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Harmonic numbers, harmonic series and zeta function

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Abstract. This paper reviews, from different points of view, results on Bernoulli numbers and polynomials, the distribution of prime numbers in connexion with the Riemann hypothesis. We give an account on the theorem of G. Robin, as formulated by J. Lagarias. The other parts are devoted to the series $\mathcal{M}i_s(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} z^n$. A significant result is that the real part f of

$$\sum \frac{\mu(n)}{n} e^{2in\pi\theta}$$

is an example of a non-trivial real-valued continuous function f on the real line which is 1-periodic, is not odd and has the property $\sum_{h=1}^{n} f(h/k) = 0$ for every positive integer k.

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1. Introduction

This is an expanded version of two lectures delivered at Chapman University, initially devoted to the theorems of Guy Robin and Jeffrey Lagarias concerning the formulation of the Riemann hypothesis in terms of harmonic numbers and the sum of divisors, but which very quickly overflowed to various topics of number theory, all or almost all of which are around the arithmetical Möbius function $\mu(n)$ and Bernoulli numbers. This is why these notes are too fragmentary and they reflect, maybe, the taste and the views of the author. To keep the paper to a reasonable size only very few proofs are given. The paper contains nine sections (some are very short). I aimed to include many topics, but I had to omit many of them because of the limitation of time and space. The first section is mainly connected with the difference equation $f(x+1) - f(x) = g(x)$. Almost every property of Bernoulli

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numbers or Bernoulli polynomials come from this equation. We will show that, conversely, Bernoulli numbers intervene well to express certain forms of solutions to the difference equation. The third and fourth sections deal with some subseries. We mention only few examples for which we give the exact sum. Some of these examples are borrowed from the theory of zeta functions of number fields. The fifth section is devoted to the so-called explicit formulas. Some properties of the Eisenstein series are presented, since they are the generating series of the arithmetical functions that are the sums of divisors. This section also contains the formulations of the theorems of G. Robin and J. Lagarias. These two results were the first motivations behind the two lectures and finally these notes. We end this section by Vacca formula and a relation giving the Euler constant γ as an integral of a very lacunary series used intensely in the dynamics of paper folding. The very short sixth section deals with some probabilistic number theory. Developing more this theme would have required voluminous additions, in time and space. The seventh section is probably the richest of this work. It uses advanced properties of the Riemann zeta function, as well as the prime number theorem. One finds in particular an interesting result on continuous functions f on $(0,1)$ of which all the sums of Riemann, corresponding to regular subdivisions are equal to zero:

$$\sum_{0<h\leq k} f\left(\frac{h}{k}\right) = 0, \quad k = 1,2,3\ldots.$$

This example is given in [3] and our theorem (7.7) gives it its right interpretation.

The eighth section deals with the basics of the so-called Leroy-Lindelof theory, which makes it possible, for example, to study the properties of power series whose coefficients are of the form $\varphi(n)$ where φ is an entire functions of zero exponential type. This section also suffered from a lack of time to be developed as it deserves. The ninth section again reviews the Bernoulli polynomials in terms of addition formulas (kubert identities) and their equivalent in the lemniscatic case.

2. Harmonic series

2.1. The formal operator $\dfrac{e^D-1}{D}$ and the difference equation. The harmonic numbers, the related Bernoulli numbers or Bernoulli polynomials and the Gamma function are all related to the difference equation. There are several approaches to this equation and we choose the method by Fourier integrals. For suitable f, we define

$$F_+(w) = \int_0^\infty f(x)e^{ixw}dx, \quad F_-(w) = \int_{-\infty}^0 f(x)e^{ixw}dx$$

Proposition 2.1. *The difference equation for given $b < a$*

$$f(x+1) - f(x) = g(x) \tag{2.1}$$

has as solution

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{ia-\infty}^{ia+\infty} \frac{\chi(w)+G_+(w)}{e^{-iw}-1}e^{-ixw}dw - \frac{1}{\sqrt{2\pi}}\int_{ib-\infty}^{ib+\infty} \frac{\chi(w)+G_-(w)}{e^{-iw}-1}e^{-ixw}dw$$

where $\chi(w)$ is holomorphic in the strip $b < \Im w < a$. The terms involving $\chi(w)$ merely represent a function of period 1, which is obviously part of the solution, a solution of the homogeneous equation. Hence

$$f(x) = p(x) + \frac{1}{\sqrt{2\pi}}\int_{ia-\infty}^{ia+\infty} \frac{G_+(w)}{e^{-iw}-1}e^{-ixw}dw - \frac{1}{\sqrt{2\pi}}\int_{ib-\infty}^{ib+\infty} \frac{G_-(w)}{e^{-iw}-1}e^{-ixw}dw$$

where $p(x)$ is any function of period 1.

All the computations are valid in L^2 if for example $g(x)e^{-c|x|} \in L^2(\mathbb{R})$ for some real $c > 0$. Furthermore solutions, for general $g(x)$ without any growth condition, were given by Guichard, Appell, Hurwitz, Picard (and many others) [45]. Some solutions, related more or less to Bernoulli polynomials, exist. To be more precise we consider the difference equation (2.1) with the second member g an entire function such that the restriction to the real line is in the Schwartz space $\mathcal{S}(\mathbb{R})$. We can solve this equation by right iteration, left iteration or localization. We denote by D the differentiation operator $\frac{d}{dx}$. By Taylor's formula the equation (2.1) takes the form

$$\left(e^{D}-1\right) f(u) = g(u)$$

or

$$\left(1-e^{-D}\right) f(u) = g(u-1).$$

The formal inverse of the operator $(1-e^{-D})$ is $\sum_{n\geq 0} e^{-nD}$ so that the equation (2.1) has the solution

$$f_1(u) = \sum_{m\geq 1} g(u-m)$$

which is in fact an entire function, the series being a series of holomorphic functions on $\mathbb{C}$ uniformly convergent on compact sets of the complex plane.
The equation (2.1) written now in the form $\left(1-e^{D}\right) f(u) = -g(u)$ can be solved formally by right iteration

$$f_2(u) = -\sum_{m\geq 0} g(u+m)$$

and we obtain an another entire solution of (2.1).
The third method to solve (2.1) is by localization. We define the operator D^{-1} by

$$D^{-1}g(u) = \int_0^u g(x)dx.$$

This is a branch of the inverse of the differentiation operator $D = \frac{d}{dx}$. The equation (2.1) takes the form

$$\left(\frac{e^{D}-1}{D}\right) f(u) = \int_0^u g(x)\,dx. \tag{2.2}$$

The function $\frac{z}{e^z-1}$ is holomorphic in the disk centered at the origin and of radius 2π. It has the power series expansion

