Abstract. In this paper, we prove the existence of entropy solutions for anisotropic elliptic unilateral problem associated to the equations of the form

\[ - \sum_{i=1}^{N} \partial_i a_i(x, u, \nabla u) - \sum_{i=1}^{N} \partial_i \phi_i(u) = f, \]

where the right hand side \( f \) belongs to \( L^1(\Omega) \). The operator \( - \sum_{i=1}^{N} \partial_i a_i(x, u, \nabla u) \) is a Leray-Lions anisotropic operator and \( \phi_i \in C^0(\mathbb{R}, \mathbb{R}) \).

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1. Introduction

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N (N \geq 2) \) with smooth boundary and let \( 1 < p_1, \ldots, p_N < +\infty \) be a \( N \) real numbers and \( \overrightarrow{p} = (p_1, \ldots, p_N) \). We consider the obstacle problem associated with the following elliptic equations

\[
\begin{cases}
    Au - \text{div}\phi(u) = f & \text{in } \Omega \\
    u = 0 & \text{on } \partial\Omega,
\end{cases}
\]
where $A$ is a Leray-Lions operator from anisotropic space $W^{1,p}(\Omega)$ into its dual $W^{-1,p'}(\Omega)$ defined by $Au = -\text{div}(x, u, \nabla u)$ and $\phi = (\phi_1, ..., \phi_N)$ belongs to $C^0(\mathbb{R}, \mathbb{R})^N$. As regards the second member, we assume that the datum $f$ belongs to $L^1(\Omega)$.

In recent years an increasing interest has turned towards anisotropic elliptic and parabolic equations. A special interest in the study of such equations is motivated by their applications to the mathematical modeling of physical and mechanical processes in anisotropic continuous medium. We refer to the recent works [4, 5, 21] where it is possible to find some references.

In [1, 7, 13] the authors proved the existence of the solutions for some unilateral nonlinear elliptic problem in the classical Sobolev space $W^{1,p}(\Omega)$ and in the Orlicz spaces with $f \in L^1(\Omega) + W^{1,p'}(\Omega)$. L. Boccardo in [12] proved the existence of solutions of some nonlinear Dirichlet problem in $L^1$ involving lower order terms in divergence form.

Boccardo et al. in [11] studied the existence of weak solutions for nonlinear elliptic problem (1.1) with $Au = -\sum_{i=1}^N \frac{\partial}{\partial x_i}(|u|^{p_i-2} \frac{\partial u}{\partial x_i})$, $\phi_i(u) = 0$ for $i = 1, ..., N$ and the right-hand side is a bounded Radon measure on $\Omega$. In the case where $Au = -\sum_{i=1}^N \frac{\partial}{\partial x_i}a_i(x, \frac{\partial u}{\partial x_i})$, $\phi_i(u) = 0$ for $i = 1, ..., N$ and the right hand side $f = (f_1, ..., f_m)^T$ is vector-valued Radon measure on $\Omega$ of finite mass, existence solutions of (1.1) is proved by Bendahma et al. in [5]. We cite some papers that have dealt with the equation (1.1) or similar problems, see [4, 5, 14, 15, 16, 21]. Note that in the isotropic case, there are large works in the direction of problem (1.2) can be found in [3, 6, 7, 8, 23].

The objective of our article is to study the anisotropic unilateral nonlinear elliptic problem associated with the nonlinear problem (1.1). More precisely, we prove the existence of entropy solutions for the following unilateral anisotropic problem.

\[
\begin{aligned}
& u \geq \psi \ \text{a.e. in } \Omega, \\
& T_k(u) \in W^{1,p}(\Omega) \ \forall k > 0, \\
& \sum_{i=1}^N \int_\Omega a_i(x, u, \nabla u) \partial_i T_k(u - v) dx + \sum_{i=1}^N \int_\Omega \phi_i(u) \partial_i T_k(u - v) dx \leq \int_\Omega f T_k(u - v) dx,
\end{aligned}
\]

where $K_\psi = \{u \in W^{1,p}(\Omega), \ u \geq \psi \ \text{a.e. in } \Omega\}$ with $\psi$ is a measurable function on $\Omega$ such that $\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and $T_k$ is the usual truncation function. Note that the existence result is proved by assuming only $\phi$ is continuous function. If we take $\psi = -\infty$, we obtain the existence results of problem (1.2) in the case of equation.

The integrals in (1.2) are well defined: Indeed under the condition (3.2), the function $a_i(x, u, \nabla u)$ is belongs to $L^p(\Omega)$ and since $\partial_i T_k(u - v)$ is belongs to $L^p(\Omega)$, the first integral in the left hand in (1.2) is well defined. For the second integral in the left hand in (1.2), since $\phi_i(u) \partial_i T_k(u - v) = 0$ on $\{|u| > \|v\|_{\infty} + k\}$ and $\phi_i \in C^0(\mathbb{R}, \mathbb{R})$, $\phi_i(u)$ is bounded in $\{|u| \leq \|v\|_{\infty} + k\}$, then the second integral is well defined. Moreover since $f \in L^1(\Omega)$ and $T_k(u - v) \in L^\infty(\Omega)$, the integral in the right hand is well defined.

Since the function $\phi_i(u)$ does not belong to $L^1_{\text{loc}}(\Omega)$ in general, the problem (1.1) does not admit weak solutions. To overcome this difficulty, we use the entropy solutions in this work which introduced for the first time by Bnial et al. in [8].

This paper is organized as follows: Section 2 is devoted to introduce some preliminary results including a brief discussion on the anisotropic Sobolev spaces. Section 3 is devoted to give some important Lemmas. Section 4 contains the main result. Section 5 will be devoted to show the principal proposition concerning the existence of solutions for approximate problems.

2. Preliminaries

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ ($N \geq 2$) with smooth boundary and let $1 < p_1, ..., p_N < \infty$ be $N$ real numbers, $p^+ = \max\{p_1, ..., p_N\}$, $p^- = \min\{p_1, ..., p_N\}$ and $p^\star = (p_1, ..., p_N)$. We denote $\partial_i = \frac{\partial}{\partial x_i}$.
The anisotropic Sobolev space (see [22])

\[ W^{1,p}(\Omega) = \left\{ u \in W^{1,1}(\Omega), \partial_i u \in L^{p_i}(\Omega), i = 1, 2, ..., N \right\} \]

is a Banach space with respect to norm

\[ \| u \|_{W^{1,p}(\Omega)} = \| u \|_{L^{1}(\Omega)} + \sum_{i=1}^{N} \| \partial_i u \|_{L^{p_i}(\Omega)} \]  \quad (2.1)\]

The space \( W^{1,p}(\Omega) \) is the closure of \( C^\infty(\Omega) \) with respect to this norm. Let us recall the Sobolev type inequalities, for \( u \in W^{1,p}(\Omega) \), there exists a constant and \( C \) (see [22]) such that

\[ \| u \|_{L^q(\Omega)} \leq C \prod_{i=1}^{N} \left( \| \partial_i u \|_{L^{p_i}(\Omega)} \right) \]  \quad (2.2)\]

where \( q = \frac{Np}{N-p} \) if \( p < N \) or \( q \in [1, +\infty[ \) if \( p \geq N \), which implies by (2.2)

\[ \| u \|_{L^q(\Omega)} \leq \frac{C}{N} \sum_{i=1}^{N} \left( \| \partial_i u \|_{L^{p_i}(\Omega)} \right) \]  \quad (2.3)\]

When \( p < N \), by (2.3), we have the continuous embedding of \( W^{1,p_0}(\Omega) \) into \( L^q(\Omega) \) for every \( q \in [1, \frac{Np}{N-p}] \). The space \( W^{1,p}(\Omega) \) is separable and reflexive Banach space which satisfies the continuous imbedding \( W^{1,p}(\Omega) \hookrightarrow W^{1,p_0}(\Omega) \) and its dual \((W^{1,p}(\Omega))^\prime\) is denoted by \( W^{-1,p}(\Omega) \).

