Moroccan J. of Pure and Appl. Anal. (MJPA)A)

Volume 4(2), 2018, Pages 62–76

ISSN: Online 2351-8227 - Print 2605-6364

DOI 10.1515/mjpaa-2018-0007

Weighted Variable Exponent Sobolev spaces on metric measure spaces

MOULAY CHERIF HASSIB^{1,a} AND YOUSSEF AKDIM^{2,b}

ABSTRACT. In this article we define the weighted variable exponent-Sobolev spaces on arbitrary metric spaces, with finite diameter and equipped with finite, positive Borel regular outer measure. We employ a Hajlasz definition, which uses a point wise maximal inequality. We prove that these spaces are Banach, that the Poincaré inequality holds and that lipschitz functions are dense. We develop a capacity theory based on these spaces. We study basic properties of capacity and several convergence results. As an application, we prove that each weighted variable exponent-Sobolev function has a quasi-continuous representative, we study different definitions of the first order weighted variable exponent-Sobolev spaces with zero boundary values, we define the Dirichlet energy and we prove that it has a minimizer in the weighted variable exponent-Sobolev spaces case.

2010 Mathematics Subject Classification. Primary: 46E35; 31B15; 46E30; 42B25; 28A80.

Key words and phrases. weighted variable exponent Sobolev spaces on metric spaces, capacity, weighted variable exponent Sobolev with zero boundary values, Dirichlet energy.

1. Introduction

The theory of Sobolev space was originally developed in the domains of $\mathbb{R}^N$ using the notion of distributional derivatives. To generalize this theory to metric spaces, P. Hajlasz showed in [7] that a p -integrable function u , $1 < p < \infty$, belongs to $W^{1,p}(\mathbb{R}^N)$ if and only if there exists a non-negative p -integrable function g such that

$$|u(x) - u(y)| \leq |x - y| |g(x) + g(y)|$$

Received February 6, 2019 - Accepted April 3, 2019.

©The Author(s) 2017. This article is published with open access by Sidi Mohamed Ben Abdallah University.

¹University Sidi Mohamed Ben Abdallah, Faculty of science and technique, Fez, Laboratory: LSI, Taza, Morocco.

²University Sidi Mohamed Ben Abdallah, Faculty of science Dhar El Mahraz, Laboratory: LAMA Fez, Morocco.

^a e-mail: cherif_hassib@yahoo.fr

^b e-mail: akdimyoussef@yahoo.fr .

for almost every $x, y \in \mathbb{R}^n$. This inequality can be stated also in metric measure spaces if $|x - y|$ is replaced by the distance between the points x and y .

This theory was generalized by N. Aissaoui in [4] in the Orlicz-Sobolev spaces; by Petteri Harjulehto, Peter Hasto Mikko Pere in [15] in the variable exponent spaces, by Takao Ohno and Tetsu Shimomura in [16] in the Musielak-Orlicz-Sobolev spaces and by M.C. Hassib, Y. Akdim, N. Aissaoui in [8] in the Musielak-Orlicz-Sobolev spaces also and by using another method. The objective of this paper is to develop this theory in the setting of weighted variable exponent -Sobolev spaces.

The paper is organized as follows:

In section 1 we recall some important definition and some results of weighted variable exponent spaces $L^{p(\cdot)}(\Omega, \omega)$, in section 2, we define the weighted variable exponent -Sobolev spaces on metric space, we show that: it is a Banach space, the Poincaré inequality holds and lipschitz continuous functions are dense. In section 3, we develop a capacity theory based on this space, we study basic properties of capacity, including monotonicity, countable subadditivity and several convergence results are studied. As an application we prove that each weighted variable exponent-Sobolev function has a quasi-continuous representative. In section 4 we study different definitions of the first order weighted variable exponent-Sobolev spaces with zero boundary values, we define the Dirichlet energy and we prove that it has a minimizer in the weighted variable exponent -Sobolev spaces on metric measure space case.

2. Preliminary

2.1. Variable exponent Lebesgue spaces on metric measure spaces.[9]. By a metric measure space we mean a triple (X, d, μ) , where X is a set, d is a metric on X and μ is a non-negative Borel regular outer measure on X which is finite in every bounded set. For simplicity, we often write X instead of (X, d, μ) . For $x \in X$ and $r \geq 0$ We call a measurable function $p : X \rightarrow [1, +\infty)$ a variable exponent.

For $A \subset X$ we define $p_A^+ = \text{ess sup}_{x \in A} p(x)$ and $p_A^- = \text{ess inf}_{x \in A} p(x)$. we use the abbreviations $p^+ = p_X^+$ and $p^- = p_X^-$.

We denote

$$C_+(X) = \{\text{continuous function } p(\cdot) : X \rightarrow [1, +\infty) / 1 < p_- \leq p(x) \leq p_+ \leq +\infty\},$$

Throughout the paper, we assume that $p \in C_+(X)$.

we say that a real-valued continuous function $p(\cdot)$ is log-Hölder continuous in X if

$$|p(x) - p(y)| \leq \frac{C}{-\log(d(x, y))} : (\forall x, y \in X) \text{ such that } d(x, y) \leq \frac{1}{2},$$

where C is a constant.

For a μ -measurable function $u : X \rightarrow \mathbb{R}$ we define the modular

$$\rho_{p(\cdot)}(u) = \int_X |u(y)|^{p(y)} d\mu(y).$$

We define the variable exponent Lebesgue space on X by

$$L^{p(\cdot)}(X) = \{u : X \rightarrow \mathbb{R}, \mu\text{-measurable} : \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) < \infty, \text{ for some } \lambda > 0\}.$$

Lemma 2.1. Under the norm $\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0, \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}$ the space $L^{p(\cdot)}(X)$ is a Banach space.

Theorem 2.1. (i) Hölder inequality For any $u \in L^{p(\cdot)}(X)$ and $v \in L^{p'(\cdot)}(X)$, we have

$$\left| \int_X u v dx \right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}.$$

- (ii) $\|u\|_{p(\cdot)} < 1$ (resp, $= 1, > 1$) $\iff \rho_{p(\cdot)}(u) < 1$ (resp, $= 1, > 1$),
 (iii) $\|u\|_{p(\cdot)} > 1 \Rightarrow \|u\|_{p(\cdot)}^{p^-} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^+}$ and $\|u\|_{p(\cdot)} < 1 \Rightarrow \|u\|_{p(\cdot)}^{p^+} \leq \rho_{p(\cdot)}(u) \leq \|u\|_{p(\cdot)}^{p^-}$,
 (iv) $\|u\|_{p(\cdot)} \rightarrow 0 \iff \rho_{p(\cdot)}(u) \rightarrow 0$ and $\|u\|_{p(\cdot)} \rightarrow \infty \iff \rho_{p(\cdot)}(u) \rightarrow \infty$.

Theorem 2.2. Let X be a locally compact doubling space and let $p^+ < 1$. The continuous functions with compact support are dense in $L^{p(\cdot)}(X)$.

Proposition 2.1. If $1 < p^- < p^+ < \infty$ and μ is atomless, then $L^{p(\cdot)}(X)$ is uniformly convex. From this it follows that $L^{p(\cdot)}(X)$ is reflexive and has the Banach-Sacks propriety, namely, if $u_i \rightharpoonup u$ weakly in $L^{p(\cdot)}(X)$ then $\frac{1}{i}(u_1 + \dots + u_i) \rightarrow u$ strongly in $L^{p(\cdot)}(X)$.

2.2. Weighted variable exponent Lebesgue spaces on metric measure spaces.[5]. Let ω be a function defined in X , ω is called a weight function in X if it's μ -measurable and strictly positive μ -almost every where in X .