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} \frac{b_n}{n!} z^n, \quad |z| < 2\pi.$$

The coefficients b_n are the Bernoulli numbers, $b_0 = 1$, $b_1 = -\frac{1}{2}$, $b_2 = \frac{1}{6}$, $b_3 = 0$, $b_4 = \frac{1}{30}$... (see (4.1) for another prsentation). A formal solution of the equation (2.1) is

$$f_3(u) = D^{-1}g(u) + \sum_{n\geq 1} \frac{b_n}{n!} D^n D^{-1} g(u).$$

The difference $f_1(u) - f_2(u)$ is a solution of the homogenous equation

$$\mathcal{H}(u+1) - \mathcal{H}(u) = 0$$

so it is a periodic function of period 1.
The harmonic numbers are defined by

$$H_n = 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{n}$$

or equivalently by

$$H_1 = 1,\ H_n = H_{n-1} + \frac{1}{n}.$$

Their generating series is

$$-\frac{\log(1-x)}{1-x} = \sum H_n x^n,\ |x| < 1.$$

The harmonic series is by definition

$$H_\infty = 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{n} + \cdots$$

There are several proofs for the divergence of this series. We retain here the observation that for positive integers m, n with $m < n$,

$$S_{m,n} = \frac{1}{m} + \frac{1}{m+1} + \cdots + \frac{1}{n}$$

is never an integer, and that $S_{n,2n}$ is a Riemann sum, so that

$$\lim_{n\to\infty} S_{n,2n} = \lim_{n\to\infty} \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n} = \int_0^1 \frac{dx}{1+x} = \log 2 = 0.69314718056\cdots$$

However the sequence $(\gamma_n)_{n\geq 1}$, $\gamma_n = H_n - \log n$ converges to the Euler constant γ, with the following expansion

$$\gamma = \gamma_n - \frac{1}{2n} + \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \cdots$$

3. Sub-series of the harmonic series

3.1. Kempner series.

An interesting sub-series of the harmonic series is

$$\sum{}' \frac{1}{n}$$

where the prime indicates that the sum is over all positive n, whose decimal expansion has no nines. The series was first studied by A. J. Kempner in 1914. The series is interesting because of the counter-intuitive result that, unlike the harmonic series, it converges. Baillie showed that, rounded to 20 decimals, the actual sum is 22.92067 66192 64150 34816.

The previous result could be related to a very open conjecture of Erdös on arithmetic progressions):

Conjecture 1 (Erdös). *Suppose that*

$$A = \{a_1 < a_2 < \cdots\}$$

is an infinite sequence of integers such that $\sum \frac{1}{a_i} = \infty$. *Then A contains arbitrarily long arithmetic progressions.*

The origin of this conjecture is a result of van der Waerden stating that if the integers are partitioned into finitely many subsets, one of these contains arithmetic progressions of arbitrary finite length. Erdös conjectured that any subset of the integers of positive asymptotic density would possess arithmetic progressions of arbitrary finite length. Roth showed that was the case for arithmetic progressions of length 3. Erdös conjecture is proved by Szemerédi in [54]. A different proof is given by Furstenberg in [21] using ergodic theory. Szemerédi's method is combinatorial and makes use of van der Waerden's theorem. Gowers [23] gave still another proof using both Fourier analysis and combinatorics (see *Remark* (5.1)).

3.2. Madelung's constants.

The electric potential V_i of all ions of a lattice felt by the ion at position r_i [13]

$$V_i = K \sum_{j\neq i} \frac{z_j}{r_{ij}}$$

where $r_{ij} = |r_i - r_j|$ is the distance between the i-th and the j-th ions. In addition, z_j is number of charges of the j-th ion and K is a physical constant. If the mutual distances r_{ij} are normalized to the nearest neighbor distance r_0 the potential may be written

$$V_i = \frac{K}{r_0} \sum_{j\neq i} \frac{z_j r_0}{r_{ij}} = \frac{K}{r_0} M_i$$

with M_i being the Madelung constant of the i-th ion

$$M_i = \sum_{j \neq i} \frac{z_j}{r_{ij}/r_0}.$$

The electrostatic energy of the ion at site r_i is the product of its charge with the potential acting at its site

$$E_{el,i} = z_i e V_i = \frac{e^2}{4\pi\epsilon_0 r_0 M_i}.$$

It can be shown [13] that for $NaCl$, the Madelung constants are

$$M_{Na} = -M_{Cl} = \sum_{j,k,l}^{\infty}{}' \frac{(-1)^{j+k+l}}{(j^2+k^2+l^2)^{1/2}} = \sum_{j,k,l}^{\infty}{}' \frac{(-1)^{j^2+k^2+l^2}}{(j^2+k^2+l^2)^{1/2}}$$

The prime indicates that the term $j = k = \ell = 0$ is omitted. We will return to the analytic properties of this series later. An alternative formulation is

$$M_{Na} = \sum_{n=1}^{\infty} (-1)^n \frac{r_3(n)}{\sqrt{n}} \tag{3.1}$$

with $r_3(n)$ the number of ways in which the integer n can be written as the sum of three squares. Legendre's three-square theorem states that a natural number can be represented as the sum of three squares of integers $n = x^2 + y^2 + z^2$ if and only if n is not of the form $n = 4^a(8b+7)$ for integers a and b [25]. The generating function for $r_3(n)$ is given by

$$\sum_{n=0}^{\infty} r_3(n) x^n = \vartheta_3^3(x) = 1 + 6x + 12x^2 + 8x^3 + 6x^4 + 24x^5 + 24x^6 + 12x^8 + 30x^9 + ...,$$

where $\vartheta_3(x)$ is one of the four Jacobi theta functions

$$\vartheta_3(x) = 1 + 2\sum_{n=0}^{\infty} x^{n^2}.$$

Theorem 3.1. (Emersleben [18]*) The series in* (3.1)

$$\sum_{n=1}^{\infty} (-1)^n \frac{r_3(n)}{\sqrt{n}}$$

diverges.

We give a proof, different from [8], supported by the following equivalent behaviors

$$\sum_{1 \le n \le x} r_3(n) = \frac{4}{3}\pi x^{\frac{3}{2}} + \mathrm{O}(x^{\frac{3}{4}}), \tag{3.2}$$

$$\sum_{1 \le n \le x} \frac{r_3(n)}{\sqrt{n}} = 2\pi x + \mathrm{O}(\sqrt{x}). \tag{3.3}$$

Indeed if the series (3.1) converges, then $\dfrac{r_3(n)}{\sqrt{n}}$ tend to 0. The Cesàro sums

$$c_n = \frac{1}{n}\sum_{1}^{n} \frac{r_3(k)}{\sqrt{k}}$$

tends also to 0. Hence

$$\frac{1}{n^{\frac{3}{2}}} \sum_{1 \le k \le n} r_3(k)$$

tend to 0 which contradicts (3.2).