**Remark 2.1.** As a consequence of the Sobolev imbedding and the continuous imbedding \( W^{1,p_0}(\Omega) \hookrightarrow W^{1,p}(\Omega) \), the imbedding \( W^{1,p_0}(\Omega) \hookrightarrow L^p(\Omega) \) is compact.

Moreover, we consider the space

\[ T^{1,p}_0(\Omega) = \{ u \text{ measurable in } \Omega, T_k(u) \in W^{1,p}_0(\Omega), \forall k > 0 \}, \]

where

\[ T_k(s) = \begin{cases} s & \text{if } |s| \leq k \\ k \frac{s}{|s|} & \text{if } |s| > k. \end{cases} \]

3. Assumptions and Lemmas:

In this section, we give the assumptions of our problem and some technical lemmas. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N (N \geq 2) \) with Lipschitz continuous boundary \( \partial \Omega \).

The functions \( a_i: \Omega \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R} \) are Carathéodory functions satisfying the following conditions, for all \( s \in \mathbb{R}, \xi \in \mathbb{R}^N, \xi' \in \mathbb{R}^N \) and a. e. in \( \Omega \),

\[ \sum_{i=1}^{N} a_i(x, s, \xi) \xi_i \geq \sum_{i=1}^{N} |\xi_i|^{p_i}, \]  \quad (3.1)\]

\[ |a_i(x, s, \xi)| \leq \beta |j_i(x) + |s|^{\frac{1}{p_i}} + |\xi_i|^{p_i-1}|, \]  \quad (3.2)\]

\[ (a_i(x, s, \xi) - a_i(x, s, \xi')) (\xi_i - \xi'_i) > 0 \text{ for } \xi_i \neq \xi'_i, \]  \quad (3.3)\]
where \(a, \beta\) are some positive constants and \(j_i\) is a positive function in \(L^p(\Omega)\).
Moreover, we suppose that
\[
\phi_i \in C_0^0(\mathbb{R}, \mathbb{R}) \quad \text{for } i = 1, ..., N. \tag{3.4}
\]
and
\[
f \in L^1(\Omega). \tag{3.5}
\]
We consider the convex set
\[
K_\psi = \{u \in W_0^{1,p}(\Omega), \ u \geq \psi \ a.e. \ in \ \Omega\}
\]
where \(\psi\) is a measurable function with values in \(\mathbb{R}\) such that
\[
\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{3.6}
\]

**Lemma 3.1.** (19) Let \(g \in L^r(\Omega)\) and let \(g_n \in L^r(\Omega), \ \|g_n\|_{L^r(\Omega)} < c, \ 1 < r < +\infty. \) If \(g_n(x) \to g(x)\) a. e. in \(\Omega\), then \(g_n \to g\) weakly in \(L^r(\Omega)\).

The following lemma generalizes lemma 5 in [13] to the anisotropic case. We utilize the method used in [2] and [13].

**Lemma 3.2.** Assume that (3.1)-(3.3) hold and let \((u_n)_n\) be a sequence in \(W_0^{1,p}(\Omega)\) such that \(u_n \to u \) in \(W_0^{1,p}(\Omega)\) and
\[
\lim_{n \to +\infty} \int_\Omega \left( a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u) \right) \nabla (u_n - u) \, dx = 0. \tag{3.7}
\]
Then \(u_n \to u\) strongly in \(W_0^{1,p}(\Omega)\) for a subsequence.

**Proof** Let \(D_n = \left[ a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u) \right] \nabla (u_n - u)\), by (3.3), \(D_n\) is a positive function and by (3.7), we have \(D_n \to 0\) in \(L^1(\Omega)\) as \(n \to +\infty. \) Since \(u_n \to u\) in \(W_0^{1,p}(\Omega)\), using Remark 2.1, we have \(u_n \to u\) strongly in \(L^p(\Omega)\). Then \(u_n \to u\) a. e. in \(\Omega\) and \(D_n \to 0\) a. e. in \(\Omega\) for a subsequence. Thus there exists a subset \(B\) of \(\Omega\), of zero measure, such that for \(x \in \Omega \setminus B, \ u(x) < +\infty, \ |\nabla u(x)| < +\infty, \ |j_i(x)| < +\infty, \ u_n(x) \to u(x)\) and \(D_n(x) \to 0\). We have
\[
D_n(x) = \sum_{i=1}^N \left[ a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u) \right] [\partial_i u_n - \partial_i u] \\
= \sum_{i=1}^N \left[ a_i(x, u_n, \nabla u_n) \partial_i u_n + a_i(x, u_n, \nabla u) \partial_i u - a_i(x, u_n, \nabla u_n) \partial_i u - a_i(x, u_n, \nabla u) \partial_i u_n \right] \\
\geq \alpha \sum_{i=1}^N |\partial_i u_n|^p_i + \alpha \sum_{i=1}^N |\partial_i u|^p_i - \beta \sum_{i=1}^N \left[ j_i(x) + |u_n|^p_i + |\partial_i u_n|^{p_i-1} \right] |\partial_i u| \\
- \beta \sum_{i=1}^N \left[ j_i(x) + |u|^p_i + |\partial_i u|^{p_i-1} \right] |\partial_i u|, \\
\geq \alpha \sum_{i=1}^N |\partial_i u_n|^p_i - c(x) \left[ 1 + \sum_{i=1}^N |\partial_i u_n|^{p_i-1} + \sum_{i=1}^N |\partial_i u_n| \right], \\
\geq \sum_{i=1}^N |\partial_i u_n|^p_i - c(x) \left[ \frac{\alpha - c(x)}{N|\partial_i u_n|^{p_i}} - \frac{c(x)}{N|\partial_i u_n| - \frac{c(x)}{N|\partial_i u_n|^{p_i-1}}} \right],
\]
where \(c(x)\) is a function which doesn’t depend on \(n\).

Since \(D_n(x) \to 0\) a. e. in \(\Omega\), the last inequality implies that \(\left( \partial_i u_n \right)_n\) is bounded uniformly with respect to \(n\).

Letting \(\xi_i^*\) be an accumulation point of \(\left( \partial_i u_n \right)_n\) for \(i = 1, ..., N\), we have \(|\xi_i^*| < +\infty\) and by the continuity of \(a_i(x, u, \nabla u)\), we obtain
\[
\left( a_i(x, u, \xi^*) - a_i(x, u, \nabla u) \right) (\xi_i^* - \partial_i u) = 0.
\]
Lemma 4.1. We consider the following approximate problems

Step1. Approximate problems.

Assume that (3.1)-(3.6) hold. Then there exists at least an entropy solution of problem (1.1).

Theorem 4.1. \( \phi \) for all \( \psi \).

Definition 4.1. Since \( \partial u \rightarrow \partial \mu \) weakly in \( L^1(\Omega) \).

Moreover, since \( u \), we set \( y_n = \frac{1}{2} a_i(x, u_n, \nabla u_n) \partial_i u_n \) and \( y' = \frac{1}{2} a_i(x, u, \nabla u) \partial_i u \), using Fatou’s lemma, we get

Then, we have

Corollary 4.1. Consequently, we conclude that \( u_n \rightarrow u \) in \( W_{0}^{1, p}(\Omega) \), the proof is complete.

Lemma 3.3. If \( u \in W_{0}^{1, p}(\Omega) \), then \( \sum_{i=1}^{N} \int_{\Omega} \partial_i u dx = 0 \).

Proof: Since \( u \in W_{0}^{1, p}(\Omega) \), there exists \( u_k \in C_{0}^{\infty}(\Omega) \) such that \( u_k \rightarrow u \) strongly in \( W_{0}^{1, p}(\Omega) \).