For a μ -measurable function $u : X \rightarrow \mathbb{R}$ we define the modular

$$\rho_{p(\cdot),\omega}(u) = \int_X \omega(y) |u(y)|^{p(y)} d\mu(y).$$

We define the variable exponent Lebesgue space on X by

$$L^{p(\cdot)}(X, \omega) = \{u : X \rightarrow \mathbb{R}, \mu - \text{measurable} : \rho_{p(\cdot),\omega}\left(\frac{u}{\lambda}\right) < \infty, \text{ for some } \lambda > 0\}.$$

Lemma 2.2. If ω is a positive measurable function, then under the norm $\|u\|_{p(\cdot),\omega} = \inf \left\{ \lambda > 0, \rho_{p(\cdot),\omega}\left(\frac{u}{\lambda}\right) \leq 1 \right\}$ the space $L^{p(\cdot)}(X, \omega)$ is a reflexive Banach space.

- Theorem 2.3.** (i) $\|u\|_{p(\cdot),\omega} < 1$ (resp, $= 1, > 1$) $\iff \rho_{p(\cdot),\omega}(u) < 1$ (resp, $= 1, > 1$),
 (ii) $\|u\|_{p(\cdot),\omega} > 1 \Rightarrow \|u\|_{p(\cdot),\omega}^{p^-} \leq \rho_{p(\cdot),\omega}(u) \leq \|u\|_{p(\cdot),\omega}^{p^+}$ and $\|u\|_{p(\cdot),\omega} < 1 \Rightarrow \|u\|_{p(\cdot),\omega}^{p^+} \leq \rho_{p(\cdot),\omega}(u) \leq \|u\|_{p(\cdot),\omega}^{p^-}$,
 (iii) $\|u\|_{p(\cdot),\omega} \rightarrow 0 \iff \rho_{p(\cdot),\omega}(u) \rightarrow 0$ and $\|u\|_{p(\cdot),\omega} \rightarrow \infty \iff \rho_{p(\cdot),\omega}(u) \rightarrow \infty$.

3. Weighted variable exponent Sobolev spaces on metric measure spaces

Let (X, d) be a metric space with finite diameter, ($\text{diam}(X) = \sup_{x,y \in X} d(x, y) < \infty$) and equipped with a finite, positive Borel regular outer, atomless measure μ . The triplet (X, d, μ) will be fixed in the sequel and will be denoted by X .

ω is a positive measurable and finite function.

$p : X \rightarrow [1, +\infty)$ is a variable exponent such that $1 < p^- < p^+ < \infty$.

Let $u : X \rightarrow [-\infty, +\infty]$ be a μ -measurable functions defined on X .

The average value of u will be denoted by, $u_X = \frac{1}{\mu(X)} \int_X u d\mu = f_X u d\mu$.

We denote by $D(u)$ the set of all μ -measurable functions $g : X \rightarrow [0, +\infty]$, such that

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y)) \quad (3.1)$$

for every $x, y \in X \setminus F, x \neq y$, with $\mu(F) = 0$. The set F is called the exceptional set for g , and g is called a Hajlasz gradient of u .

Note that the right-hand side of (3.1) is always defined for $x \neq y$. For the points $x, y \in X, x \neq y$ such that the left-hand side of (3.1) is undefined, we may assume that the left-hand side is $+\infty$.

Definition 3.1. The Dirichlet- weighted variable exponent Sobolev spaces $L_{p(\cdot),\mu}^1(X,\omega)$ is the space of all μ -measurable functions u such that $D(u) \cap L^{p(\cdot)}(X,\omega) \neq \emptyset$. This space is equipped with the semi-norm

$$\|u\|_{L_{p(\cdot),\mu}^1(X,\omega)} = \inf\{\|g\|_{L^{p(\cdot)}(X,\omega)} : g \in D(u) \cap L^{p(\cdot)}(X,\omega)\}.$$

The weighted variable exponent Sobolev spaces on Arbitrary Metric space X is

$$M_{p(\cdot),\mu}^1(X,\omega) = L^{p(\cdot)}(X) \cap L_{p(\cdot),\mu}^1(X,\omega),$$

equipped with the norm

$$\|u\|_{M_{p(\cdot),\mu}^1(X,\omega)} = \|u\|_{L^{p(\cdot)}(X)} + \|u\|_{L_{p(\cdot),\mu}^1(X,\omega)}.$$

Lemma 3.1. If $(\forall y \in X) \omega(y) \geq 1$, then for every $u \in M_{p(\cdot),\mu}^1(X,\omega)$ there is a unique $g \in D(u) \cap L^{p(\cdot)}(X,\omega)$; denoted by g_u such that

$$\|g_u\|_{L^{p(\cdot)}(X,\omega)} = \|u\|_{L_{p(\cdot),\mu}^1(X,\omega)}.$$

Proof. Let $(g_i) \subset D(u) \cap L^{p(\cdot)}(X,\omega)$ a sequence such that

$$\|u\|_{L_{p(\cdot),\mu}^1(X,\omega)} = \lim_{i \rightarrow +\infty} \|g_i\|_{L^{p(\cdot)}(X,\omega)}$$

since $L^{p(\cdot)}(X,\omega)$ is reflexive, there is a sub-sequence denoted again by (g_i) , and a function $g_u \in L^{p(\cdot)}(X,\omega)$ such that $g_i \rightharpoonup g_u$ weakly in $L^{p(\cdot)}(X,\omega)$. By Mazur lemma there exists a convex combinations $f_k = \sum_{i=k}^{m_k} \alpha_{k_i} g_i$, such that $f_k \rightarrow g_u$ strongly in $L^{p(\cdot)}(X,\omega)$, where $\alpha_{k_i} \geq 0$ and $\sum_{i=k}^{m_k} \alpha_{k_i} = 1$.

Since every $g_i \in D(u) \cap L^{p(\cdot)}(X,\omega)$, we have:

$$\begin{aligned} |u(x) - u(y)| &= \sum_{i=k}^{i=m_k} \alpha_{k_i} |u(x) - u(y)| \\ &\leq \sum_{i=k}^{i=m_k} \alpha_{k_i} d(x;y) (g_i(x) + g_i(y)) \\ &= d(x;y) (f_k(x) + f_k(y)), \quad \text{for } \mu - \text{almost all } x, y \in X. \end{aligned}$$

Therefore $f_k \in D(u) \cap L^{p(\cdot)}(X,\omega)$, for all k .

On the other hand, there exists a subsequence of (f_k) which converges to g_u point wise μ -almost everywhere in X , thus $g_u \in D(u) \cap L^{p(\cdot)}(X,\omega)$.

Let $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that

$$\|g_i\|_{L^{p(\cdot)}(X,\omega)} \leq \|u\|_{L_{p(\cdot),\mu}^1(X,\omega)} + \varepsilon, \quad \forall i \geq k.$$

For this k we have

$$\|f_k\|_{L^{p(\cdot)}(X,\omega)} \leq \sum_{i=k}^{m_k} \alpha_{k_i} \|g_i\|_{L^{p(\cdot)}(X,\omega)}.$$

Therefore

$$\|f_k\|_{L^{p(\cdot)}(X,\omega)} \leq \|u\|_{L_{p(\cdot),\mu}^1(X,\omega)} + \varepsilon.$$

Thus

$$\|g_u\|_{L^{p(\cdot)}(X,\omega)} = \|u\|_{L_{p(\cdot),\mu}^1(X,\omega)}.$$

Now suppose that there exists two minimal g_1 and g_2 with $g_1 \neq g_2$, then

$$\|g_1 - g_2\|_{L^{p(\cdot)}(X,\omega)} > 0.$$

Let $G_1 = \frac{g_1}{\|g_1\|_{L^{p(\cdot)}(X,\omega)}}$ and $G_2 = \frac{g_2}{\|g_2\|_{L^{p(\cdot)}(X,\omega)}}$. By the uniform convexity of the norm we have

$$\|G_1 + G_2\|_{L^{p(\cdot)}(X,\omega)} < 2$$

then

$$\|\frac{1}{2}g_1 + \frac{1}{2}g_2\|_{L^{p(\cdot)}(X,\omega)} < \|g_1\|_{L^{p(\cdot)}(X,\omega)}$$

which is the contradiction since $(\frac{1}{2}g_1 + \frac{1}{2}g_2) \in D(u) \cap L^{p(\cdot)}(X,\omega)$.