The problem of the computation of the number $r_k(n)$ of representations of n by k squares lies at the crossroads of many ideas, analytic, algebraic and geometric (confer section (6)). The sequence $r_2(n)$ is particularly elegant. Similar to the relations (3.2) and (3.3) we have for $r_2(n)$ [27], [4];

$$\sum_{1\leq n\leq x} r_2(n) = \pi x + \sqrt{x}\sum_{n=1}^{\infty}\frac{r_2(n)}{\sqrt{n}}J_1\left(2\pi\sqrt{nx}\right) \tag{3.4}$$

$$\sum_{1\leq n\leq x}\frac{r_2(n)}{\sqrt{x-n}} = 2\pi x\sqrt{x} + \sum_{n=1}^{\infty}\frac{r_2(n)}{\sqrt{n}}\sin\left(2\pi\sqrt{nx}\right) \tag{3.5}$$

The ordinary Bessel function $J_\nu(z)$ of order ν is defined by

$$J_\nu(z) = \sum_{n=0}^{\infty}\frac{(-1)^m(z/2^{2m+\nu}}{m!\Gamma(m+1+\nu)}, \quad |z| < \infty.$$

The presence of Bessel's functions is simply fascinating and mysterious.

4. Some known formulas and Hilbert polynomials

Some known (and less known) series are

$$\begin{aligned}
\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots &= \frac{\pi^2}{6}\\
\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots &= \frac{\pi^4}{96}\\
\frac{1}{1^3}-\frac{1}{2^3}+\frac{1}{3^3}-\cdots &= \frac{\pi^3}{32}\\
\frac{1}{1^5}-\frac{1}{2^5}+\frac{1}{3^5}-\cdots &= \frac{5\pi^5}{1536}.
\end{aligned}$$

These values are all related to the analytic properties of zeta functions of number fields, finite degree field extensions of the field of rational numbers $\mathbb{Q}$, even that the equalities

$$\sum_{k=1}^{\infty}\frac{1}{k^{2n}} = (-1)^{n-1}\frac{(2\pi)^{2n}}{2(2n)!}B_{2n}, \tag{4.1}$$

where B_{2n} are the Bernoulli Numbers, given by

$$\frac{z}{e^z-1}+\frac{z}{2}-1 = \sum_{n=1}^{\infty}B_{2n}\frac{z^{2n}}{(2n)!}, \quad |z| < 2\pi, \tag{4.2}$$

can be merely shown by classical Fourier series methods. A very interesting arithmetical property of Bernoulli Numbers is given by

Theorem 4.1 (Von Staudt-Clausen)**.**

$$B_{2n} + \sum_{\substack{p\,\text{prime}\\ p-1|2n}}\frac{1}{p} \in \mathbb{Z}.$$

We give an overview of the method used to prove these formulas. We choose, as an example, the third. Let N be a natural number and $(\mathbb{Z}/N\mathbb{Z})^\times$ the group of units in the ring $\mathbb{Z}/N\mathbb{Z}$. A Dirichlet character χ, modulo N, is by definition a multiplicative groups homomorphism

$$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times.$$

We define the Dirichlet L-function, with respect to χ by

$$L(s,\chi) = \sum_{n=1}^{\infty}\frac{\chi(n)}{n^s},$$

where $\chi(n)$ is defined as $\chi(n \bmod N)$ if n and N are relatively prime, and 0 otherwise. The third formula can be written

$$\frac{1}{1^3} - \frac{1}{2^3} + \frac{1}{3^3} - \cdots = L(3, \chi_4) = \frac{\pi^3}{32}$$

where

$$\chi_4 : (\mathbb{Z}/4\mathbb{Z})^\times = \{1 \bmod 4,\ \ 3 \bmod 4\} \longrightarrow \mathbb{C}^\times$$

$$\chi_4(1 \bmod 4) = 1, \quad \chi_4(3 \bmod 4) = -1.$$

With this character (the only non-trivial Dirichlet character modulo 4) we get the classical Dirichlet beta function

$$L(s, \chi_4) = \sum_{n=0} \frac{(-1)^n}{(2n+1)^s} = \beta(s).$$

We define a sequence of rational functions $h_n(t)_{n \geq 1}$ by

$$h_1(t) = \frac{1+t}{2(1-t)}, \tag{4.3}$$

$$h_n(t) = \left(t\frac{d}{dt}\right)^{n-1} h_1(t), \quad n \geq 1, \tag{4.4}$$

The operator $\vartheta = t\dfrac{d}{dt}$, sometimes called Theta operator or Boole's operator, is very interesting and it is possible to make explicit the powers ϑ^n as differential operators in $\dfrac{d}{dt}$, with monic coefficients, that is as elements of $\mathbb{C}[t][\dfrac{d}{dt}]$ [51]. Actually we have:

Proposition 4.1. *For every smooth function f of the variable t*

$$\begin{aligned}
\vartheta^n f =& t\frac{df}{dt} + \Big[\frac{1^{n-1}}{1-2} + \frac{2^{n-1}}{2-1}\Big] t^2 \frac{d^2 f}{dt^2} \\
&+ \Big[\frac{1^{n-1}}{(1-2)(1-3)} + \frac{2^{n-1}}{(2-1)(2-3)} + \frac{3^{n-1}}{(3-1)(3-2)}\Big] t^3 \frac{d^3 f}{dt^3} \\
&+ \Big[\frac{1^{n-1}}{(1-2)(1-3)(1-4)} + \frac{2^{n-1}}{(2-1)(2-3)(2-4)} + \frac{3^{n-1}}{(3-1)(3-2)(3-4)} + \\
&+ \frac{4^{n-1}}{(4-1)(4-2)(4-3)}\Big] t^4 \frac{d^4 f}{dt^4} \\
&+ \cdots \\
&+ \Big[\frac{1^{n-1}}{(1-2)(1-3)\cdots(1-n)} + \frac{2^{n-1}}{(2-1)(2-3)\cdots(2-n)} + \frac{3^{n-1}}{(3-1)(3-2)(3-4)\cdots(3-n)} \\
&\qquad + \frac{n^{n-1}}{(n-1)(n-2)\cdots(n-(n-1))}\Big] t^n \frac{d^n f}{dt^n}
\end{aligned}$$