Moreover, since \( u_k \in C_{0}^{\infty}(\Omega) \), by Green’s Formula, we have

Since \( \partial_i u_k \rightarrow \partial_i \mu \) strongly in \( L^p(\Omega) \), we have \( \partial_i u_k \rightarrow \partial_i \mu \) strongly in \( L^1(\Omega) \).

We pass to limit in (3.8), we conclude that \( \sum_{i=1}^{N} \int_{\Omega} \partial_i u dx = 0 \).

4. Main result

Definition 4.1. A function \( u \in T_{0}^{1, p}(\Omega) \) such that \( u \geq \psi \) a. e. in \( \Omega \) is an entropy solution of the problem (1.1) if

for all \( \varphi \in K_{\psi}(\Omega) \cap L^{\infty}(\Omega) \).

Theorem 4.1. Assume that (3.1)-(3.6) hold. Then there exists at least an entropy solution of problem (1.1).

Proof:

Step1. Approximate problems. We consider the following approximate problems

\[
\begin{align*}
\begin{cases}
    u_n \in K_\psi, \\
    \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i (u_n - v) dx + \sum_{i=1}^{N} \int_{\Omega} \phi_i'(u_n) \partial_i (u_n - v) dx \leq \int_{\Omega} f_n(u_n - v) dx,
\end{cases}
\end{align*}
\]

(4.1)

where \( f_n = T_n(f) \) and \( \phi_i'(s) = \phi_i(T_n(s)) \).

Lemma 4.1. We consider the operator \( \Phi_n : K_\psi \rightarrow W^{-1, p'}(\Omega) \) defined by

\[
< \Phi_n u, v > = \sum_{i=1}^{N} \int_{\Omega} \phi_i(T_n(u)) \partial_i v dx \quad \text{for all } u \in K_\psi \text{ and } v \in W_{0}^{1, p}(\Omega).
\]
The operator $B_n = A + \Phi_n$ is pseudo-monotone and coercive in the following sense; there exists $v_0 \in K_\phi$ such that 
\[
\frac{\langle B_n v, v - v_0 \rangle}{\|v\|^p_{W_0^1, p}(\Omega)} \to +\infty \quad \text{if} \quad \|v\|_{W_0^1, p}(\Omega) \to +\infty \quad \text{and} \quad v \in K_\phi.
\]

For the proof of Lemma 4.1, (see Appendix).

**Proposition 4.1.** Under the conditions (3.1)-(3.6), there exists at least one solution of the problem (4.1).

**Proof.** Thanks to Lemma 4.1 and Theorem 8.2 chapter 2 in [19], there exists at least one solution to the problem (4.1).

**Step 2. A priori estimate.**

**Proposition 4.2.** Assume that (3.1)-(3.6) hold and if $u_n$ is a solution of the approximate problem (4.1). Then there exists a constant $C$ such that

\[
\sum_{i=1}^{N} \int_{\Omega} |\partial_i T_k(u_n)|^p dx \leq C(k + 1) \quad \forall k > 0.
\]

**Proof.** Let $v = u_n - \eta T_k(u_n^+ - \psi^+)$ where $\eta \geq 0$. Since $v \in W_0^1, p(\Omega)$ and for all $\eta$ small enough, we have $v \in K_\phi$. We take $v$ as test function in problem (4.1), we have

\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n^+ - \psi^+) dx + \sum_{i=1}^{N} \int_{\Omega} \phi^i_n(u_n) \partial_i T_k(u_n^+ - \psi^+) dx \leq \int_{\Omega} f_n T_k(u_n^+ - \psi^+) dx.
\]

Which implies that

\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n^+ - \psi^+) dx < \int_{\Omega} f_n T_k(u_n^+ - \psi^+) dx + \sum_{i=1}^{N} \int_{\Omega} |\phi^i_n(u_n)| |\partial_i T_k(u_n^+ - \psi^+)| dx.
\]

Since $\partial_i T_k(u_n^+ - \psi^+) = 0$ on the set $\{u_n^+ - \psi^+ > k\}$, we have

\[
\sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ < k\}} a_i(x, u_n, \nabla u_n) \partial_i (u_n^+ - \psi^+) dx < \int_{\Omega} f_n T_k(u_n^+ - \psi^+) dx
\]

thus, we can write

\[
\sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ < k\}} a_i(x, u_n, \nabla u_n) \partial_i u_n^+ dx \leq \int_{\Omega} f_n T_k(u_n^+ - \psi^+) dx + \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ < k\}} |\phi^i_n(u_n)| |\partial_i u_n^+| dx
\]

\[
+ \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ < k\}} |\phi^i_n(u_n)| |\partial_i \psi^+| dx + \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ < k\}} a_i(x, u_n, \nabla u_n) \partial_i \psi^+ dx
\]

Thanks to Young’s inequalities, we obtain

\[
\sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ < k\}} a_i(x, u_n, \nabla u_n) \partial_i u_n^+ dx \leq \int_{\Omega} f_n T_k(u_n^+ - \psi^+) dx
\]

\[
+ C_1(\alpha) \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ < k\}} |\phi^i_n(T_k + \|\psi\|_\infty(u_n))| |\partial_i u_n^+| dx + \frac{\alpha}{\lambda} \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ < k\}} |\partial_i u_n^+|^p dx
\]

\[
+ \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ < k\}} |\phi^i_n(T_k + \|\psi\|_\infty(u_n))| |\partial_i \psi^+| dx
\]

\[
+ \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ < k\}} \beta |j_i(x)| + |u_n^+|^{p-1} + |\partial_i u_n^+|^p \beta |\partial_i \psi^+| dx
\]

Thanks to (3.2), we have

\[
\sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ < k\}} a_i(x, u_n, \nabla u_n) \partial_i u_n^+ dx \leq \int_{\Omega} f_n T_k(u_n^+ - \psi^+) dx
\]
\[
+C_2(\alpha) \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^l(T_k + \|\psi\|_\infty (u_n))|^p dx + \frac{\alpha}{6} \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\partial_i u_n^+|^p dx \\
+ \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^u(T_k + \|\psi\|_\infty (u_n))||\partial_i \phi^+|^p dx + \beta \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} j_i(x) |\partial_i \phi^+|^p dx \\
+ \beta N \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |u_n^+|^p |\partial_i \phi^+|^p dx + \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\partial_i u_n^+|^p |\partial_i \phi^+|^p dx
\]

Using Young’s inequality, we get
\[
\sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} a_i(x, u_n, \nabla u_n) \partial_i u_n^+ dx \leq \int_{\Omega} f_n T_k (u_n^+ - \psi^+) dx
\]
\[
+ C_3(\alpha) \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^l(T_k + \|\psi\|_\infty (u_n))|^p dx + \frac{\alpha}{6} \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\partial_i u_n^+|^p dx \\
+ \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\phi_i^u(T_k + \|\psi\|_\infty (u_n))||\partial_i \phi^+|^p dx + \beta \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} j_i(x) |\partial_i \phi^+|^p dx \\
+ \beta C_4 \sum_{i=1}^{N} \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\partial_i u_n^+|^p dx + C_5(\alpha) \sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\partial_i \phi^+|^p dx
\]

Using (3.1), (3.4), (3.5) and (3.6), we get
\[
\sum_{i=1}^{N} \int_{\{u_n^+ - \psi^+ \leq k\}} |\partial_i u_n^+|^p dx \leq Ck + C'
\]  
(4.2)

Since \( \{x \in \Omega, u^+ \leq k\} \subset \{x \in \Omega, u^+ - \psi^+ \leq k + \|\psi^+\|_\infty\} \), then
\[
\sum_{i=1}^{N} \int_{\{u^+ \leq k\}} |\partial_i T_k(u_n^+)|^p dx = \sum_{i=1}^{N} \int_{\{u^+ \leq k\}} |\partial_i u_n^+|^p dx \leq \sum_{i=1}^{N} \int_{\{u^+ - \psi^+ \leq k + \|\psi^+\|_\infty\}} |\partial_i u_n^+|^p dx.
\]