With simple verification, we have the following lemma.

Lemma 3.2. Let $g_1 \in D(u_1)$, $g_2 \in D(u_2)$ and $\alpha, \beta \in \mathbb{R}$. If $g \geq \alpha|g_1| + \beta|g_2|$, μ -a.e. Then $g \in D(\alpha u_1 + \beta u_2)$.

Lemma 3.3. Let (u_n) and (g_n) be two sequences of functions such that,

$$\forall n \in \mathbb{N} : g_n \in D(u_n).$$

If $u_n \rightarrow u$ in $L^{p(\cdot)}(X)$ and $g_n \rightarrow g$ in $L^{p(\cdot)}(X,\omega)$, then $g \in D(u)$.

Proof. There are two subsequences of (u_n) and (g_n) which we denote again by (u_n) and (g_n) such that $u_n \rightarrow u$ and $g_n \rightarrow g$ μ -a.e. For all $n \in \mathbb{N}$, let F_n be the exceptional set for g_n , H a set of measure zero such that $u \rightarrow u$ and $g_n \rightarrow g$ on H^c , and $F = H \cup (\bigcup_n F_n)$. We have $\mu(F) = 0$ and

$$|u(x) - u(y)| \leq d(x, y)|g(x) + g(y)|, \forall x, y \in F.$$

Therefore $g \in D(u)$.

Theorem 3.1. $(M_{p(\cdot),\mu}^1(X,\omega), \|\cdot\|_{M_{p(\cdot),\mu}^1(X,\omega)})$ is a Banach space.

Proof. It is clear that $M_{p(\cdot),\mu}^1(X,\omega)$ is a vector space.

Let $(u_n)_n$ be a Cauchy sequence in $M_{p(\cdot),\mu}^1(X,\omega)$, then $(u_n)_n$ is a Cauchy sequence in $L^{p(\cdot)}(X)$, thus there exists a function $u \in L^{p(\cdot)}(X)$ such that

$u_n \rightarrow u$ in $L^{p(\cdot)}(X)$. We can choose a subsequence denoted again $(u_n)_n$ such that $\|u_{n+1} - u_n\|_{M_{p(\cdot),\mu}^1(X,\omega)} < 2^{-n}$ and $u_n \rightarrow u$ μ -a.e in X . There exists a non-negative function $g_n \in L^{p(\cdot)}(X,\omega)$ such that $\|g_n\|_{L^{p(\cdot)}(X,\omega)} < 2^{-n}$ and

$$|(u_{n+1} - u_n)(x) - (u_{n+1} - u_n)(y)| \leq d(x, y)(g_n(x) + g_n(y)) \quad \mu\text{-a.e in } X.$$

By adding the last inequalities, we have for $n > m$,

$$|(u_n - u_m)(x) - (u_n - u_m)(y)| \leq d(x, y) \left(\sum_{k=m}^{\infty} g_k(x) + \sum_{k=m}^{\infty} g_k(y) \right) \quad \mu\text{-a.e in } X.$$

Letting $n \rightarrow \infty$ yields

$$|(u - u_m)(x) - (u - u_m)(y)| \leq d(x, y) \left(\sum_{k=m}^{\infty} g_k(x) + \sum_{k=m}^{\infty} g_k(y) \right) \quad \mu\text{-a.e in } X.$$

Since

$$\left\| \sum_{k=m}^{\infty} g_k \right\|_{L^{p(\cdot)}(X,\omega)} \leq \sum_{k=m}^{\infty} 2^{-k} = 2^{-m+1}$$

the previous inequality implies that $(u - u_m) \in M_{p(\cdot),\mu}^1(X,\omega)$ and therefore $u \in M_{p(\cdot),\mu}^1(X,\omega)$.

On the other hand, we have

$$\|u - u_m\|_{L_{p(\cdot),\mu}^1(X,\omega)} \leq \left\| \sum_{k=m}^{\infty} g_k \right\| \leq 2^{-m+1},$$

then $u_m \rightarrow u$ in $M_{p(\cdot),\mu}^1(X,\omega)$.

Theorem 3.2. $M_{p(\cdot),\mu}^1(X, \omega)$ is reflexive.

Proof. By proposition 2.1 We have $L^{p(\cdot)}(X)$ is a reflexive Banach space and by theorem 3.1 $M_{p(\cdot),\mu}^1(X, \omega)$ is closed, thus $M_{p(\cdot),\mu}^1(X, \omega)$ is reflexive.

Lemma 3.4. The function u belongs to $M_{p(\cdot),\mu}^1(X, \omega)$ if and only if $u \in L^{p(\cdot)}(X)$ and there exists two sequences $(u_n) \subset L^{p(\cdot)}(X)$ and $(g_n) \subset D(u_n) \cap L^{p(\cdot)}(X, \omega)$, such that $u_n \rightarrow u$ μ -a.e and $g_n \rightarrow g$ μ -a.e for some $g \in L^{p(\cdot)}(X, \omega)$.

Proof. The same as in [11] lemma 2.5.

Lemma 3.5. Let $u_1, u_2 \in M_{p(\cdot),\mu}^1(X, \omega)$. If $g_1 \in D(u_1)$ and $g_2 \in D(u_2)$, then :

- (i) $u = \max(u_1, u_2) \in M_{p(\cdot),\mu}^1(X, \omega)$ and $g = \max(g_1, g_2) \in D(u) \cap L^{p(\cdot)}(X, \omega)$.
- (ii) $v = \min(u_1, u_2) \in M_{p(\cdot),\mu}^1(X, \omega)$ and $h = \max(g_1, g_2) \in D(v) \cap L^{p(\cdot)}(X, \omega)$.

Proof. We only prove the case (i), the proof of (ii) is similar.

Let F_1 and F_2 the exceptional sets for g_1 and g_2 , respectively. Clearly $u \in L^{p(\cdot)}(X)$ and $g \in L^{p(\cdot)}(X, \omega)$. It remains to prove that $g \in D(u)$. Let $G = \{x \in X \setminus (F_1 \cup F_2) : u_1(x) \geq u_2(x)\}$.

If $x, y \in G$, then

$$|u(x) - u(y)| = |u_1(x) - u_1(y)| \leq d(x, y)(g_1(x) + g_1(y)).$$

Analogously, for $x, y \in X \setminus G$, we obtain $|u(x) - u(y)| \leq d(x, y)(g_2(x) + g_2(y))$.

For the remaining cases, let $x \in G$ and $y \in X \setminus G$:

If $u_1(x) \geq u_2(y)$, then

$$|u(x) - u(y)| = u_1(x) - u_2(y) \leq u_1(x) - u_1(y) \leq d(x, y)(g_1(x) + g_1(y)).$$

If $u_1(x) < u_2(y)$, then

$$|u(x) - u(y)| = u_2(y) - u_1(x) \leq u_2(y) - u_2(x) \leq d(x, y)(g_2(x) + g_2(y)).$$

The case $x \in X \setminus G$ and $y \in G$ follows by symmetry and hence

$$|u(x) - u(y)| \leq d(x, y)(g(x) + g(y))$$

for all $x, y \in X \setminus (F_1 \cup F_2)$, with $\mu((F_1 \cup F_2)) = 0$.

Theorem 3.3. (The Poincare inequality).