This is a local problem and by the Stone-Weierstrass theorem, we restrict ourselves to monic functions $f(x) = x^k$, $k \in \mathbb{Z}^+$ as test functions. It is equivalent to show the following lemma:

Lemma 4.1. *For each real a and positive integer n:*

$$
\begin{aligned}
a^n =& a + \Big[\frac{1^{n-1}}{1-2} + \frac{2^{n-1}}{2-1}\Big] a(a-1) \\
&+ \Big[\frac{1^{n-1}}{(1-2)(1-3)} + \frac{2^{n-1}}{(2-1)(2-3)} + \frac{3^{n-1}}{(3-1)(3-2)}\Big] a(a-1)(a-2) \\
&+ \Big[\frac{1^{n-1}}{(1-2)(1-3)(1-4)} + \frac{2^{n-1}}{(2-1)(2-3)(2-4)} + \frac{3^{n-1}}{(3-1)(3-2)(3-4)} + \\
&+ \frac{4^{n-1}}{(4-1)(4-2)(4-3)}\Big] a(a-1)(a-2)(a-3) \\
&+ \cdots \\
&+ \Big[\frac{1^{n-1}}{(1-2)(1-3)\cdots(1-n)} + \frac{2^{n-1}}{(2-1)(2-3)\cdots(2-n)} + \frac{3^{n-1}}{(3-1)(3-2)(3-4)\cdots(3-n)} \\
&\qquad + \frac{n^{n-1}}{(n-1)(n-2)\cdots(n-(n-1))}\Big] a(a-1)(a-2)\cdots(a-(n-1)).
\end{aligned}
$$

The Hilbert polynomials $H_j(X)$ are extensions of binomial coefficients [47]:

$$H_0(X) = 1, \quad H_j(X) = \binom{X}{j} = \frac{X(X-1)\cdots(X-j+1)}{j!}. \tag{4.5}$$

The family $(H_j(X))_{0\le j\le n}$ is an algebraic basis of the space $\mathbb{R}_n[X]$ of polynomials of degree at most n. For each $n \in \mathbb{Z}^+$ we have

$$X^n = \sum_{j\ge 0} A_{nj} H_j(X) \tag{4.6}$$

with

$$A_{nj} = 0 \quad j > n.$$

Several formulas for A_{kj} are known, in particular [47] (p.2-4)

$$A_{nj} = \sum_{l_i>0, l_1+l_2+\cdots l_j = n} \frac{n!}{l_1! l_2! \cdots l_j!} \tag{4.7}$$

or

$$A_{nj} = \sum_{m=0}^{j} (-1)^m \binom{j}{m} (j-m)^n = \Delta^j X^n{}_{|X=0} \tag{4.8}$$

where Δ is the difference operator $\Delta f(x) = f(x+1) - f(x)$. This ends the proof of the lemma (4.1).
It is possible to give another approach to the problem of the calculation of $(xD)^n f$ by integral representations, at least for the class of analytic functions of exponential type, in order to explain and to fix our ideas. We recall the following theorem of Wigert type [28].

Theorem 4.2. *In order that $f(z)$ should be an integral function of order 1 and type γ it is necessary and sufficient that $f(z)$ should be of the form*

$$f(z) = \frac{1}{2i\pi} \int_{\mathcal{C}} e^{zu} \phi(u)\, du \tag{4.9}$$

where $\mathcal{C}$ is a contour which includes the circle $\{|u| = \gamma\}$ and ϕ is holomorphic in the open disc $\{|u| < \gamma\}$. In this case we have the inversion formula

$$\phi(u) = \int_0^\infty f(x) e^{xu}\, dx, \quad \gamma < u. \tag{4.10}$$

Remark 4.1. *An example of this situation is given by the following integral*

$$f(z) = \int_\Gamma e^{z\sigma} \Phi(\sigma)\, d\sigma,$$

where Γ is a curve of finite length and Φ is continuous on Γ. The $f(z)$ is an entire function of exponential type at most one and its Borel transform is

$$\phi(u) = \int_\Gamma \frac{\Phi(\sigma)}{u-\sigma}\, d\sigma.$$

Hence the Borel transform of $f(z)$ is the Cauchy transform of $\Phi(\sigma)$. It is holomorphic outside Γ and vanishing at ∞.

The formula (4.9) gives, with $D_z = \dfrac{d}{dz}$:

$$(zD_z)^n f(z) = \frac{1}{2i\pi} \int_C (zD_z)^n e^{zu} \phi(u)\, du. \tag{4.11}$$

This integral representation has the advantage to reduce the problem to the computation of $(zD_z)^n e^{zu}$ which is a classical one. We introduce the exponential polynomials Φ_n by

$$(xD_x)^n e^x = \Phi_n(x) e^x.$$

They admit the generating series:

$$e^{x(e^z-1)} = \sum_{n=0}^{\infty} \Phi_n(x) \frac{z^n}{n!}. \tag{4.12}$$

These polynomials satisfy many identities, through derivations of (4.12) with respect to x or to z. The equality (4.11) becomes the following integral representation

$$(zD_z)^n f(z) = \frac{1}{2i\pi} \int_C \Phi_n(zu) e^{zu} \phi(u)\, du$$

which is a kind of a multiplicative convolution.

Remark 4.2. *If we were interested by Cauchy type representation formula, we could use*

$$(zD_z)^n \frac{1}{u-z} = \sum_{k=1}^{\infty} \frac{k^n}{u^{k+1}} z^k.$$

A particular case is, for $n \in \mathbb{Z}^+$

$$(zD_z)^n \frac{z}{1-z} = \sum_{k=1}^{\infty} k^n z^k. \tag{4.13}$$

The inverses of the Theta operator $\vartheta = x\dfrac{d}{dx}$ are enough interesting. A possible one is given by

$$\vartheta^{-1} F(x) = \int_0^x F(u) \frac{du}{u}$$

acting on functions F for which the integral converges. An easy induction shows that for $n \in \mathbb{Z}^+$

$$(zD_z)^{-n} \frac{z}{1-z} = \Phi(z, n) \tag{4.14}$$

where

$$\Phi(z, s) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

is the classical polylogarithm function. The formulas (7.1) *and* (7.2) *extended by defining $(zD_z)^0$ as the identity operator give $(zD_z)^n \dfrac{z}{1-z}$ for each $n \in \mathbb{Z}$.*

On the other hand if we could obtain for the operator $\left(z^3 \dfrac{d}{dz}\right)^p$ *the equivalent of the proposition* (4.1), *then we could obtain the same result for the operator* $\left(\dfrac{d}{zdz}\right)^p$ *by using the following relation, due to Glaisher* [56], *(p.109):*

$$z^{p+1}\left(\frac{d}{z\,dz}\right)^p = \frac{1}{z^{p+1}}\left(z^3\frac{d}{dz}\right)^p \frac{1}{z^{2p-2}}$$