Thus, by (4.2), we obtain
\[
\sum_{i=1}^{N} \int_{\Omega} |\partial_i T_k(u_n^+)|^p dx \leq (k + \|\psi^+\|_\infty)C + C' \quad \forall k > 0.
\]  
(4.3)

Similarly, taking \( v = u_n + T_k(u_n) \) as test function in approximate problem (4.1), we obtain
\[
\sum_{i=1}^{N} \int_{\Omega} |\partial_i T_k(u_n^+)|^p dx \leq C'(k + 1).
\]  
(4.4)

Combining (4.3) and (4.4), we get
\[
\sum_{i=1}^{N} \int_{\Omega} |\partial T_k(u_n)|^p dx \leq (k + \|\psi^+\|_\infty + 1)C' \quad \forall k > 0.
\]

**Step 3. Strong convergence of truncations.**

**Proposition 4.3.** If \( u_n \) is a solution of approximate problem (4.1). Then there exists a measurable function \( u \) and a subsequence of \( u_n \) such that
\[
T_k(u_n) \to T_k(u) \text{ strongly in } W^{1,p}_0(\Omega).
\]

**Proof.** Using Proposition 4.1, we obtain
\[
\|T_k(u_n)\|_{W^{1,p}_0(\Omega)} \leq C(k + \|\psi^+\|_\infty + 1)^{\frac{1}{p'}}.
\]  
(4.5)
Now, we will prove that \( (u_n)_n \) is a Cauchy sequence in measure in \( \Omega \). For all \( \lambda > 0 \), we have
\[
\{ |u_n - u_m| > \lambda \} \subset \{ |u_n| > k \} \cup \{ |u_m| > k \} \cup \{ |T_k(u_n) - T_k(u_m)| > \lambda \}
\]
which implies that
\[
\text{meas} \{ |u_n - u_m| > \lambda \} \leq \text{meas} \{ |u_n| > k \} + \text{meas} \{ |u_m| > k \} + \text{meas} \{ |T_k(u_n) - T_k(u_m)| > \lambda \}.
\]
(4.6)

By Hölder’s inequality, Remark 2.1 and (4.5), we have
\[
k \cdot \text{meas} \{ |u_n| > k \} = \int_{\{ |u_n| > k \}} |T_k(u_n)|\,dx \leq \int_{\Omega} |T_k(u_n)|\,dx
\]
\[
\leq \frac{1}{\text{meas}(\Omega)^{\frac{1}{p'}}} \|T_k(u_n)\|_{L^p(\Omega)}
\]
\[
\leq C(\text{meas}(\Omega))^{\frac{1}{p'}} \|T_k(u_n)\|_{W^{1,p}_0(\Omega)}
\]
\[
\leq C(k + \|\psi^+\|_{L^\infty} + 1)^{\frac{1}{p'}}.
\]
Then \( \text{meas} \{ |u_n| > k \} \leq C \left( \frac{1}{k^{1+1/p'}} + \frac{1 + \|\psi^+\|_{L^\infty}}{k^p} \right)^{\frac{1}{p'}} \to 0 \) as \( k \to +\infty \). Which implies that, for all \( \varepsilon > 0 \), there exists \( k_0 \) such that \( \forall k > k_0 \), we have
\[
\text{meas} \{ |u_n| > k \} \leq \frac{\varepsilon}{3} \text{ and } \text{meas} \{ |u_m| > k \} \leq \frac{\varepsilon}{3}.
\]
(4.7)

Moreover, since the sequence \( (T_k(u_n))_n \) is bounded in \( W^{1,p}_0(\Omega) \), there exists a subsequence \( (T_k(u_n))_n \) such that \( T_k(u_n) \) converges to \( v_k \) a.e. in \( \Omega \), weakly in \( W^{1,p}_0(\Omega) \) and strongly in \( L^p(\Omega) \) as \( n \) tends to \( +\infty \). Then the sequence \( (T_k(u_n))_n \) is a Cauchy sequence in measure in \( \Omega \), thus \( \forall \lambda > 0 \), there exists \( n_0 \) such that
\[
\text{meas} \{ |T_k(u_n) - T_k(u_m)| > \lambda \} \leq \frac{\varepsilon}{3}, \quad \forall n, m \geq n_0.
\]
(4.8)

Combining (4.6), (4.7) and (4.8), then for all \( \lambda > 0 \) and for all \( \varepsilon > 0 \), we have
\[
\text{meas} \{ |u_n - u_m| > \lambda \} \leq \varepsilon \quad \forall n, m \geq n_0.
\]

Then \( (u_n)_n \) is a Cauchy sequence in measure in \( \Omega \), then there exists a subsequence denoted by \( (u_n)_n \) such that \( u_n \) converges to a measurable function \( u \) a.e. in \( \Omega \) and
\[
T_k(u_n) \to T_k(u) \text{ weakly in } W^{1,p}_0(\Omega) \text{ and a.e. in } \Omega \quad \forall k > 0.
\]
(4.9)

It remains to prove that
\[
\lim_{n \to \infty} \sum_{i=1}^N \int_{\Omega} \left( a_i(x, T_k(u_n)) - a_i(x, T_k(u)) \right) \left( \partial_i T_k(u_n) - \partial_i T_k(u) \right) \,dx = 0.
\]
(4.10)

Let us take \( v = u_n + T_1(u_n - T_m(u_n))^- \) as test function in approximate problem (4.1), we obtain
\[
- \sum_{i=1}^N \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_1(u_n - T_m(u_n))^- \,dx - \sum_{i=1}^N \int_{\Omega} \phi_i^m(u_n) \partial_i T_1(u_n - T_m(u_n))^- \,dx
\]
\[
\leq - \int_{\Omega} f_i T_1(u_n - T_m(u_n))^- \,dx.
\]
Then
\[
\sum_{i=1}^N \int_{\{m+1 \leq u_n \leq m\}} a_i(x, u_n, \nabla u_n) \partial_i u_n \,dx + \sum_{i=1}^N \int_{\{m+1 \leq u_n \leq -m\}} \phi_i(u_n) \partial_i u_n \,dx
\]
\[
\leq - \int_{\Omega} f_i T_1(u_n - T_m(u_n))^- \,dx.
\]
(4.11)
Using (3.1), (4.12), (4.13) and Lebesgue’s theorem, we obtain

\[
\sum_{i=1}^{N} \int_{\{-(m+1) \leq u_n \leq -m\}} \phi_i(t) \chi_{\{-(m+1) \leq u_n \leq -m\}} dt. \]

Then by Green’s formula, we have

\[
\sum_{i=1}^{N} \int_{\Omega} \phi_i(u_n) \partial_i u_n dx = \sum_{i=1}^{N} \int_{\Omega} \partial_i \Phi_i^m(u_n) dx = 0. \]

Then, we get

\[
\sum_{i=1}^{N} \int_{\{-(m+1) \leq u_n \leq -m\}} a_i(x, u_n, \nabla u_n) \partial_i u_n dx \leq - \int_{\Omega} f_n T_1(u_n - T_m(u_n))^- dx
\]

By Lebesgue’s theorem, we have

\[
\lim_{m \to +\infty} \limsup_{n \to +\infty} \int_{\Omega} f_n T_1(u_n - T_m(u_n))^- dx = 0
\]

Then, we have

\[
\lim_{m \to +\infty} \limsup_{n \to +\infty} \sum_{i=1}^{N} \int_{\{-(m+1) \leq u_n \leq -m\}} a_i(x, u_n, \nabla u_n) \partial_i u_n dx = 0. \tag{4.12}
\]

Similarly, taking \( v = u_n - \eta T_1(u_n - T_m(u_n))^+ \) as test function in approximate problem (4.1), we get

\[
\lim_{m \to +\infty} \limsup_{n \to +\infty} \sum_{i=1}^{N} \int_{\{m \leq u_n \leq m+1\}} a_i(x, u_n, \nabla u_n) \partial_i u_n dx = 0. \tag{4.13}
\]

We consider the following function of one real variable:

\[
h_m(s) = \begin{cases} 
1 & \text{if } |s| \leq m \\
0 & \text{if } |s| \geq m + 1 \\
m + 1 - |s| & \text{if } m \leq |s| \leq m + 1,
\end{cases}
\]

where \( m > k \).