If $p(\cdot)$ is log-Hölder continuous, then exists $C > 0$, depending only of p^-, p^+ and $\mu(X)$ such that

$$(\forall u \in M_{p(\cdot),\mu}^1(X, \omega)) (\forall g \in D(u) \cap L^{p(\cdot)}(X, \omega)) : \|u - u_X\|_{p(\cdot),\omega} \leq C \text{diam}(X) \|g\|_{p(\cdot),\omega}.$$

Proof. Integrating the inequality $|u(x) - u(y)| \leq d(x, y)(g(x) + g(y))$ over y we obtain

$$|u(x) - \int_X u(y) d\mu(y)| \leq \int_X |u(x) - u(y)| d\mu(y) \leq \text{diam}(X)(g(x) + \int_X g(y) d\mu(y)).$$

Thus,

$$|u(x) - u_X| \leq \text{diam}(X)(g(x) + \frac{C' \|1\|_{p'(\cdot),\omega}}{\mu(X)} \|g\|_{p(\cdot),\omega}).$$

Therefore

$$\|u - u_X\|_{p(\cdot),\omega} \leq C \text{diam}(X) \|g\|_{L^{p(\cdot)}(X, \omega)},$$

where $C = C(p^-, p^+, \mu(X))$.

Theorem 3.4. Let $p(\cdot)$ a log-Hölder continuous function. If $(\forall y \in X) \omega(y) \geq 1$, and $\omega \in L^{p'(\cdot)}(X)$ then for every $u \in M_{p(\cdot),\mu}^1(X, \omega)$ and every $\varepsilon > 0$ there exists a Lipschitz function $h \in M_{p(\cdot),\mu}^1(X, \omega)$ such that

- (i) $\mu(\{x \in X : u(x) \neq h(x)\}) \leq \varepsilon$
(ii) $\|u - h\|_{M_{p(\cdot),\mu}^1(X,\omega)} \leq \varepsilon$.

Proof. We fix $u \in M_{p(\cdot),\mu}^1(X,\omega)$ and denote by $g \in L^{p(\cdot)}(X,\omega)$ a Hajlasz gradient of u . We write $E_n = \{x \in X : |u(x)| \leq n \text{ and } g(x) \leq n\}$.
We have

$$\forall y \in E_n^c : n \leq |u(y)| + |g(y)|.$$

Thus

$$\mu(E_n^c) \leq \frac{2^{p^+-1}}{n^{p^+}} \int_{E_n^c} (|u(y)|^{p(y)} + |g(y)|^{p(y)}) d\mu(y)$$

therefore

$$\mu(E_n^c) \leq \frac{2^{p^+-1}}{n^{p^+}} \left(\int_{E_n^c} |u(y)|^{p(y)} d\mu(y) + \int_{E_n^c} \omega(y) |g(y)|^{p(y)} d\mu(y) \right).$$

Thus

$$\mu(E_n^c) \leq \frac{2^{p^+-1}}{n^{p^+}} (\rho_{p(\cdot)}(u) + \rho_{p(\cdot),\omega}(g)),$$

then

$$\lim_{n \rightarrow +\infty} \mu(E_n^c) = 0.$$

The restriction $u|_{E_n}$ is Lipschitz with the constant $2n$. We can extend $u|_{E_n}$ by Mcshane extension [12], to all of X as a Lipschitz function by defining

$$\bar{u}(x) = \inf_{y \in E_n} \{u(y) + 2n \operatorname{dist}(x, y)\}.$$

We put:

$$u_n = (\operatorname{sing} \bar{u}) \min(|\bar{u}|, n).$$

It is clear that u_n is Lipschitz with constant $2n$, $u|_{E_n} = u_n|_{E_n}$, $|u_n| \leq n$ and for all $n \geq 1$

$$\mu(\{x : u(x) \neq u_n(x)\}) \leq \mu(E_n^c) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

On the other hand, we have

$$\rho_{p(\cdot)}(u_n - u) = \int_X |u_n(y) - u(y)|^{p(y)} d\mu(y) = \int_{E_n^c} |u_n(y) - u(y)|^{p(y)} d\mu(y).$$

Thus

$$\rho_{p(\cdot)}(u_n - u) \leq 2^{p^+-1} \left(\int_{E_n^c} |u_n(y)|^{p(y)} d\mu(y) + \int_{E_n^c} |u(y)|^{p(y)} d\mu(y) \right).$$

We have

$$\int_{E_n^c} |u_n(y)|^{p(y)} d\mu(y) \leq \int_{E_n^c} n^{p(y)} d\mu(y) \leq 2^{p^+-1} \int_{E_n^c} (|u(y)|^{p(y)} + |g(y)|^{p(y)}) d\mu(y).$$

Therefore

$$\rho_{p(\cdot)}(u_n - u) \leq 2^{p^+-1} (2^{p^+-1} \int_{E_n^c} (|u(y)|^{p(y)} + \omega(y) |g(y)|^{p(y)}) d\mu(y) + \int_{E_n^c} |u(y)|^{p(y)} d\mu(y)).$$

Since $\lim_{n \rightarrow +\infty} \mu(E_n^c) = 0$, then

$$\lim_{n \rightarrow +\infty} \rho_{p(\cdot)}(u_n - u) = 0.$$

Thus

$$\lim_{n \rightarrow +\infty} \|u_n - u\|_{L^{p(\cdot)}(X)} = 0.$$

It remains to estimate the gradient, let

$$g_n = \begin{cases} 0 & \text{for } x \in E_n, \\ g(x) + 3n & \text{for } x \in E_n^c. \end{cases}$$

It is easy to check that

$$|(u - u_n)(x) - (u - u_n)(y)| \leq d(x, y)(g_n(x) + g_n(y)), \text{ for } \mu - \text{almost all } x, y \in X.$$

We have

$$\int_X \omega(y) |g_n(y)|^{p(y)} d\mu(y) = \int_{E_n^c} \omega(y) |g_n(y)|^{p(y)} d\mu(y).$$

Thus

$$\int_X \omega(y) |g_n(y)|^{p(y)} d\mu(y) \leq 2^{p^+-1} \left(\int_{E_n^c} \omega(y) |g(y)|^{p(y)} d\mu(y) + \int_{E_n^c} \omega(y) (3n)^{p(y)} d\mu(y) \right).$$

We have

$$\int_{E_n^c} \omega(y) (3n)^{p(y)} d\mu(y) \leq \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) (3n)^{p^+} \mu(E_n^c) \|\omega\|_{L^{p'(\cdot)}(X)}.$$

On the other hand we have

$$\mu(E_n^c) \leq \int_{E_n^c} \left(\frac{|u(y)| + |g(y)|}{n} \right)^{p(y)} d\mu(y),$$

thus

$$n^{p^+} \mu(E_n^c) \leq \int_{E_n^c} \left(|u(y)| + |g(y)| \right)^{p(y)} d\mu(y).$$

Hence

$$n^{p^+} \mu(E_n^c) \leq 2^{p^+-1} \left(\int_{E_n^c} |u(y)|^{p(y)} d\mu(y) + \int_{E_n^c} \omega(y) |g(y)|^{p(y)} d\mu(y) \right),$$

which implies

$$n^{p^+} \mu(E_n^c) \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Then

$$\int_{E_n^c} \omega(y) |g(y)|^{p(y)} d\mu(y) \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Thus $\|g_n\|_{L^{p(\cdot)}(X, \omega)} \rightarrow 0$, as $n \rightarrow \infty$. Therefore

$$\lim_{n \rightarrow +\infty} \|u_n - u\|_{M_{p(\cdot), \mu}^1(X, \omega)} = 0.$$

4. Capacity on Musielac- Orlicz-Sobolev space on metric space

Definition 4.1. let T the classe of Borel sets in X , and a function $C : T \rightarrow [0, +\infty]$.