For each positive integer n we have in (4.4):

$$h_r(t) \in \mathbb{Q}[t, \frac{1}{1-t}].$$

If $x \in \mathbb{C} \setminus \mathbb{Z}$ and $t = e^{2i\pi x}$ we have for (4.3)

$$h_1(t) = -\frac{1}{2}\frac{1}{2i\pi}\lim_{N\to+\infty}\sum_{-N}^{N}\left(\frac{1}{x+n}+\frac{1}{x-n}\right) = \frac{i}{4}\cot \pi x, \tag{4.15}$$

and differentiating r times, $r \geq 2$

$$h_r(t) = (r-1)!\left(-\frac{1}{2i\pi}\right)^r \sum_{n\in\mathbb{Z}} \frac{1}{(x+n)^r}. \tag{4.16}$$

An application of (4.15) and (4.16) gives [35] (p.86):

Theorem 4.3. *Let $N \geq 1$ be a natural number, χ a Dirichlet character modulo N, and r a natural number. Assume that $\chi(-1) = (-1)^r$. If we put $\zeta_N = e^{2i\pi/N}$ then we have*

$$L(r,\chi) = \frac{1}{(r-1)!}\left(\frac{-2i\pi}{N}\right)^r \frac{1}{2}\sum_{a\in(\mathbb{Z}/N\mathbb{Z})^\times} \chi(a) h_r(\zeta_N^a).$$

This gives the desired equality

$$\frac{1}{1^3} - \frac{1}{2^3} + \frac{1}{3^3} - \cdots = \frac{1}{(3-1)!}\left(\frac{-2i\pi}{4}\right)^3 \left(h_3(i) - h_3(i^3)\right) = \frac{\pi^3}{32}.$$

4.1. Analytic methods. The Riemann zeta function is the holomorphic function defined by the absolutely convergent Dirichlet series for $\Re s > 1$

$$\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s}.$$

It is known that the zeta function extends to a meromorphic function on all of $\mathbb{C}$, with a unique simple pole at $s = 1$ with residue 1. The Laurent expansion of the Riemann zeta function $\zeta(s)$ at its pole $s = 1$ can be written in the form

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\gamma_n (s-1)^n$$

with

$$\gamma_n = \lim_{N\to\infty}(\sum_{k=1}^{N}\frac{\log^n k}{k} - \frac{\log^{n+1} N}{n+1}).$$

$\gamma_0 = \gamma$ is the well known Euler constant, and, for $n \geqq 1$, γ_n is sometimes called generalized Euler constants ([5], Entry 13, p.80). Moreover, the analytic continuation of $\zeta(s)$ satisfies the functional equation

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s).$$

The function $(s-1)\zeta(s)$ is entire and has the Hadamard's factorization [55] (p. 30):

$$(s-1)\zeta(s) = \frac{e^{(\log 2\pi - 1 - \frac{1}{2}\gamma)s}}{2\Gamma(\frac{s}{2}+1)}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}} = \frac{1}{2}e^{(\log 2\pi - 1)s}\prod_{n>0}\left(1+\frac{s}{2n}\right)e^{-\frac{s}{2n}}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}},$$

where ρ describes the non trivial zeros of the function ζ. We have the Dirichlet series expansion for its inverse

$$\frac{1}{\zeta(s)} = \sum_{n\geq 1} \frac{\mu(n)}{n^s},$$

where the Möbius arithmetical function is defined as follows:

$$\mu(n) = \begin{cases} 1 \text{ if } n = 1 \\ \\ 0 \text{ if } n \text{ has one or more repeated prime factors} \\ \\ (-1)^k \text{ if } n \text{ is the product of } k \text{ prime factors.} \end{cases}$$

The prime number theorem, in its simplest form, asserts that

$$\pi(x) = \sharp\{p \text{ prime}, p < x\} \sim \frac{x}{\log x}.$$

For example consider $x = 10^{21}$. We know from a computation of X. Gourdon based on earlier work of M. Deléglise, J. Rivat, and P. Zimmermann [14] (p.11 and the references therein) that

$$\pi(10^{21}) = 21127269486018731928.$$

The prime Number theorem is equivalent to a property on the harmonic series

$$\sum_{n\geq 1} \frac{\mu(n)}{n} = 0. \tag{4.17}$$

The Riemann zeta function plays an important role in prime number theory, arising because of the famous Euler product formula, which expresses $\zeta(s)$ as a product over primes, in the right half plane $\{\Re s > 1\}$. In fact

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1}$$

There are several formulas for $\pi(x)$. For instance in 1898 H. Laurent [38] gave the following formula for the counting function:

$$\pi(n) = 2 + \sum_{k=5}^{n} \frac{e^{2i\pi\frac{(k-1)!}{k}} - 1}{e^{-2i\pi\frac{1}{k}} - 1},$$

but this formula does not seem to be of any help in giving an asymptotic of $\pi(x)$.

The prime zeta function is an analogue of the Riemann zeta function, studied by Glaisher [22], [20]. It is defined as the following infinite series, which converges for $\Re(s) > 1$

$$P(s) = \sum_{p \in \text{primes}} \frac{1}{p^s} = \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \cdots$$

The series converges absolutely for for $\Re s > 1$. It can be shown that $P(s)$ can be continued analytically to the strip $0 < \Re s < 1$. As has been shown by Landau and Walfisz [37] the function $P(s)$ cannot be continued beyond the imaginary axis $\Re s = 0$. This is due to the fact that we have a clustering of singular points along the imaginary axis emanating from the non-trivial zeros of the zeta function on the critical line $\Re s = \frac{1}{2}$. On the real axis we have singular points for $s = \frac{1}{k}$ where k runs through all positive integers without a square factor, that is for

$$s = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{10}, \frac{1}{11}, \frac{1}{13} \cdots .$$