Let \( \varphi = u_n - \eta (T_k(u_n) - T_k(u))^+ h_m(u_n) \) as test function in approximate problem (4.1), we obtain

\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i (T_k(u_n) - T_k(u))^+ h_m(u_n) dx \\
+ \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla u_n) (T_k(u_n) - T_k(u))^+ \partial_i h_m(u_n) dx \\
+ \sum_{i=1}^{N} \int_{\Omega} \phi_i^m(u_n) \partial_i (T_k(u_n) - T_k(u))^+ h_m(u_n) dx \\
+ \sum_{i=1}^{N} \int_{\Omega} \phi_i^m(u_n) \partial_i u_n (T_k(u_n) - T_k(u))^+ h_m(u_n) dx
\]

\[
\leq \int_{\Omega} f_n (T_k(u_n) - T_k(u))^+ h_m(u_n) dx \tag{4.14}
\]

By (4.12) and (4.13), we have the second integral in (4.14) converges to zero as \( n \) and \( m \) tend to \(+\infty\).

Since \( h_m(u_n) = 0 \) if \( |u_n| > m + 1 \), we have

\[
\sum_{i=1}^{N} \int_{\Omega} \phi_i^m(u_n) \partial_i (T_k(u_n) - T_k(u))^+ h_m(u_n) dx = \sum_{i=1}^{N} \int_{\Omega} \phi_i (T_{m+1}(u_n))^+ h_m(u_n) \partial_i (T_k(u_n) - T_k(u))^+ dx.
\]

Using Lebesgue’s theorem, we have \( \phi_i(T_{m+1}(u_n))^+ h_m(u_n) \to \phi_i(T_{m+1}(u))^+ h_m(u) \) in \( L^p(\Omega) \) and \( \partial_i T_k(u_n) \to \partial_i T_k(u) \) weakly in \( L^p(\Omega) \) as \( n \) tends to \(+\infty\), then the third integral in (4.14) converges to zero as \( n \) and \( m \) tend to \(+\infty\). Using (3.1), (4.12), (4.13) and Lebesgue’s theorem, we obtain

\[
\lim_{m \to +\infty} \limsup_{n \to +\infty} \sum_{i=1}^{N} \int_{\{-(m+1) \leq u_n \leq -m\}} |\partial_i u_n|^p (T_k(u_n) - T_k(u))^+ dx = 0 \tag{4.15}
\]
Moreover, we have
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \sum_{i=1}^{N} \int_{\{|u_n| \leq m+1\}} |\partial_i u_n|^p (T_k(u_n) - T_K(u))^+ dx = 0. \tag{4.16}
\]
We deduce that
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i (T_k(u_n) - T_K(u))^+ h_m(u_n) dx \leq 0,
\]
which implies that
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \sum_{i=1}^{N} \int_{\{|u_n| \leq m\}} a_i(x, u_n, \nabla u_n) \partial_i (T_k(u_n) - T_K(u))^+ h_m(u_n) dx
\]
\[
- \lim_{m \to +\infty} \lim_{n \to +\infty} \sum_{i=1}^{N} \int_{\{|u_n| > m\}} a_i(x, u_n, \nabla u_n) \partial_i T_k(u) h_m(u_n) dx \leq 0.
\]
Since \(h_m(u_n) = 0\) in \(\{|u_n| > m + 1\}\), we have
\[
\sum_{i=1}^{N} \int_{\{|u_n| \leq m\}} a_i(x, u_n, \nabla u_n) \partial_i T_k(u) h_m(u_n) dx
\]
\[
= \sum_{i=1}^{N} \int_{\{|u_n| > m\}} a_i(x, u_n, \nabla u_n) \partial_i T_k(u) h_m(u_n) dx.
\]
Since \((a_i(x, T_m+1(u_n), \nabla T_m+1(u_n)))_{n \geq 0}\) is bounded in \(L^p(\Omega)\), we have \(a_i(x, T_m+1(u_n), \nabla T_m+1(u_n))\) converges to \(X_m^i\) weakly in \(L^p(\Omega)\). Then
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \sum_{i=1}^{N} \int_{\{|u_n| \geq m\}} a_i(x, T_m+1(u_n), \nabla T_m+1(u_n)) \partial_i T_k(u) h_m(u_n) dx
\]
\[
= \lim_{m \to +\infty} \sum_{i=1}^{N} \int_{\{|u_n| > m\}} X_m^i \partial_i T_k(u) h_m(u_n) dx = 0,
\]
which implies
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \sum_{i=1}^{N} \int_{\{|u_n| \geq m\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) \partial_i (T_k(u_n) - T_K(u))^+ h_m(u_n) dx \leq 0. \tag{4.17}
\]
Moreover, we have \(a_i(x, T_k(u_n), \nabla T_k(u_n)) h_m(u_n) \to a_i(x, T_k(u), \nabla T_k(u)) h_m(u)\) in \(L^p(\Omega)\)
and \(\partial_i (T_k(u_n) - T_k(u)) \to 0\) weakly in \(L^p(\Omega)\), then
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \sum_{i=1}^{N} \int_{\{|u_n| \geq m\}} a_i(x, T_k(u_n), \nabla T_k(u_n)) \partial_i (T_k(u_n) - T_k(u))^+ h_m(u_n) dx = 0. \tag{4.18}
\]
Combining (3.3), (4.17) and (4.18), we deduce
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \sum_{i=1}^{N} \int_{\{|u_n| \geq m\}} \left( a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u), \nabla T_k(u)) \right) \partial_i (T_k(u_n) - T_k(u))^+ h_m(u_n) dx = 0. \tag{4.19}
\]
Similarly, we take \(\varphi = u_n + (T_k(u_n) - T_k(u))^+ h_m(u_n)\) as test function in approximate problem (4.1), we obtain,
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \sum_{i=1}^{N} \int_{\{|u_n| \leq m\}} \left( a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u), \nabla T_k(u)) \right) \partial_i (T_k(u_n) - T_k(u))^+ h_m(u_n) dx = 0. \tag{4.20}
\]
Combining (4.19) and (4.20) we get
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \sum_{i=1}^{N} \int_{\Omega} \left( a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)) \right) \partial_i(T_k(u_n) - T_k(u)) h_m(u_n) dx = 0.
\]
(4.21)

Now, we prove \( \lim_{m \to +\infty} \lim_{n \to +\infty} \sum_{i=1}^{N} \int_{\Omega} \left( a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)) \right) \partial_i(T_k(u_n) - T_k(u))(1 - h_m(u_n)) dx = 0. \)
(4.22)

Let \( \varphi = u_n + T_k(u_n)(1 - h_m(u_n)) \) as test function in approximate problem (4.1), we obtain
\[
- \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n) (1 - h_m(u_n)) dx + \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i u_n T_k(u_n) - h_m'(u_n) dx
\]
\[
- \sum_{i=1}^{N} \int_{\Omega} \phi_i(u_n) \partial_i T_k(u_n) (1 - h_m(u_n)) dx + \sum_{i=1}^{N} \int_{\Omega} \phi_i'(u_n) \partial_i u_n T_k(u_n) - h_m'(u_n) dx
\]
\[
\leq - \sum_{i=1}^{N} \int_{\Omega} f_i T_k(u_n) (1 - h_m(u_n)) dx
\]
(4.23)