1) C is called capacity if the following axioms are satisfied:

i) $C(\emptyset) = 0$

ii) $E \subset F \Rightarrow C(E) \leq C(F)$, for all E and F in T .

iii) for all sequences $(E_n) \subset T$:

$$C\left(\bigcup_n E_n\right) \leq \sum_n C(E_n).$$

2) C is called outer capacity if for all $E \in T$:

$$C(E) = \inf\{C(O) : O \supset E, O \text{ open}\}$$

3) A property that holds true except perhaps on a set of capacity zero is said to be true C -quasieverywhere, (abbreviated C -q.e.).

4) f and (f_n) are real-valued function finite C -q.e. We say that (f_n) converges to f in C -capacity if:

$$\forall \varepsilon > 0, \lim_{n \rightarrow +\infty} C(\{x : |f_n(x) - f(x)| > \varepsilon\}) = 0.$$

5) f and (f_n) are real-valued function finite C -q.e. We say that (f_n) converges to f C -quasi uniformly, (abbreviated C -q.u) if :

$$\forall \varepsilon > 0, \exists X \in T : C(X) < \varepsilon \text{ and } (f_n) \text{ converges to } f \text{ uniformly, on } X^c$$

Definition 4.2. The Sobolev $C_{p(\cdot),\omega}$ -capacity of the set $E \subset X$ is defined by :

$$C_{p(\cdot),\omega}(E) = \inf_{u \in A_{p(\cdot),\omega}(E)} \|u\|_{M_{p(\cdot),\mu}^1(X,\omega)}$$

Where

$$A_{p(\cdot),\omega}(E) = \{u \in M_{p(\cdot),\mu}^1(X,\omega) : u \geq 1 \text{ on an open set containing } E \text{ and } u \geq 0\}.$$

If $A_{p(\cdot),\omega}(E) = \emptyset$ we set $C_{p(\cdot),\omega}(E) = \infty$. Functions belonging to $A_{p(\cdot),\omega}(E)$ are called admissible functions for E .

Remark 4.1. In the definition of $C_{p(\cdot),\omega}(E)$, we can restrict ourselves to those admissible functions u for which $0 \leq u \leq 1$.

Proof. if $A'_{p(\cdot),\omega}(E) = \{u \in A_{p(\cdot),\omega}(E) : 0 \leq u \leq 1\}$, then $A'_{p(\cdot),\omega}(E) \subset A_{p(\cdot),\omega}(E)$ implies

$$C_{p(\cdot),\omega}(E) \leq \inf_{u \in A'_{p(\cdot),\omega}(E)} \|u\|_{M_{p(\cdot),\mu}^1(X,\omega)}.$$

For the reverse inequality, let $\varepsilon > 0$ and take $u \in A_{p(\cdot),\omega}$ such that

$$\|u\|_{M_{p(\cdot),\mu}^1(X,\omega)} < C_{p(\cdot),\omega}(E) + \varepsilon.$$

Then by lemma 3.5, we have $v = \max(0, \min(u, 1))$ belongs to $A'_{p(\cdot),\omega}(E)$. Thus

$$\inf_{f \in A'_{p(\cdot),\omega}(E)} \|f\|_{M_{p(\cdot),\mu}^1(X,\omega)} \leq \|v\|_{M_{p(\cdot),\mu}^1(X,\omega)} \leq C_{p(\cdot),\omega}(E) + \varepsilon.$$

Letting $\varepsilon \rightarrow 0$, we obtain

$$\inf_{f \in A'_{p(\cdot),\omega}(E)} \|f\|_{M_{p(\cdot),\mu}^1(X,\omega)} \leq C_{p(\cdot),\omega}(E).$$

This complete the proof.

Theorem 4.1. $C_{p(\cdot),\omega}$ is an outer capacity.

Proof. It is obvious that $C_{p(\cdot),\omega}(\emptyset) = 0$ and $C_{p(\cdot),\omega}(E) \leq C_{p(\cdot),\omega}(F)$ if $E \subset F$.

Let $(E_i) \subset X$, so that $\sum_i C_{p(\cdot),\omega}(E_i) < +\infty$, then $C_{p(\cdot),\omega}(E_i) < +\infty; \forall i \in \mathbb{N}$.

Therefore $\forall \varepsilon > 0, \exists u_i \in A_{p(\cdot),\omega}(E_i)$ and $g_{u_i} \in D(u_i) \cap L_\varphi(X)$ so that

$$\|u_i\|_{L^{p(\cdot)}(X)} + \|g_{u_i}\|_{L^{p(\cdot)}(X,\omega)} \leq C_{p(\cdot),\omega}(E_i) + \varepsilon 2^{-i}.$$

We show that $v = \sup_i u_i$ is an admissible function for $\bigcup_{i=1}^\infty E_i$ and $g = \sup_i g_{u_i} \in D(v) \cap L^{p(\cdot)}(X,\omega)$. Let $v_k = \max_{1 \leq i \leq k} u_i$, by lemma 3.5 the function $g_{v_k} = \max_{1 \leq i \leq k} g_{u_i}$ belongs to $D(v_k) \cap L^{p(\cdot)}(X,\omega)$. Since $v_k \rightarrow v$ $\mu.a.e$ and $g_{v_k} \rightarrow g$ $\mu.a.e$, lemma 3.4 gives $v \in M_{p(\cdot),\mu}^1(X,\omega)$.

Clearly $v \geq 1$ in a neighbourhood of $\bigcup_{i=1}^\infty E_i$. Then

$$C_{p(\cdot),\omega}\left(\bigcup_{i=1}^\infty E_i\right) \leq \|v\|_{M_{p(\cdot),\mu}^1(X,\omega)} \leq \sum_{i=1}^\infty (\|u_i\|_{L^{p(\cdot)}(X)} + \|g_{u_i}\|_{L^{p(\cdot)}(X,\omega)}) \leq \sum_{i=1}^\infty C_{p(\cdot),\omega}(E_i) + \varepsilon.$$

The claim follows by letting $\varepsilon \rightarrow 0$. Hence $C_{p(\cdot),\omega}$ is a capacity.

It remains to prove that $C_{p(\cdot),\omega}$ is outer.

Indeed, $C_{p(\cdot),\omega}(E) \leq \inf\{C_{p(\cdot),\omega}(O) : E \subset O, O \text{ is open}\}$. For the reverse inequality, let $\varepsilon > 0$ and $u \in A_{p(\cdot),\omega}(E)$ such that $\|u\|_{M_{p(\cdot),\mu}^1(X,\omega)} \leq C_{p(\cdot),\omega}(E) + \varepsilon$. Since $u \in A_{p(\cdot),\omega}(E)$, there is an open set $O, E \subset O$, such that $u \geq 1$ on O .

Then $C_{p(\cdot),\omega}(O) \leq \|u\|_{M_{p(\cdot),\mu}^1(X,\omega)} \leq C_{p(\cdot),\omega}(E) + \varepsilon$.

The claim follows by letting $\varepsilon \rightarrow 0$.

Theorem 4.2. *If (K_n) is a decreasing sequence of compacts and $K = \bigcap_n K_n$, then*

$$\lim_{n \rightarrow +\infty} C_{p(\cdot),\omega}(K_n) = C_{p(\cdot),\omega}(K).$$

Proof. First, we observe that $\lim_{n \rightarrow +\infty} C_{p(\cdot),\omega}(K_n) \geq C_{p(\cdot),\omega}(K)$.

Let O be an open set containing K . By the compactness of K , $K_i \subset O$ for all sufficiently large i . Thus $\lim_{n \rightarrow +\infty} C_{p(\cdot),\omega}(K_n) \leq C_{p(\cdot),\omega}(O)$ and since $C_{p(\cdot),\omega}$ is an outer capacity, we obtain the claim by taking infimum over all open set O containing K .

