}^{t_1} [u(s) - u(x)] df(s) + \int_{t_1}^b [u(s) - u(b)] df(s) \right| \\ & \leq \left| \int_a^{t_0} [u(s) - u(a)] df(s) \right| + \left| \int_{t_0}^{t_1} [u(s) - u(x)] df(s) \right| + \left| \int_{t_1}^b [u(s) - u(b)] df(s) \right| \\ & \leq \sup_{s \in [a, t_0]} |u(s) - u(a)| \cdot \bigvee_a^{t_0}(f; p) + \sup_{s \in [t_0, t_1]} |u(s) - u(x)| \cdot \bigvee_{t_0}^{t_1}(f; p) \\ & \quad + \sup_{s \in [t_1, b]} |u(t_1) - u(b)| \cdot \bigvee_{t_1}^b(f; p). \end{aligned} \quad (3.3)$$

As u is of r -H–Hölder type, we have

$$\begin{aligned} \sup_{s \in [a, t_0]} |u(s) - u(a)| &\leq \sup_{s \in [a, t_0]} [H |s - a|^r] = H (t_0 - a)^r, \\ \sup_{s \in [t_0, t_1]} |u(s) - u(x)| &\leq \sup_{s \in [t_0, t_1]} [H |s - x|^r] \\ &= H \max \{ (t_1 - x)^r, (x - t_0)^r \} \\ &= H [\max \{ (t_1 - x), (x - t_0) \}]^r \\ &= H \left[\frac{t_1 - t_0}{2} + \left| x - \frac{t_0 + t_1}{2} \right| \right]^r, \end{aligned}$$

and

$$\sup_{s \in [t_1, b]} |u(s) - u(b)| \leq \sup_{s \in [t_1, b]} [H |s - b|^r] = H (b - t_1)^r,$$

Therefore, by (3.3), we have

$$\begin{aligned}
 & \left| [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \right| \\
 & \leq H(t_0 - a)^r \cdot \bigvee_a^{t_0}(f; p) + H \left[\frac{t_1 - t_0}{2} + \left| x - \frac{t_0 + t_1}{2} \right| \right]^r \cdot \bigvee_{t_0}^{t_1}(f; p) + H(b - t_1)^r \cdot \bigvee_{t_1}^b(f; p) \\
 & \leq H \max \left\{ (t_0 - a)^r, \left[\frac{t_1 - t_0}{2} + \left| x - \frac{t_0 + t_1}{2} \right| \right]^r, (b - t_1)^r \right\} \cdot \bigvee_a^b(f; p) \\
 & = H \max \left\{ (t_0 - a), \left[\frac{t_1 - t_0}{2} + \left| x - \frac{t_0 + t_1}{2} \right| \right], (b - t_1) \right\}^r \cdot \bigvee_a^b(f; p).
 \end{aligned}$$

To prove the sharpness of the constant 1 for any $r \in (0, 1]$, assume that (3.1) holds with a constant $C > 0$, that is,

$$\begin{aligned}
 & \left| [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \right| \\
 & \leq C \max \left\{ (t_0 - a), \left(\frac{t_1 - t_0}{2} + \left| x - \frac{t_0 + t_1}{2} \right| \right), (b - t_1) \right\}^r \cdot \bigvee_a^b(f; p). \quad (3.4)
 \end{aligned}$$

Choose $u(t) = t^r$, $r \in (0, 1]$, $t \in [0, 1]$ and $f : [0, 1] \rightarrow [0, \infty)$ given by

$$f(t) = \begin{cases} 0, & t \in (0, 1] \\ 1, & t = 0 \end{cases}$$

As

$$|u(x) - u(y)| = |x^r - y^r| \leq |x - y|^r, \quad \forall x \in [0, 1], \quad r \in (0, 1],$$

it follows that u is r -H-Hölder type with the constant $H = 1$.

By using the integration by parts formula for Riemann-Stieltjes integrals, we have:

$$\int_0^1 f(t) du(t) = f(1)u(1) - f(0)u(0) - \int_0^1 u(t) df(t) = 0,$$

and $\bigvee_0^1(f; p) = 1$. Consequently, by (3.4), we get

$$|t_0^r| \leq C \max \left\{ t_0, \left(\frac{t_1 - t_0}{2} + \left| x - \frac{t_0 + t_1}{2} \right| \right), (1 - t_1) \right\}^r, \quad \forall t_0, t_1 \in [0, 1], \text{ with } t_0 \leq t_1.$$

Assume first

$$\max \left\{ t_0, \left(\frac{t_1 - t_0}{2} + \left| x - \frac{t_0 + t_1}{2} \right| \right), (1 - t_1) \right\}^r = t_0^r$$

so that we get $1 \leq C$.

Now, assume that

$$\max \left\{ t_0, \left(\frac{t_1 - t_0}{2} + \left| x - \frac{t_0 + t_1}{2} \right| \right), (1 - t_1) \right\}^r = (1 - t_1)^r.$$

choose $t_1 = 1 - t_0$, so that we get $1 \leq C$.