The Euler product expansion for the Riemann zeta function $\zeta(s)$ implies that

$$\log \zeta(s) = \sum_{n>0} \frac{P(ns)}{n},$$

which by Möbius inversion gives

$$P(s) = \sum_{n>0} \mu(n) \frac{\log \zeta(ns)}{n}.$$

According to (4.1) the values $P(2k)$ are given series involving Bernoulli numbers. For example [15]:

$$\begin{aligned} P(2) &= \sum_{n>0} \mu(n) \frac{\log \zeta(2n)}{n} = \sum_{n>0} \frac{\mu(n)}{n} \log \left(\frac{(2\pi)^{2n}}{2(2n)!} |B_{2n}| \right) \\ &= 0.452247420041065498506543364832247\ldots, \end{aligned}$$

whence

$$\zeta(2) = \frac{\pi^2}{6} = 1.644934066848226436472415166646025\cdots$$

When s goes to 1, we have

$$P(s) \sim \log \zeta(s) \sim \log \left(\frac{1}{s-1} \right).$$

More precisely [20]

$$P(1+\epsilon) = \log \frac{1}{\epsilon} + C + \mathrm{O}(\epsilon)$$

with

$$C = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log \zeta(n) = -0.315718452\cdots$$

Actually there exists a constant $C \in \mathbb{R}$ such that, for any $x \geq 3$, we have the Mertens formula

$$\sum_{1 \leq p \leq x} \frac{1}{p} = \log \log x + \mathrm{O}(x)$$

or, more precisely

$$\sum_{1 \leq p \leq x} \frac{1}{p} = \log \log x + C + \mathrm{O}((\log x)^{-2}).$$

Several properties of the function $P(s)$ are given in [9]. It is denoted there by $\zeta_P(s)$. Let $\mathcal{P}$ the set of all prime numbers. It is an infinite set. Then

$$\pi(x) := \sharp \{p \in \mathcal{P},\ p < x\}$$

is the number of primes less than x with $\frac{1}{2}$ added when x is prime. The n-th prime number p_n is asymptotically $n \log n$. For, if p_n is the n-th prime number and π is the prime-counting function we have $\pi(p_n) = n$. Now, the Prime number theorem implies that

$$n \sim \frac{p_n}{\log p_n},$$

which shows that $\log p_n \sim \log n$ and hence $p_n \sim n \log n$. The probability for an integer n to be prime is $\frac{1}{\log n}$.

5. Explicit formula

5.1. According to Riemann. The following formula, due to Riemann (1860), involving the integral logarithm function $\mathrm{Li}(x) = \int_0^x \frac{dt}{\log t}$ relates the distribution of prime numbers to the non-trivial zeros ρ of the zeta function. More precisely Riemann considered another weighted prime counting function, denoted by $\Pi(x)$, related to the harmonic series and defined by

$$\Pi(x) = \sum_{\substack{p^r < x, \\ p \text{ prime}}} \frac{1}{r}.$$

Then

$$\Pi(x) = \sum_{n>0} \frac{1}{n} \pi(x^{\frac{1}{n}}), \tag{5.1}$$

the sum being finite. By inversion

$$\pi(x) = \sum_{n>0} \frac{\mu(n)}{n} \Pi(x^{\frac{1}{n}}).$$

Furthermore

$$\lim_{\epsilon \to 0} \frac{\Pi(x+\epsilon) + \Pi(x-\epsilon)}{2} = \mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) + \int_x^{\infty} \frac{1}{t^2-1} \frac{dt}{t \log t} - \log 2.$$

The integral logarithm function admits the following expansion

$$\mathrm{Li}(x) \sim \frac{x}{\log x} \sum_{k \geq 0} \frac{k!}{(\log x)^k}$$

The prime number theorem can be stated as $\pi(x) \sim \mathrm{Li}(x)$. In 1899, La Vallée Poussin proved that

$$\pi(x) \sim \mathrm{Li}(x) + \mathrm{O}\left(x e^{-a\sqrt{\log x}} \right)$$

A precise statement can be given for H. von Koch showed in 1901 that, if and only if the Riemann hypothesis is true, the error term in the above relation can be improved to

$$\pi(x) = \mathrm{Li}(x) + O\left(\sqrt{x} \log x\right).$$

Another explicit formula, perhaps more tractable, is given connected to the logarithmic derivative of the Riemann zeta function.

$$Z(s) = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

where the von Mangoldt function Λ is given by

$$\Lambda(n) = \begin{cases} \log p \text{ if } n = p^k \text{ for some prime } p \text{ and } k \geq 1 \\ \\ 0 \text{ otherwise.} \end{cases} \tag{5.2}$$

It is neither multiplicative nor additive. We have by the Möbius inversion formula:

$$\log n = \sum_{d|n} \Lambda(d), \quad \Lambda(n) = \sum_{d|n} \mu(d) \log d.$$

We define the second Chebyshev function

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{p^{\nu} \leq x} \log p$$

and

$$\psi_0(x) = \lim_{\epsilon \to 0} \frac{\psi(x+\epsilon) + \psi(x-\epsilon)}{2} = \begin{cases} \psi(x), \ x \neq p^{\nu} \\ \psi(x) - \frac{1}{2} \log p, \ x = p^{\nu}. \end{cases}$$

The second explicit formula is [17](ch.5, th. 5.9):

$$\psi_0(x) = x + Z(0) - \frac{1}{2}\log(1-x^{-2}) - \lim_{T\to+\infty} \sum_{|\Im\rho|<T} \frac{x^\rho}{\rho}.$$

So we have again a correlation between primes numbers (here their statistics) and the location of non trivial zeros of the Riemann zeta function.
There are several extended forms of the previous explicit formula. We present a version due to A.Weil, with a variation in the definition to include a remark on Eisenstein series. We also give a well known version due to A.P. Guinand, valid under the Riemann hypothesis.

5.2. According to Weil.

Definition 5.1. *A function $F(x)$ is residual if the three following conditions are fulfilled*

(1) *It is defined and differentiable for $0 < x < \infty$ and for some $\gamma > 0$, we have $F(x) = O(x^{-\gamma})$ as x tends to 0 and $F(x) = O(x^\gamma)$ as x tends to $+\infty$, so that we may introduce for complex s the Mellin transforms*

$$\xi_r(s) = \int_0^1 F(x)x^{s-1}dx, \quad \xi_l(s) = \int_1^{+\infty} F(x)x^{s-1}dx$$

ξ_r is defined in a right half plane and ξ_l in a left half plane.

(2) *These two functions can be continued into one another in a domain D which is the exterior of a bounded set S.*
(3) *If we denote the joint value of the continued function by $\xi_0(s)$, $s = \sigma + i\tau$, then we have*

$$\lim_{|\tau|\mapsto+\infty} \xi_0(\sigma+i\tau) = 0$$

uniformly on finite intervals $\sigma_1 \le \sigma \le \sigma_2$.