Theorem 4.3. *Let $E \subset X$. If there exists $f \in M_{p(\cdot),\mu}^1(X,\omega)$ such that $f = +\infty$ on an open set containing E then, $C_{p(\cdot),\omega}(E) = 0$.*

Proof. If there exists $f \in M_{p(\cdot),\mu}^1(X,\omega)$ such that $f = +\infty$ on an open set O containing E , then $f \geq \alpha$ on O for all $\alpha > 0$. Therefore,

$$\forall \alpha > 0 : C_{p(\cdot),\omega}(E) \leq \frac{1}{\alpha} \|f\|_{M_{p(\cdot),\mu}^1(X,\omega)}.$$

Letting $\alpha \rightarrow +\infty$, we obtain $C_{p(\cdot),\omega}(E) = 0$.

Theorem 4.4. *Let's consider the following propositions:*

- i) $f_n \rightarrow f$ in $M_{p(\cdot),\mu}^1(X,\omega)$ strongly.
 - ii) $f_n \rightarrow f$ in $C_{p(\cdot),\omega}$ - capacity.
 - iii) There is a subsequence (f_{n_j}) such that : $f_{n_j} \rightarrow f$, $C_{p(\cdot),\omega} - q.u.$
 - iv) $f_{n_j} \rightarrow f$, $C_{p(\cdot),\omega} - q.e.$
- We have i) $\Rightarrow$ ii) $\Rightarrow$ iii) $\Rightarrow$ iv).

Proof. Let show that i) $\Rightarrow$ ii).

By theorem 4.3, we have f and f_n are finites for every n ; $C_{p(\cdot),\omega} - q.e.$

Let $\varepsilon > 0$; then

$$C_{p(\cdot),\omega}(\{x : |f_n - f|(x) > \varepsilon\}) \leq \frac{\|f - f_n\|_{M_{p(\cdot),\mu}^1(X,\omega)}}{\varepsilon}.$$

Let show that ii) $\Rightarrow$ iii).

Let $\varepsilon > 0 \exists f_{n_j}$ such that

$$C_{p(\cdot),\omega}(\{x : |f_{n_j} - f|(x) > 2^{-j}\}) < \varepsilon 2^{-j}.$$

We put

$$E_j = \{x : |f_{n_j} - f|(x) > 2^{-j}\} \text{ and } G_m = \bigcup_{j \geq m} E_j.$$

We have $C_{p(\cdot),\omega}(G_m) \leq \sum_{j \geq m} \varepsilon 2^{-j} < \varepsilon$.

On the other hand :

$$\forall x \in (G_m)^c, \forall j \geq m : |f_{n_j} - f|(x) \leq 2^{-j}.$$

Thus $f_{n_j} \rightarrow f$ $C_{p(\cdot),\omega} - q.u.$

Let show that iii) $\Rightarrow$ iv).

We have $\forall j \in \mathbb{N}, \exists E_j \subset X : C_{p(\cdot),\omega}(E_j) \leq \frac{1}{j}$ and $f_{n_j} \rightarrow f$ on $(E_j)^c$.

We put $E = \bigcap_j E_j$, then $C_{p(\cdot),\omega}(E) = 0$ and $f_{n_j} \rightarrow f$ on E^c .

Theorem 4.5. If $f_n, f \in M_{p(\cdot),\mu}^1(X, \omega)$ such that $f_n \rightharpoonup f$ weakly in $M_{p(\cdot),\mu}^1(X, \omega)$, then:

$$\liminf f_n \leq f \leq \limsup f_n, \quad C_{p(\cdot),\omega} - q.e.$$

Proof. By theorem 3.2 $M_{p(\cdot),\mu}^1(X, \omega)$ is reflexive. By the Banach-Saks theorem, there is a subsequence denoted again (f_n) such that the sequence $g_n = \frac{1}{n} \sum_{i=1}^n f_i$ converges to f strongly in $M_{p(\cdot),\mu}^1(X, \omega)$. By theorem 4.4, there is a sub-sequence of (g_n) denoted again (g_n) such that

$$\lim_{n \rightarrow +\infty} g_n = f \quad C_{p(\cdot),\omega} - q.e.$$

On the other hand,

$$\liminf f_n \leq \lim_{n \rightarrow +\infty} g_n.$$

Therefore,

$$\liminf f_n \leq f \quad C_{p(\cdot),\omega} - q.e.$$

For the second inequality, it suffices to replace f_n by $(-f_n)$ in the first inequality.

Theorem 4.6. Assuming that ω is a positive measurable and finite function. Let (O_n) an increasing sequence of open subsets of X and $O = \bigcup_n O_n$, then

$$\lim_{n \rightarrow +\infty} C_{p(\cdot),\omega}(O_n) = C_{p(\cdot),\omega}(O).$$

Proof. We have $\lim_{n \rightarrow +\infty} C_{p(\cdot),\omega}(O_n) \leq C_{p(\cdot),\omega}(O)$. For the reverse Inequality, if $\lim_{n \rightarrow +\infty} C_{p(\cdot),\omega}(O_n) = +\infty$ there is nothing to show.

Assuming that $\lim_{n \rightarrow +\infty} C_{p(\cdot),\omega}(O_n) < +\infty$, we have

$$\forall n \in \mathbb{N}, \exists f_n \in M_{p(\cdot),\mu}^1(X, \omega) : f_n \geq 1 \text{ on } O_n \text{ and } \|f_n\|_{M_{p(\cdot),\mu}^1(X, \omega)} \leq C_{p(\cdot),\omega}(O_n) + \frac{1}{n}.$$

Then

$$\|f_n\|_{L^{p(\cdot)}(X)} + \|g_n\|_{L^{p(\cdot)}(X, \omega)} \leq C_{p(\cdot),\omega}(O_n) + \frac{1}{n}.$$

Where $g_n \in D(f_n) \cap L^{p(\cdot)}(X, \omega)$. Now (f_n) and (g_n) are two bounded sequences in $L^{p(\cdot)}(X)$ and $L^{p(\cdot)}(X, \omega)$ respectively, hence there exists two sub-sequences, which we denote again by (f_n) and (g_n) , such that $f_n \rightharpoonup f$ weakly in $L^{p(\cdot)}(X)$ and $g_n \rightharpoonup g$, weakly in $L^{p(\cdot)}(X, \omega)$. Thus

$$\|f\|_{L^{p(\cdot)}(X)} \leq \liminf_n \|f_n\|_{L^{p(\cdot)}(X)} \text{ and } \|g\|_{L^{p(\cdot)}(X, \omega)} \leq \liminf_n \|g_n\|_{L^{p(\cdot)}(X, \omega)}.$$

Hence

$$\|f\|_{M_{p(\cdot),\mu}^1(X, \omega)} \leq \lim_{n \rightarrow +\infty} C_{p(\cdot),\omega}(O_n).$$

On the other hand we have

$$\forall n \in \mathbb{N} : f_n \geq 1 \text{ on } O_n$$

thus, by theorem 4.5, $f \geq 1$ on O $C_{p(\cdot),\omega} - q.e.$

Let Y be a subset of O where $f \geq 1$, then $C_{p(\cdot),\omega}(O) = C_{p(\cdot),\omega}(Y)$.

Thus,

$$C_{p(\cdot),\omega}(O) \leq \lim_n (C_{p(\cdot),\omega}(O_n)).$$

Theorem 4.7. We have

$$(\exists \alpha > 0) (\forall E \subset X), \mu(E) \leq \alpha \cdot C_{p(\cdot),\omega}(E).$$

Proof. Let $E \subset X$ and $f \in A_{p(\cdot),\omega}(E)$, then $f \geq 1$ on an open set O containing E and $f \geq 0$. We have

$$\mu(E) \leq \mu(O) \leq \int_O f(x) d\mu(x) \leq \int_X f(x) d\mu(x).$$

By Holder inequality we have

$$\int_X f(x) d\mu(x) \leq 2 \|f\|_{L^{p(\cdot)}(X)} \|\mathbf{1}_X\|_{L^{p'(\cdot)}(X)}.$$

Therefore

$$\mu(E) \leq 2 (\mu(X))^\gamma \|f\|_{M_{p(\cdot),\mu}^1(X,\omega)},$$

where

$$\gamma = \begin{cases} p'^-, & \text{if } \|\mathbf{1}_X\|_{L^{p'(\cdot)}(X)} \geq 1, \\ p'^+, & \text{if } \|\mathbf{1}_X\|_{L^{p'(\cdot)}(X)} < 1. \end{cases}$$

We obtain the claim by passing to inf on $f \in A_{p(\cdot),\omega}$.