Finally, we assume that

$$\max \left\{ t_0, \left(\frac{t_1 - t_0}{2} + \left| x - \frac{t_0 + t_1}{2} \right| \right), (1 - t_1) \right\}^r = \left(\frac{t_1 - t_0}{2} + \left| x - \frac{t_0 + t_1}{2} \right| \right)^r.$$

Define $f : [0, 1] \rightarrow [0, \infty)$ given by

$$f(t) = \begin{cases} 0, & t \in (0, 1) \\ 1, & t = 0, 1 \end{cases}$$

Clearly, $V_0^1(f; p) = 2$. Therefore, for $t_0 = 0$ and $t_1 = 1$, so that we get $1 \leq C \left(\frac{1}{2} + \left| x - \frac{1}{2} \right| \right)^r 2^{1/p}$. Choosing $x = \frac{1}{2}$ and $r = \frac{1}{p}$ or $p = \frac{1}{r}$, it follows that $1 \leq C \left(\frac{1}{2} \right)^r 2^r$, i.e., $C \geq 1$. Hence, the inequality (3.1) is sharp, and the theorem is completely proved.

Theorem 3.2. Let $1 \leq p < \infty$. Let $f, u : [a, b] \rightarrow \mathbb{R}$ be such that $u \in L^p[a, b]$ and f has a Lipschitz property on $[a, b]$. If u is r -Hölder continuous, then the inequality

$$\begin{aligned} & \left| \mathcal{R}^{[a, b]}(f, u; t_0, x, t_1) \right| \\ & \leq LH \begin{cases} \frac{(t_0 - a)^{r+1}}{(rp+1)^{\frac{1}{p}}} + (t_1 - t_0)^{1-\frac{1}{p}} \left(\frac{(t_1 - x)^{rp+1} - (t_0 - x)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} + \frac{(b - t_1)^{r+1}}{(rp+1)^{\frac{1}{p}}}, & a \leq x \leq t_0 \leq t_1 \leq b \\ \frac{(t_0 - a)^{r+1}}{(rp+1)^{\frac{1}{p}}} + (t_1 - t_0)^{1-\frac{1}{p}} \left(\frac{(x - t_0)^{rp+1} + (t_1 - x)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} + \frac{(b - t_1)^{r+1}}{(rp+1)^{\frac{1}{p}}}, & a \leq t_0 \leq x \leq t_1 \leq b \\ \frac{(t_0 - a)^{r+1}}{(rp+1)^{\frac{1}{p}}} + (t_1 - t_0)^{1-\frac{1}{p}} \left(\frac{(x - t_0)^{rp+1} - (x - t_1)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} + \frac{(b - t_1)^{r+1}}{(rp+1)^{\frac{1}{p}}}, & a \leq t_0 \leq t_1 \leq x \leq b \end{cases} \quad (3.5) \end{aligned}$$

holds for all $p > 1$ and $r \in (0, 1]$ with constant $H > 0$.

Proof. As in the proof of Theorem 3.1, we have by Lemma 1.2

$$\begin{aligned} & \left| [u(x) - u(a)] f(t_0) + [u(b) - u(x)] f(t_1) - \int_a^b f(s) du(s) \right| \\ & = \left| \int_a^{t_0} [u(s) - u(a)] df(s) + \int_{t_0}^{t_1} [u(s) - u(x)] df(s) + \int_{t_1}^b [u(s) - u(b)] df(s) \right| \\ & \leq \left| \int_a^{t_0} [u(s) - u(a)] df(s) \right| + \left| \int_{t_0}^{t_1} [u(s) - u(x)] df(s) \right| + \left| \int_{t_1}^b [u(s) - u(b)] df(s) \right| \\ & \leq L \left[(t_0 - a)^{1-\frac{1}{p}} \left(\int_a^{t_0} |u(s) - u(a)|^p ds \right)^{\frac{1}{p}} + (t_1 - t_0)^{1-\frac{1}{p}} \left(\int_{t_0}^{t_1} |u(s) - u(x)|^p ds \right)^{\frac{1}{p}} \right. \\ & \quad \left. + (b - t_1)^{1-\frac{1}{p}} \left(\int_{t_1}^b |u(s) - u(b)|^p ds \right)^{\frac{1}{p}} \right] \\ & \leq LH \left[(t_0 - a)^{1-\frac{1}{p}} \left(\int_a^{t_0} |s - a|^{rp} ds \right)^{\frac{1}{p}} + (t_1 - t_0)^{1-\frac{1}{p}} \left(\int_{t_0}^{t_1} |s - x|^{rp} ds \right)^{\frac{1}{p}} \right. \\ & \quad \left. + (b - t_1)^{1-\frac{1}{p}} \left(\int_{t_1}^b |s - b|^{rp} ds \right)^{\frac{1}{p}} \right]. \end{aligned}$$