By (4.12) and (4.13), we have \( \lim_{m \to +\infty} \lim_{n \to +\infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i u_n T_k(u_n) - h_m'(u_n) dx = 0. \) Then the second integral in (4.23) converges to zero as \( n \) and \( m \) tends to \(+\infty\). Since \( \partial_i T_k(u_n) \to \partial_i T_k(u) \) in \( L^p(\Omega) \) and \( \phi_i(T_k(u_n))(1 - h_m(u_n)) \to \phi_i(T_k(u))(1 - h_m(u)) \) strongly in \( L^p(\Omega) \), we have
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \sum_{i=1}^{N} \int_{\Omega} \phi_i(u_n) \partial_i T_k(u_n) (1 - h_m(u_n)) dx = \lim_{m \to +\infty} \sum_{i=1}^{N} \int_{\Omega} \phi_i(T_k(u)) \partial_i T_k(u) (1 - h_m(u)) dx.
\]

By Lebesgue’s theorem, we get
\[
\lim_{m \to +\infty} \sum_{i=1}^{N} \int_{\Omega} \phi_i(T_k(u)) \partial_i T_k(u) (1 - h_m(u)) dx = 0.
\]

Then the third integral in (4.23) converges to zero as \( n \) and \( m \) tends to \(+\infty\).

We set \( \Phi_i(t) = \int_{t}^{\infty} \phi_i(s) T_k(s) - h_m'(s) ds \), by Green’s Formula, we have
\[
\sum_{i=1}^{N} \int_{\Omega} \phi_i(u_n) \partial_i u_n T_k(u_n) - h_m'(u_n) dx = \sum_{i=1}^{N} \int_{\Omega} \partial_i \Phi_i(u_n) dx = 0.
\]

Then the fourth integral in (4.23) converges to zero as \( n \) and \( m \) tend to \(+\infty\). By the Lebesgue’s theorem, we have the integral on the right hand in (4.23) converges to zero as \( n \) and \( m \) tend to \(+\infty\). We deduce
\[
\lim_{m \to +\infty} \lim_{n \to +\infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n)(1 - h_m(u_n)) dx = 0.
\]
(4.24)

Besides this, for \( \eta \) small enough, we take \( \varphi = u_n - \eta T_k(u_n^+ - \psi^+)(1 - h_m(u_n)) \) as test function in approximate problem (4.1), we obtain
\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i T_k(u_n^+ - \psi^+)(1 - h_m(u_n)) dx - \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_n, \nabla u_n) \partial_i u_n T_k(u_n^+ - \psi^+ h_m'(u_n)) dx
\]
\[
+ \sum_{i=1}^{N} \int_{\Omega} \phi_i(u_n) \partial_i T_k(u_n^+ - \psi^+)(1 - h_m(u_n)) dx - \sum_{i=1}^{N} \int_{\Omega} \phi_i(u_n) \partial_i u_n T_k(u_n^+ - \psi^+ h_m'(u_n)) dx
\]
Furthermore, we have

\[ \leq \int_\Omega f_n T_k (u_n^+ - \psi^+) (1 - h_m(u_n)) \, dx \]  

By Hölder’s inequality, (3.1), (4.12) and (4.13), we have

\[ \lim \limits_{m \to +\infty} \lim \limits_{n \to +\infty} \sum_{i=1}^{N} \int_\Omega \phi_i^p (u_n) \partial_i u_n T_k (u_n^+ - \psi^+) h_m(u_n) \, dx = 0. \]

Using Young’s inequality, we obtain

\[ \sum_{i=1}^{N} \int_\Omega a_i (x, u_n, \nabla u_n) \partial_i T_k (u_n^+ - \psi^+) (1 - h_m(u_n)) \, dx \leq \]

\[ = \sum_{i=1}^{N} \int_\Omega \phi_i^p (u_n) \partial_i u_n^+ (1 - h_m(u_n)) \, dx + \sum_{i=1}^{N} \int_\Omega \phi_i^p (u_n) \partial_i \psi^+ (1 - h_m(u_n)) \, dx \]

By (4.12), we have the first integral on the right hand converges to zero as \( n \) and \( m \) tend to \( +\infty \). Using the Lebesgue’s theorem, we obtain the second integral in the right hand converges to zero as \( n \) and \( m \) tend to \( +\infty \). Since

\[ \sum_{i=1}^{N} \int_{u_n^+ - \psi^+ \leq k} \phi_i^p (u_n) \partial_i u_n^+ (1 - h_m(u_n)) \, dx \]

\[ = \sum_{i=1}^{N} \int_\Omega \phi_i^p (T_k (\| \psi^+ \|_{L^\infty(\Omega)}) (u_n)) \partial_i T_k (\| \psi^+ \|_{L^\infty(\Omega)}) (u_n^+)(1 - h_m(u_n)) \, dx. \]

Since \( \partial_i T_k (\| \psi^+ \|_{L^\infty(\Omega)}) (u_n^+) \rightharpoonup \partial_i T_k (\| \psi^+ \|_{L^\infty(\Omega)}) (u^+) \) weakly in \( L^p(\Omega) \) and \( \phi_i^p (T_k (\| \psi^+ \|_{L^\infty(\Omega)}) (u_n)) (1 - h_m(u_n)) \to \phi_i (T_k (\| \psi^+ \|_{L^\infty(\Omega)}) (u))(1 - h_m(u_n)) \) strongly in \( L^p(\Omega) \), we have

\[ \sum_{i=1}^{N} \int_\Omega \phi_i^p (T_k (\| \psi^+ \|_{L^\infty(\Omega)}) (u)) \partial_i T_k (\| \psi^+ \|_{L^\infty(\Omega)}) (u_n^+)(1 - h_m(u))^dx \]

\[ = \sum_{i=1}^{N} \int_\Omega \phi_i (T_k (\| \psi^+ \|_{L^\infty(\Omega)}) (u)) \partial_i T_k (\| \psi^+ \|_{L^\infty(\Omega)}) (u)(1 - h_m(u)) \, dx + \varepsilon(n). \]

By Lebesgue’s theorem, we have

\[ \lim \limits_{m \to +\infty} \sum_{i=1}^{N} \int_\Omega \phi_i (T_k (\| \psi^+ \|_{L^\infty(\Omega)}) (u)) \partial_i T_k (\| \psi^+ \|_{L^\infty(\Omega)}) (u)(1 - h_m(u)) \, dx = 0. \]

Then, we have the third integral converges to zero as \( n \) and \( m \) tend to \( +\infty \). Similarly as (4.24), we obtain

\[ \lim \limits_{m \to +\infty} \lim \limits_{n \to +\infty} \sum_{i=1}^{N} \int_{u_n > 0} a_i (x, u_n, \nabla u_n) \partial_i T_k (u_n)(1 - h_m(u_n)) = 0. \]  

Combining (4.24) and (4.27), we get

\[ \lim \limits_{m \to +\infty} \lim \limits_{n \to +\infty} \sum_{i=1}^{N} \int_\Omega a_i (x, u_n, \nabla u_n) \partial_i T_k (u_n)(1 - h_m(u_n)) \, dx = 0. \]

Furthermore, we have

\[ \sum_{i=1}^{N} \int_\Omega \left( a_i (x, T_k (u_n), \nabla T_k (u_n)) - a_i (x, T_k (u_n), \nabla T_k (u_n)) \right) \left( \partial_i T_k (u_n) - \partial_i T_k (u) \right) \, dx \]
which implies that, 