Corollary 4.1. *If $(f_n)_n$ is a sequence which converges to f in $M_{p(\cdot),\mu}^1(X,\omega)$, then there exists a subsequence of $(f_n)_n$ which converges to f , μ -a.e.*

Proof. It is an immediate consequence of theorem 4.4 and theorem 4.7.

Theorem 4.8. *For each $f \in M_{p(\cdot),\mu}^1(X,\omega)$, there is a $C_{p(\cdot),\omega}$ -quasi continuous function $g \in M_{p(\cdot),\mu}^1(X,\omega)$ such that $f = g$, μ -a.e.*

Proof. Let $f \in M_{p(\cdot),\mu}^1(X,\omega)$. By theorem 3.4, there exists a sequence (f_n) of lipschitz function, such that $f_n \rightarrow f$ in $M_{p(\cdot),\mu}^1(X,\omega)$.

By theorem 4.4, there exists a subsequence of (f_n) denoted again by (f_n) such that $f_n \rightarrow f$ $C_{p(\cdot),\omega}$ -q.u. The claim follows by theorem 4.7.

Theorem 4.9. *We have the following results of the $C_{p(\cdot),\omega}$ capacity :*

1) *If O is an open set of X and $E \subset X$ such that $\mu(E) = 0$, then*

$$C_{p(\cdot),\omega}(O) = C_{p(\cdot),\omega}(O - E).$$

2) *Let u and v are $C_{p(\cdot),\omega}$ -quasicontinuous functions in X , we have*

i) *if $u = v$, almost everywhere in an open set $O \subset X$, then*

$$u = v \quad C_{p(\cdot),\omega} - \text{quasieverywhere in } O,$$

ii) *if $u \leq v$, almost everywhere in an open set $O \subset X$, then*

$$u \leq v \quad C_{p(\cdot),\omega} - \text{quasieverywhere in } O.$$

Proof. 1) It obvious that $C_{p(\cdot),\omega}(O) \geq C_{p(\cdot),\omega}(O - E)$. Let $u \in A_{p(\cdot),\omega}(O - E)$ thus $u \geq 1$ in an open set containing $O - E$. Let the function f define as

$$\begin{cases} f(x) = u(x) & , \text{ if } x \in X - E \\ f(x) = 1 & , \text{ if } x \in E. \end{cases}$$

We have $f \in A_{p(\cdot),\omega}(O)$ and $\|f\|_{M_{p(\cdot),\mu}^1(X,\omega)} = \|u\|_{M_{p(\cdot),\mu}^1(X,\omega)}$, thus

$$C_{p(\cdot),\omega}(O) \leq \|u\|_{M_{p(\cdot),\mu}^1(X,\omega)},$$

and by passing to inf we get $C_{p(\cdot),\omega}(O) \leq C_{p(\cdot),\omega}(O - E)$.

2) Since $C_{p(\cdot),\omega}$ is an outer capacity we get the results by [?].

5. Weighted variable exponents Sobolev spaces with zero boundary values

Let (X, d) be a metric space with finite diameter, ($\text{diam}(X) = \sup_{x,y \in X} d(x, y) < \infty$) and equipped with a finite, positive Borel regular outer, atomless measure μ . The triplet (X, d, μ) will be fixed in the sequel and will be denoted

by X .

ω is a positive measurable and finite function.

$p : X \longrightarrow [1, +\infty)$ is a variable exponent such that $1 < p^- < p^+ < \infty$.

Definition 5.1. Let E be a subset of X , we say that a function u belongs to the weighted variable exponents Sobolev spaces with zero boundary values, and denoted $u \in B_{p(\cdot)}^{1,0}(E, \omega)$, if there is a $C_{p(\cdot), \omega}$ -quasicontinuous functions $\tilde{u} \in M_{p(\cdot), \mu}^1(X, \omega)$ such that $\tilde{u} = u$ almost everywhere in E and $\tilde{u} = 0$ $C_{p(\cdot), \omega}$ -quasi-everywhere in $X \setminus E$.

The space $B_{p(\cdot)}^{1,0}(E, \omega)$ is endowed with the norm

$$\|u\|_{B_{p(\cdot)}^{1,0}(E, \omega)} = \|\tilde{u}\|_{M_{p(\cdot), \mu}^1(X, \omega)}.$$

Remark 5.1. The norm of $B_{p(\cdot)}^{1,0}(E, \omega)$ does not depend on the choice of the $C_{p(\cdot), \omega}$ -quasicontinuous representative since $C_{p(\cdot), \omega}(D) = 0$ implies that $\mu(D) = 0$ for every $D \subset X$.

Theorem 5.1. $B_{p(\cdot)}^{1,0}(E, \omega)$ is a Banach spaces.

Proof. Let $(u_i)_i$ be a Cauchy sequence in $B_{p(\cdot)}^{1,0}(E, \omega)$, for every i there is a $C_{p(\cdot), \omega}$ -quasicontinuous function $\tilde{u}_i \in M_{p(\cdot), \mu}^1(X, \omega)$ such that $\tilde{u}_i = u_i$ almost everywhere in E and $\tilde{u}_i = 0$, $C_{p(\cdot), \omega}$ -quasi-everywhere in $X \setminus E$.

Since $M_{p(\cdot), \mu}^1(X, \omega)$ is a Banach space, there is a function $u \in M_{p(\cdot), \mu}^1(X, \omega)$ such that $\tilde{u}_i \rightarrow u$ in $M_{p(\cdot), \mu}^1(X, \omega)$ as $i \rightarrow \infty$. By theorem 4.4, u is $C_{p(\cdot), \omega}$ -quasicontinuous and by theorem 4.5 we have $u = 0$ $C_{p(\cdot)}$ -quasi-everywhere in $X \setminus E$. Thus $u \in B_{p(\cdot)}^{1,0}(E, \omega)$ and the spaces $B_{p(\cdot)}^{1,0}(E, \omega)$ is complete.

Corollary 5.1. The space $B_{p(\cdot)}^{1,0}(E, \omega)$ is reflexive.

Proof. The space $M_{p(\cdot), \mu}^1(X, \omega)$ is reflexive Banach, by theorem 5.1 the space $B_{p(\cdot)}^{1,0}(E, \omega)$ is closed, and therefore $B_{p(\cdot)}^{1,0}(E, \omega)$ is reflexive.

Lemma 5.1. Let $u \in B_{p(\cdot)}^{1,0}(E, \omega)$ and $v \in M_{p(\cdot), \mu}^1(X, \omega)$ are bounded functions. If v is $C_{p(\cdot), \omega}$ -quasicontinuous, then $uv \in B_{p(\cdot)}^{1,0}(E, \omega)$.