Simple computations yield that

$$\int_a^{t_0} |s - a|^{rp} ds = \int_a^{t_0} (s - a)^{rp} ds = \frac{(t_0 - a)^{rp+1}}{rp+1},$$

$$\int_{t_0}^{t_1} |s-x|^{rp} ds = \begin{cases} \int_{t_0}^{t_1} (s-x)^{rp} ds, & a \leq x \leq t_0 \\ \int_{t_0}^x (x-s)^{rp} ds + \int_x^{t_1} (s-x)^{rp} ds, & t_0 \leq x \leq t_1 \\ \int_{t_0}^{t_1} (x-s)^{rp} ds, & t_1 \leq x \leq b \end{cases}$$

$$= \begin{cases} \frac{(t_1-x)^{rp+1} - (t_0-x)^{rp+1}}{rp+1} & a \leq x \leq t_0 \\ \frac{(x-t_0)^{rp+1} + (t_1-x)^{rp+1}}{rp+1}, & t_0 \leq x \leq t_1 \\ \frac{(x-t_0)^{rp+1} - (x-t_1)^{rp+1}}{rp+1}, & t_1 \leq x \leq b \end{cases} ,$$

and

$$\int_{t_1}^b |s-b|^{rp} ds = \int_{t_1}^b (b-s)^{rp} ds = \frac{(b-t_1)^{rp+1}}{rp+1}.$$

Combining these equalities with the last inequality above we get the required result.

Corollary 3.1. *Let $1 \leq p < \infty$. Let $f, u : [a, b] \rightarrow \mathbb{R}$ be such that $u \in L^p[a, b]$ and f has a Lipschitz property on $[a, b]$. If u is r -Hölder continuous, then the inequality*

$$\left| (x-a)f(t_0) + (b-x)f(t_1) - \int_a^b s^{r-1}f(s) ds \right|$$

$$\leq LH \begin{cases} \frac{(t_0-a)^{r+1}}{(rp+1)^{\frac{1}{p}}} + (t_1-t_0)^{1-\frac{1}{p}} \left(\frac{(t_1-x)^{rp+1} - (t_0-x)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} + \frac{(b-t_1)^{r+1}}{(rp+1)^{\frac{1}{p}}}, & a \leq x \leq t_0 \leq t_1 \leq b \\ \frac{(t_0-a)^{r+1}}{(rp+1)^{\frac{1}{p}}} + (t_1-t_0)^{1-\frac{1}{p}} \left(\frac{(x-t_0)^{rp+1} + (t_1-x)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} + \frac{(b-t_1)^{r+1}}{(rp+1)^{\frac{1}{p}}}, & a \leq t_0 \leq x \leq t_1 \leq b \\ \frac{(t_0-a)^{r+1}}{(rp+1)^{\frac{1}{p}}} + (t_1-t_0)^{1-\frac{1}{p}} \left(\frac{(x-t_0)^{rp+1} - (x-t_1)^{rp+1}}{rp+1} \right)^{\frac{1}{p}} + \frac{(b-t_1)^{r+1}}{(rp+1)^{\frac{1}{p}}}, & a \leq t_0 \leq t_1 \leq x \leq b \end{cases} \quad (3.6)$$

holds for all $p > 1$ and $r \in (0, 1]$ with constant $H > 0$.

Proof. Setting $u(t) = t^r$, $t \in [a, b]$, $r \in (0, 1]$, in Theorem 3.2 we get the required result.

Corollary 3.2. *Let $1 \leq p < \infty$. Let $f, u : [a, b] \rightarrow \mathbb{R}$ be such that $u \in L^p[a, b]$ and f has a Lipschitz property on $[a, b]$. If u is K -Lipschitz continuous on $[a, b]$, then the inequality*

$$\left| (x-a)f(t_0) + (b-x)f(t_1) - \int_a^b f(s) ds \right|$$

$$\leq LK \begin{cases} \frac{(t_0-a)^2}{(p+1)^{\frac{1}{p}}} + (t_1-t_0)^{1-\frac{1}{p}} \left(\frac{(t_1-x)^{p+1} - (t_0-x)^{p+1}}{p+1} \right)^{\frac{1}{p}} + \frac{(b-t_1)^2}{(p+1)^{\frac{1}{p}}}, & a \leq x \leq t_0 \leq t_1 \leq b \\ \frac{(t_0-a)^2}{(p+1)^{\frac{1}{p}}} + (t_1-t_0)^{1-\frac{1}{p}} \left(\frac{(x-t_0)^{p+1} + (t_1-x)^{p+1}}{p+1} \right)^{\frac{1}{p}} + \frac{(b-t_1)^2}{(p+1)^{\frac{1}{p}}}, & a \leq t_0 \leq x \leq t_1 \leq b \\ \frac{(t_0-a)^2}{(p+1)^{\frac{1}{p}}} + (t_1-t_0)^{1-\frac{1}{p}} \left(\frac{(x-t_0)^{r+1} - (x-t_1)^{r+1}}{r+1} \right)^{\frac{1}{p}} + \frac{(b-t_1)^2}{(p+1)^{\frac{1}{p}}}, & a \leq t_0 \leq t_1 \leq x \leq b \end{cases} \quad (3.7)$$

holds for all $p > 1$ and constant $K > 0$.

Proof. Setting $r = 1$ in Corollary 3.1, we get the required result.

Remark 3.1. The inequalities (3.6) and (3.7) generalize the recent result(s) in [2].

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