This definition is motivated by Hecke's theory on modular forms [57]. The following theorem of Bochner [7] gives a characterization

Theorem 5.2. *A function $F(x)$ defined for $0 < x < \infty$ is residual if and only if it can be represented as a series*

$$F(x) = \sum_{n\ge0} \frac{c_n}{n!}\left(\log\frac{1}{x}\right)^n, \quad \gamma_0 = \limsup |c_n|^{\frac{1}{n}} < \infty.$$

The function $\xi_0(s)$ is the Borel transform $\xi_0(s) = \sum_{n\ge0} \frac{c_n}{s^{n+1}}$ of the entire function of exponential type $F(e^{-y}) = \sum_{n\ge0} \frac{c_n}{n!}y^n$.

We slightly modify the previous definition. Consider the class $\mathcal{W}$ of complex-valued functions $f(x)$ on the positive half-line $\mathbb{R}_+$, continuous and continuously differentiable except for finitely many points at which both $f(x)$ and $f'(x)$ have at most a discontinuity of the first kind, and at which the value of $f(x)$ and $f'(x)$ are defined as the average of the right and left limits. Suppose also that there is $\delta > 0$ such that $f(x) = \mathrm{O}(x^\delta)$ as $x \to 0+$ and $f(x) = \mathrm{O}(x^{-1-\delta})$ as $x \to +\infty$. Let $f(s)$ be the Mellin transform

$$\tilde{f}(s) = \int_0^\infty f(x)x^s\frac{dx}{x}.$$

This function is analytic in the strip $-\delta < \Re s < 1+\delta$.

Theorem 5.3 (Weil). *For $f \in \mathcal{W}$ we have*

$$\tilde{f}(0) - \sum_\rho \tilde{f}(\rho) + \tilde{f}(1) = \sum_{n=1}^\infty \Lambda(n)\left\{f(n) + \frac{1}{n}f(\frac{1}{n})\right\} + (\log 4\pi + \gamma)f(1)$$

$$+ \int_1^\infty \left\{f(x) + \frac{1}{x}f(\frac{1}{x}) - \frac{2}{x}f(1)\right\}\frac{dx}{x - x^{-1}}.$$

The first sum ranges over all nontrivial zeros of $\zeta(s)$ and is understood as

$$\lim_{T\to+\infty} \sum_{|\Im\rho|<T} \tilde{f}(\rho),$$

and Λ being the Mangoldt function (5.2) *and γ is the Euler constant.*

5.3. According to Guinand. In [26] Guinand obtained under the Riemann hypothesis a general summation formula connecting prime numbers and non-trivial zeros of the Riemann zeta-functiom. His result can be stated as follows:

Theorem 5.4.

$$\lim_{T\to\infty}\left\{\sum_{0<m\log p<T} \frac{\log p}{p^{\frac{1}{m}}} f\left(m\log p\right) - \int_0^T f(t)e^{\frac{t}{2}}dt\right\} - \frac{1}{2}\int_0^\infty f(t)\left(\frac{1}{t} - \frac{e^{\frac{3t}{2}}}{\sinh t}\right)dt$$

$$-(2\pi)^{\frac{1}{2}}\lim_{T\to\infty}\left\{\sum_{0<\rho<T} g(\rho) - \frac{1}{2\pi}\int_0^T g(t)\log\frac{t}{2\pi}dt\right\},$$

where

$$g(x) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\int_0^\infty f(t)\cos xt\,dt,$$

p runs through the prime numbers, m through the positive integers, and $\frac{1}{2}+\rho$ through the non trivial zeros of the Riemann zeta function.

5.4. According to Bost. If we denote the characteristic function of prime numbers by $\nu(n)$, then [20]:

$$P(s) = \sum_{n\geq 1}\frac{\nu(n)}{n^s}, \quad P(s)\zeta(s) = \sum_{n\geq 1}\frac{\omega(n)}{n^s}$$

where $\omega(n)$ is the number of different prime factors in n

$$\omega(n) = \sum_{d|n}\nu(d)$$

which, by the Möbius inversion formula, gives

$$\nu(n) = \sum_{d|n}\mu(d)\omega(\frac{n}{d}).$$

Remark 5.1. *It is proved in* [24] *that the primes contain arbitrarily long arithmetic progressions and if*

$$\mu = \sum_{p\in\mathcal{P}}\frac{1}{p}\delta_{\log p},$$

then μ is a positive measure whose Fourier transform $\hat{\mu}$ coincide with the distribution defined by the locally integrable function $l(t) = P(1+2i\pi t)$, that is for every infinitely differentiable on the real line with compact support ϕ, we have [9]

$$\int_{\mathbb{R}}\phi(t)P(1+2i\pi t)\,dt = \sum_{p\in\mathcal{P}}\frac{1}{p}\hat{\phi}(\log p).$$

5.5. Cramer's Dirichlet series. There exist many versions of explicit formulas in which summation is taken over half of the zeroes of the Riemann zeta function [55] (p. 320), [40], using the functions

$$\sum_{\Im\rho>0} \frac{e^{i\rho z}}{\rho}, \quad \sum_{\Im\rho>0} \frac{e^{i\rho z}}{\rho\zeta'(\rho)}, \quad \Re z > 0.$$

For example

Theorem 5.5. *If*

$$V(w) = \sum_{\Im\rho>0} e^{i\rho w}$$

This series converges absolutely for $\Im w > 0$ and

(1) *$V(w)$ admits a meromorphic continuation to the whole plane $\mathbb{C}$ cut from 0 to $-\infty$, and has there first order poles at $w = \pm \log p^m$ with respective principal parts*

$$\frac{\log p}{2i\pi} \frac{1}{w - \log p^m}, \quad \frac{\log p}{2i\pi} \frac{1}{w + \log p^m}$$

(2) *Near $w = 0$, we have*

$$V(w) = \frac{1}{2i\pi} \left(\frac{\log w}{1 - e^{-iw}} + \frac{\gamma + \log 2\pi - \frac{i\pi}{2}}{w} \right) + \frac{3}{4} + \eta i + wW(w)$$

where $\eta > 0$ and $W(w)$ is single valued and regular for $|w| < \log 2$.

(3) *The function*

$$U(w) = e^{-\frac{1}{2}iw} V(iw) + \frac{1}{4\pi} \frac{\log w}{\sin \frac{w}{2}}$$

admits a single valued analytic continuation satisfying the functional equation

$$U(w) + U(-w) = 2\cos\frac{w}{2} - \frac{1}{4\cos\frac{w}{2}}$$

5.6. Prime numbers theorem as a Tauberian theorem. We follow here the presentation of N.Wiener [58]. In general the singularities or the behaviors of a Dirichlet series

$$f(s) = \sum_{n=0}^{\infty} a_n e^{\lambda_n s}, \ \ \lambda_0 < \lambda_1 < \dots$$

depend on the coefficients (a_n) and the exponents (λ_n). However there are circumstances where only the exponents determine the behavior of the sum or the singularities of the series on the boundary of the domain of convergence. An example of a such situation is given by the outstanding Tauberian theorem of Hardy and Littelwood [28] (Theorem 114, p.173):

Theorem 5.6. (High-Indices Theorem) *If $f(x) = \sum_{k=0}^{\infty} a_k x^{n_k}$ converges for $0 < x < 1$ where the exponents n_k are positive integers satisfying the lacunarity condition*

$$\frac{n_{k+1}}{n_k} \geq q > 1$$

and if $f(x) \to A$ as $x \to 1^-$, then the infinite series $\sum_{k=0}^{\infty} a_k$ converges and its sum is A.