Proof. Let $\tilde{u} \in M_{p(\cdot), \mu}^1(X, \omega)$ be the $C_{p(\cdot), \omega}$ -quasicontinuous representative of u . we have $\tilde{u}v$ is $C_{p(\cdot), \omega}$ -quasicontinuous in X . Let $D = \{x \in X \setminus E : \tilde{u}v \neq 0\}$, we have $D = G \cup H$ where $G = \{x \in X \setminus E : \tilde{u} \neq 0\}$ and $H = \{x \in X \setminus E : v = \infty\}$. It obvious that $C_{p(\cdot), \omega}(G) = 0$ and by theorem 4.3 we have $C_{p(\cdot), \omega}(H) = 0$, thus $C_{p(\cdot), \omega}(D) = 0$ and $\tilde{u}v = 0$ $C_{p(\cdot), \omega}$ -quasi-everywhere in $X \setminus E$. Since $\tilde{u}v = uv$ almost everywhere in E , then $uv \in B_{p(\cdot)}^{1,0}(E, \omega)$.

Theorem 5.2. Let $G \subset E$ such that $C_{p(\cdot), \omega}(G) = 0$, we have

$$B_{p(\cdot)}^{1,0}(E, \omega) = B_{p(\cdot)}^{1,0}(E \setminus G, \omega).$$

Proof. It obvious that $B_{p(\cdot)}^{1,0}(E \setminus G, \omega) \subset B_{p(\cdot)}^{1,0}(E, \omega)$. Let $u \in B_{p(\cdot)}^{1,0}(E, \omega)$ there is a $C_{p(\cdot), \omega}$ -quasicontinuous function $\tilde{u} \in M_{p(\cdot), \mu}^1(X, \omega)$ such that $\tilde{u} = u$ almost everywhere in E and $\tilde{u} = 0$ $C_{p(\cdot), \omega}$ -quasi-everywhere in $X \setminus E$. Since $C_{p(\cdot), \omega}(G) = 0$, we have $\tilde{u} = 0$ $C_{p(\cdot), \omega}$ -quasi-everywhere in $X \setminus (E \setminus G)$. Thus

$$u \in B_{p(\cdot)}^{1,0}(E \setminus G, \omega).$$

Remark 5.2. If $C_{p(\cdot), \omega}(\partial E) = 0$, then $B_{p(\cdot)}^{1,0}(\text{int } E, \omega) = B_{p(\cdot), \omega}^{1,0}(\bar{E}, \omega)$.

Theorem 5.3. *Let O be an open subset of X . Then $B_{p(\cdot)}^{1,0}(E, \omega) = B_{p(\cdot)}^{1,0}(E \setminus O, \omega)$. if and only if $C_{p(\cdot), \omega}(E \cap O) = 0$.*

Proof. It is the same as that given in [14], Theorem 4.8.

6. The Dirichlet Energy Integral Minimisers

In this section, (X, d) is a metric space with finite diameter, $(\text{diam}(X) = \sup_{x, y \in X} d(x, y) < \infty)$ and equipped with a finite, positive Borel regular outer, atomless measure μ . The triplet (X, d, μ) will be fixed in the sequel and will be denoted by X .

ω is a positive measurable and finite function.

$p : X \rightarrow [1, +\infty)$ is a variable exponent such that $1 < p^- < p^+ < \infty$.

Definition 6.1. *Let $h \in M_{p(\cdot), \mu}^1(X, \omega)$. For all $u \in B_{p(\cdot)}^{1,0}(E, \omega)$ we put*

$$I_{E, h}^{p(\cdot), \omega}(u) = \int_E (|u(x)|^{p(x)} + \omega(x)((g_u(x))^{p(x)} + (g_h(x))^{p(x)})) d\mu(x),$$

$I_{E, h}^{p(\cdot), \omega}$ is call the energy operator corresponding to the boundary value function u .

Lemma 6.1. [?] *Let E be a reflexive Banach space. If $I : E \rightarrow \mathbb{R}$ is lower semicontinuous and coercive operator, then there is an element in E that minimizes I .*

Theorem 6.1. *There exists a function $u \in B_{p(\cdot)}^{1,0}(E, \omega)$ such that*

$$I_{E, h}^{p(\cdot), \omega}(u) = \inf_{v \in B_{p(\cdot)}^{1,0}(E, \omega)} I_{\Omega, h}^{p(\cdot), \omega}(v). \quad (6.1)$$

Proof. We have $B_{p(\cdot)}^{1,0}(E, \omega)$ is a reflexive Banach space.

Since $x \rightarrow x^p$ is convex for every fixed $1 < p < \infty$, then $I_{E, h}^{p(\cdot), \omega}$ is convex.

Let (u_i) be a sequence of functions in $B_{p(\cdot)}^{1,0}(E, \omega)$ converging to $u \in B_{p(\cdot)}^{1,0}(E, \omega)$. Then $u_i \rightarrow u$ in $L^{p(x)}(X)$ and $g_{u_i} \rightarrow g_u$ in $L^{p(x)}(X, \omega)$, thus $u_i \rightarrow u$ and $g_{u_i} \rightarrow g_u$ μ almost every where in X and by Fatou's theorem we have $I_{\Omega, h}^{p(\cdot), \omega}$ is lower semicontinuous.

If $\|u\|_{B_{p(\cdot)}^{1,0}(E, \omega)} \rightarrow \infty$ then $I_{\Omega, h}^p(u) \rightarrow \infty$, and therefore the operator $I_{\Omega, h}^{p(\cdot), \omega}$ is coercive, and the theorem follows by lemma 6.1.

References

- [1] Adams, D.R. and Hedberg, L.I. Function spaces and potential theory, Springer (1999).
- [2] Aissaoui, N. and Benkirane, A. Capacité dans les espaces d'Orlicz, *Ann. Sci. Math. Québec* **18** (1) (1994), 1 - 23.
- [3] Aissaoui, N. and Benkirane, A. Potentiel non lineaire dans les espaces d'Orlicz. *Ann. Sci. Math. Québec* **18** (2) (1994), 105 - 118.
- [4] N. Aissaoui, *Strongly nonlineaire potentiel*, Abstract and Applied Analysis, 2002, 357-374.
- [5] David Cruz-Uribe, Lars Diening, HÄSTÖ. *The maximal operator on weighted variable Lebesgue spaces*, Georgian Mathematical Journal 15(4) û January (2008).
- [6] Diening, L., Harjulehto, P., HÄSTÖ, P. and Rudicka, M. Lebesgue and Sobolev spaces with variable exponents. *Lecture Notes in Mathematics*, vol, 2017. Springer, Berlin (2011).
- [7] P. Hajlasz, *Sobolev space on arbitry metric space*, Kluwer academic publishers, 1996, 403-415.
- [8] M.C. Hassib, Y. Akdim, N. Aissaoui and. Musielak-Orlicz-Sobolev spaces on arbitrary metric space. *Commentationes Mathematicae*. vol.

- [9] P. Harjulehto, P. HÄSTÖ and M. Pere, Variable exponent Lebesgue spaces on metric spaces, the Hardy-Littlewood maximal operator, *Real Anal. Exchange* 30 (2005), no. 1, 87-104.
- [10] Ismail Aydin. *Journal of Function Spaces and Applications*. Volume 2012, 17 pages.
- [11] J. Kinnunen and O. Martio, *The Sobolev capacity on metric spaces*, *Annales Academic Scientiarum Fennicae*, 1996, 367-382.
- [12] E. J. McShane, *Extension of range of functions*, *Bull. Am. Math. Soc*, 1934, 837-842.
- [13] Ismail Aydin. *Journal of Function Spaces and Applications*. Volume 2012, 17 pages.
- [14] T. Kilpeläinen, J. Kinnunen, and O. Martio, Sobolev spaces with zero boundary values on metric spaces, *Potential Analysis*, 12 (2000), 233-247.
- [15] Petteri Harjulehto, Peter Hasto and Mikko Pere. *Variable exponent Sobolev spaces on metric measure spaces*, *Funct. Approx. Comment. Math.* 36, 2006, 79-94.
- [16] Takao Ohno and Tetsu Shimomura. Musielak-Orlicz-Sobolev spaces on metric measure spaces. *Czechoslovak mathematical journal*, Volume 65, (2015), Issue 2, pp 435-474.