**Step 4. Passing to the limit.**

Now, let 

\[ \varphi \in K_{P} \cap L^{\infty}(\Omega), \]

we take \( v = u_{n} - T_{k}(u_{n} - \varphi) \) as test function in approximate problem (4.1), we obtain

\[
\sum_{i=1}^{N} \int_{\Omega} a_{i}(x, u_{n}, \nabla T_{k}(u_{n})) \partial_{i} T_{k}(u_{n} - \varphi) \, dx + \sum_{i=1}^{N} \int_{\Omega} f_{i}(u_{n}) \partial_{i} T_{k}(u_{n} - \varphi) \, dx
\]

\[ \leq \int_{\Omega} f_{n} T_{k}(u_{n} - \varphi) \, dx, \]

which implies that,

\[
\sum_{i=1}^{N} \int_{\Omega} a_{i}(x, T_{k+\|\varphi\|_{\infty}(u_{n}), \nabla T_{k+\|\varphi\|_{\infty}(u_{n}))} \partial_{i} T_{k}(u_{n} - \varphi) \, dx + \sum_{i=1}^{N} \int_{\Omega} f_{i}(T_{k+\|\varphi\|_{\infty}(u_{n})) \partial_{i} T_{k}(u_{n} - \varphi) \, dx
\]

\[ \leq \int_{\Omega} f_{n} T_{k}(u_{n} - \varphi) \, dx. \]

Since \( T_{k}(u_{n}) \rightarrow T_{k}(u) \) strongly in \( W_{0}^{1, \frac{n}{n}}(\Omega) \) and a.e. in \( \Omega \) \( \forall k > 0 \), we have

\[ a_{i}(x, T_{k+\|\varphi\|_{\infty}(u_{n}), \nabla T_{k+\|\varphi\|_{\infty}(u_{n}))} \rightarrow a_{i}(x, T_{k+\|\varphi\|_{\infty}(u), \nabla T_{k+\|\varphi\|_{\infty}(u))} \text{ weakly in } L^{p}(\Omega), \]

\[ f_{i}(T_{k+\|\varphi\|_{\infty}(u_{n})) \rightarrow f_{i}(T_{k+\|\varphi\|_{\infty}(u)) \text{ strongly in } L^{p}(\Omega) \text{ and } \partial_{i} T_{k}(u_{n} - \varphi) \rightarrow \partial_{i} T_{k}(u - \varphi) \text{ strongly in } L^{p}(\Omega) \]

we can pass to limit in

\[
\begin{align*}
\forall \varphi \in K_{P} & \cap L^{\infty}(\Omega) \text{ and } \forall k > 0, \\
\sum_{i=1}^{N} \int_{\Omega} a_{i}(x, u_{n}, \nabla u_{n}) \partial_{i} T_{k}(u_{n} - \varphi) \, dx + \sum_{i=1}^{N} \int_{\Omega} f_{i}(u_{n}) \partial_{i} T_{k}(u_{n} - \varphi) \, dx & \leq \int_{\Omega} f_{n} T_{k}(u_{n} - \varphi) \, dx,
\end{align*}
\]

this completes the proof of theorem 4.1.
5. Appendix

In this section, we will show that the operator $B_u = A + \Phi_n$ is coercive and pseudo-monotone.

**Proof of Lemma 4.1.** We consider the operator $\Phi_n : K_\varphi \to W^{-1, p'}(\Omega)$ defined by

$$< \Phi_n u, v > = \sum_{i=1}^{N} \int_{\Omega} \phi_i(u) \partial_i v \, dx.$$ By Hölder’s inequality, we have, for all $u, v \in X$,

$$| < \Phi_n u, v > | \leq \sum_{i=1}^{N} \left( \int_{\Omega} |\phi_i(T_n(u))|^p \, dx \right)^\frac{1}{p} \left( \int_{\Omega} |\partial_i v|^p \, dx \right)^\frac{1}{p}$$

$$\leq \sum_{i=1}^{N} \left( \int_{\Omega} |\phi_i(s)|^p \, dx \right)^\frac{1}{p} \left( \int_{\Omega} |\partial_i v|^p \, dx \right)^\frac{1}{p}$$

$$\leq \sum_{i=1}^{N} \left( \int_{\Omega} |\phi_i(s)| + 1 \right)^p \left( \int_{\Omega} |\partial_i v|^p \, dx \right)^\frac{1}{p}$$

$$\leq \sum_{i=1}^{N} (\sup_{|s| \leq n} |\phi_i(s)| + 1) \left( \int_{\Omega} |\partial_i v|^p \, dx \right)^\frac{1}{p}$$

$$\leq \max_{1 \leq i \leq N} (\sup_{|s| \leq n} |\phi_i(s)| + 1)(\text{meas}(\Omega) + 1)^\frac{1}{p} \sum_{i=1}^{N} \left( \int_{\Omega} |\partial_i v|^p \, dx \right)^\frac{1}{p}$$

$$\leq C(n) ||v||_{W_0^{1,p}(\Omega)}$$

which implies that $| < \Phi_n u, v > | \leq C(n)$. Moreover, let $v_0 \in K_\varphi$, thanks to Hölder’s inequality and (3.2), we have

$$| < A v, v_0 > | \leq \sum_{i=1}^{N} \left( \int_{\Omega} |a_i(x, v, \nabla v)|^p \, dx \right)^\frac{1}{p} \left( \int_{\Omega} |\partial_i v_0|^p \, dx \right)^\frac{1}{p}$$

$$\leq \beta \sum_{i=1}^{N} \left( \int_{\Omega} (f_i(x) + |v|^p + |\partial_i v|^p) \, dx \right)^\frac{1}{p} \left( \int_{\Omega} |\partial_i v_0|^p \, dx \right)^\frac{1}{p}$$

$$\leq \beta \sum_{i=1}^{N} \left( C_1 + \int_{\Omega} |\partial_i v|^p + \int_{\Omega} |\partial_i v|^p \partial_i v \, dx \right)^\frac{1}{p} \left( \int_{\Omega} |\partial_i v_0|^p \, dx \right)^\frac{1}{p}$$

$$\leq \beta \sum_{i=1}^{N} C_1 \left( 1 + \frac{2}{C_1} \int_{\Omega} |\partial_i v|^p \partial_i \, dx \right)^\frac{1}{p} \left( \int_{\Omega} |\partial_i v_0|^p \, dx \right)^\frac{1}{p}$$

$$\leq \beta C_2 \sum_{i=1}^{N} \left( 1 + \frac{2}{C_1} \int_{\Omega} |\partial_i v|^p \partial_i \, dx \right)^\frac{1}{p} \left( \int_{\Omega} |\partial_i v_0|^p \, dx \right)^\frac{1}{p}$$

$$\leq \beta C_2 \left( 1 + C_3 \left( \sum_{i=1}^{N} |\partial_i v|^p \partial_i \right)^\frac{1}{p} \right) \sum_{i=1}^{N} \left( \int_{\Omega} |\partial_i v_0|^p \, dx \right)^\frac{1}{p}$$

$$\leq \beta C_2 \left( 1 + C_3 \left( \sum_{i=1}^{N} |\partial_i v|^p \partial_i \right)^\frac{1}{p} \right) ||v_0||_{W_0^{1,p}(\Omega)}.$$
We will prove that

\[ <A_n v, v - v_0> \leq \frac{1}{n} \sum_{i=1}^{N} \int_{\Omega} \left| \sum_{i=1}^{N} \frac{\partial |v|^p}{p} \right| dx \leq \frac{1}{n} \int_{\Omega} \left| \sum_{i=1}^{N} \frac{\partial |v|^p}{p} \right| dx \\|v\|_{W_0^{1,p}(\Omega)} \leq \frac{1}{n} \int_{\Omega} \left| \sum_{i=1}^{N} \frac{\partial |v|^p}{p} \right| dx \\|v\|_{W_0^{1,p}(\Omega)} \]

Additionally, by Jensen’s inequality, we have

\[ \|v\|_{1,\frac{p}{p}}^p = \left( \sum_{i=1}^{N} \int_{\Omega} \left| \partial_i \|v\|^p \right| dx \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{N} \int_{\Omega} \left| \partial_i \|v\|^p \right| dx \right)^{\frac{1}{p}} \leq C \sum_{i=1}^{N} \int_{\Omega} \left| \partial_i \|v\|^p \right| dx, \]

where

\[ p^+ = \begin{cases} p^- & \text{if } \|\partial_i v\|_{L^p(\Omega)} \geq 1 \\ p^+ & \text{if } \|\partial_i v\|_{L^p(\Omega)} < 1. \end{cases} \]

Then

\[ \sum_{i=1}^{N} \int_{\Omega} \left| \partial_i \|v\|^p \right| dx \rightarrow +\infty \text{ and } \sum_{i=1}^{N} \int_{\Omega} \left| \partial_i \|v\|^p \right| dx \rightarrow +\infty \text{ as } \|v\|_{W_0^{1,p}(\Omega)} \rightarrow +\infty. \]

Using (5.1), we get

\[ <A_n v, v - v_0> \rightarrow +\infty \text{ as } \|v\|_{1,\frac{p}{p}} \rightarrow +\infty. \]

Since \( <\Phi_n v, v \rangle \text{ and } <\Phi_n v, v \rangle \text{ are bounded, then we have} \)

\[ <B_n v, v - v_0> = \frac{1}{\|v\|_{W_0^{1,p}(\Omega)}} <A_n v, v - v_0> + \frac{1}{\|v\|_{W_0^{1,p}(\Omega)}} <\Phi_n v, v - v_0> \rightarrow +\infty \text{ as } \|v\|_{W_0^{1,p}(\Omega)} \rightarrow +\infty. \]

We deduce that the operator \( B_n = A + \Phi_n \) is coercive. It remains to prove that the operator \( B_n \) is pseudo-monotone. Let \((u_k)_k \) be a sequence in \( W_0^{1,p}(\Omega) \) such that

\[ \begin{cases} u_k \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega) \\ B_n u_k \rightharpoonup \chi \text{ weakly in } W^{-1,p'}(\Omega) \\ \limsup_{k \rightarrow +\infty} <B_n u_k, u_k > \leq <\chi, u>. \end{cases} \]

We will prove that \( \chi = B_n u \) and \( <B_n u_k, u_k > \rightarrow <\chi, u > \text{ as } k \rightarrow +\infty. \) Since \( W_0^{1,p}(\Omega) \hookrightarrow L^{p'}(\Omega), \) then \( u_k \rightarrow u \) strongly in \( L^{p'}(\Omega) \) and a.e. in \( \Omega \) for a subsequence denoted again \((u_k)_k \). Since \((u_k)_k \) is bounded in \( W_0^{1,p}(\Omega), \) by (3.2), we have \((a_i(x, u_k, \nabla u_k))_k \) is bounded in \( L^{p'}(\Omega). \) Then there exists a function \( \varphi_i \in L^{p'}(\Omega) \) such that

\[ a_i(x, u_k, \nabla u_k) \rightarrow \varphi_i \text{ as } k \rightarrow +\infty. \]

What is more, since \( (\varphi_i^p(u_k))_k \) is bounded in \( L^{p'}(\Omega) \) and \( \varphi_i^p(u_k) \rightarrow \varphi_i^p(u) \) a.e. in \( \Omega, \) we have

\[ \varphi_i^p(u_k) \rightarrow \varphi_i^p(u) \text{ strongly in } L^{p'}(\Omega) \text{ as } k \rightarrow +\infty. \]
For all \( v \in W^{1,p}_0(\Omega) \), using (5.2) and (5.3), we obtain
\[
<\chi_v, v> = \lim_{k \to +\infty} <B_n u_k, v>
\]
\[
= \lim_{k \to +\infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i v dx + \lim_{k \to +\infty} \sum_{i=1}^{N} \int_{\Omega} \phi_i^+(u_k) \partial_i v dx
\]
\[
= \sum_{i=1}^{N} \int_{\Omega} \phi_i \partial_i v dx + \sum_{i=1}^{N} \int_{\Omega} \phi_i^+(u) \partial_i v dx.
\]
Hence, we have
\[
\limsup_{k \to +\infty} <B_n u_k, u_k> = \limsup_{k \to +\infty} \left[ \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i u_k dx + \sum_{i=1}^{N} \int_{\Omega} \phi_i^+(u_k) \partial_i u_k dx \right]
\]
\[
\leq <\chi, u>
\]
\[
= \sum_{i=1}^{N} \int_{\Omega} \phi_i \partial_i u dx + \sum_{i=1}^{N} \int_{\Omega} \phi_i^+(u) \partial_i u dx
\]
which implies that
\[
\limsup_{k \to +\infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i u_k dx \leq \sum_{i=1}^{N} \int_{\Omega} \phi_i \partial_i u dx. \quad (5.4)
\]
By (3.3), we have \( \sum_{i=1}^{N} \int_{\Omega} \left( a_i(x, u_k, \nabla u_k) - a_i(x, u_k, \nabla u) \right) (\partial_i u_k - \partial_i u) dx > 0. \)

Then
\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i u_k dx \geq \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u) \partial_i u_k dx + \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i u dx.
\]

Using (5.2), we get
\[
\liminf_{k \to +\infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i u_k dx \geq \sum_{i=1}^{N} \int_{\Omega} \phi_i \partial_i u dx. \quad (5.5)
\]
Combining (5.4) and (5.5), we obtain
\[
\lim_{k \to +\infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i u_k dx = \sum_{i=1}^{N} \int_{\Omega} \phi_i \partial_i u dx. \quad (5.6)
\]
\[
\lim_{k \to +\infty} <B_n u_k, u_k> = \lim_{k \to +\infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i u_k dx + \lim_{k \to +\infty} \sum_{i=1}^{N} \int_{\Omega} \phi_i^+(u_k) \partial_i u_k dx
\]
\[
= \sum_{i=1}^{N} \int_{\Omega} \phi_i \partial_i u dx + \sum_{i=1}^{N} \int_{\Omega} \phi_i^+(u) \partial_i u dx
\]
\[
= <\chi, u>.
\]
In addition to this, since \( a_i(x, u_k, \nabla u) \) converges to \( a_i(x, u, \nabla u) \) strongly in \( L^{p_i}(\Omega) \), by (5.6), we obtain
\[
\sum_{i=1}^{N} \int_{\Omega} \left( a_i(x, u_k, \nabla u_k) - a_i(x, u_k, \nabla u) \right) (\partial_i u_k - \partial_i u) dx = 0.
\]
Using lemma 3.2, we get \( u_k \) converges to \( u \) strongly in \( W^{1,\overline{p}}_0(\Omega) \) and a.e. in \( \Omega \), then \( a_i(x, u_k, \nabla u) \) converges to \( a_i(x, u, \nabla u) \) weakly in \( L^{p_i}(\Omega) \) and \( \phi_i^+(u_k) \) converges to \( \phi_i^+(u) \) strongly in \( L^{p_i}(\Omega) \). Then for all \( v \in W^{1,\overline{p}}_0(\Omega) \), we have
\[ < \chi, v > = \lim_{k \to +\infty} < B_n u_k, v > \]
\[ = \lim_{k \to +\infty} \sum_{i=1}^{N} \int_{\Omega} a_i(x, u_k, \nabla u_k) \partial_i v dx + \lim_{k \to +\infty} \sum_{i=1}^{N} \int_{\Omega} \phi_i(u_k) \partial_i v dx \]
\[ = \sum_{i=1}^{N} \int_{\Omega} a_i(x, u, \nabla u) \partial_i v dx + \sum_{i=1}^{N} \int_{\Omega} \phi_i(u) \partial_i v dx \]
\[ = < B_n u, v > \]
which implies that \( B_n u = \chi \).

References