5.7. Ramanujan Differential system. The divisor functions that we are going to define constitute an excellent laboratory where the Fourier analysis, arithmetic and spectral theory combine in a harmonious and fertile mixture We define for $s \in \mathbb{C}$

$$\sigma_s(n) = \sum_{d|n} d^s.$$

The generating functions of the divisor sums $\sigma_s(n)$ satisfy various identities, usually proved by making extensive use of the fact that they are modular forms. These generating functions, in case $s = 1, 2, 3$ are the Eisenstein series

$$E_2(\tau) = 1 - 24 \sum_{n \geq 1} \sigma_1(n) e^{2i\pi n\tau}, \quad \Im\tau > 0$$

together with

$$E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n), e^{2i\pi nz},$$

and

$$E_6(z) \ = \ 1 - 504 \sum_{n \geq 1} \sigma_5(n)\, e^{2i\pi nz}$$

for which we have the following important Ramanujan differential system [48], [46]:

$$\begin{cases} \dfrac{1}{2\pi i}\dfrac{d}{dz}\, E_2(z) & = \frac{1}{12}(E_2^2 - E_4). \\ \dfrac{1}{2\pi i}\dfrac{d}{dz}\, E_4(z) & = \frac{1}{3}(E_2 E_4 - E_6) \\ \dfrac{1}{2\pi i}\dfrac{d}{dz}\, E_6(z) & = \frac{1}{2}(E_2 E_6 - E_4^2), \end{cases}$$

Furthermore the Eisenstein series $y = E_2(z)$ satisfies de Chazy equation [59]

$$D^3 y - y D^2 y + \frac{3}{2}(Dy)^2 = 0, \quad Dy = \frac{1}{2i\pi}\frac{dy}{dz}. \tag{5.3}$$

The Chazy equation is an example of an ordinary differential equation with the general solution having moving natural boundary (Painlevé property) and $\mathrm{SL}_2(\mathbb{C})$-invariance [12]. It also arose as a reduction of the self-dual Yang-Mills equation in [2]. It is remarkable that so many arithmetical relations between the divisor functions are consequences of the the Ramanujan differential system or the Chazy equation. For example it is readily seen from (5.3) that we have the following recursion relation for the sums of divisors of natural numbers $\sigma_1(n) = \sum_{d|n} d$:

$$\sigma_1(n) = \frac{12}{n^2(n-1)} \sum_{k=1}^{n-1} k(3n - 5k)\sigma_1(k)\sigma_1(n-k).$$

The series $E_2(z)$ is a holomorphic function on $\mathfrak{H} = \{z \in \mathbb{C}, \Im z > 0\}$, the upper half plane, and if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$E_2\left(\frac{az+b}{cz+d}\right) \ = \ (cz+d)^2\, E_2(z) \ + \ \frac{6c}{i\pi}(cz+d), \tag{5.4}$$

that is to say that E_2 is a quasi-modular form of weight 2. Meanwhile, if we define

$$E_2^*(z) \ = \ E_2(z) - \frac{3}{\pi y}\,, \quad y = \Im z\,, \tag{5.5}$$

then E_2^* is a (not holomorphic) modular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$. Moreover, E_2 is the logarithmic derivative of the discriminant function $\Delta(z)$, the classical cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$

$$\Delta(z) \;=\; \eta(z)^{24} \;=\; q \prod_{n=1}^{\infty} (1-q^n)^{24}\,, \quad q = \mathrm{e}^{2\pi i z}.$$

In fact, we have

$$E_2(z) = \frac{1}{2i\pi} \frac{\Delta'(z)}{\Delta(z)}. \tag{5.6}$$

Ingham (1930) proved the prime number theorem using the particularly elegant identity of Ramanujan:

$$\sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}.$$

More generally we define

$$E_{2k}(z) = 1 + \gamma_{2k} \sum_{k=1}^{\infty} (-1)^{k+1} \sigma_{2k-1}(n) e^{2i\pi n z}, \tag{5.7}$$

with

$$\gamma_{2k} = (-1)^k \frac{4k}{B_{2k}}.$$

For $k > 1$, the Eisenstein series $E_{2k}(z)$ can be written in more analytic form

$$E_{2k}(z) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} (cz+d)^{-2k} \tag{5.8}$$

The series converges absolutely and uniformly on compact sets of $\mathfrak{H}$ if $k > 1$, and E_{2k} is a modular form of weight $2k$, that is

$$E_{2k}(\gamma z) = (cz+d)^{2k} E_{2k}(z)$$

for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma = \mathrm{SL}_2(\mathbb{Z})$. The sum over $(c,d) = 1$ is the same as the sum over $\gamma \in \Gamma_\infty \backslash \Gamma$, where

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, n \in \mathbb{Z} \right\}.$$

The case of $k = 1$ is of a particular interest. It gives rise to the classical non-holomorphic Eisenstein series. Let

$$E(z,s) = \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{y^s}{|cz+d|^{2s}} = \frac{1}{2} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \Im(\gamma z)^s \tag{5.9}$$

The normalized Eisenstein series is

$$E^\star(z,s) = \frac{1}{2}\pi^{-s}\Gamma(s)\zeta(2s)E(z,s) = \frac{1}{2}\pi^{-s}\Gamma(s) \sum_{(m,n)\neq(0,0)} \frac{y^s}{|mz+n|^{2s}}.$$

This series converges for $\Re s > 1$ and it is an eigenfunction of the hyperbolic Laplacian $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$. More precisely

$$\Delta E(z,s) = s(s-1)E(z,s).$$

Moreover $E^\star(z,s)$ has the Fourier expansion

$$E^\star(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{s-1}\Gamma(1-s)\zeta(2-2s)y^{1-s} + 2\sum_{n\neq 0} |n|^{s-\frac{1}{2}} \sigma_{1-2s}(|n|)\sqrt{y} K_{s-\frac{1}{2}}(2\pi|n|y) e^{2i\pi n x}.$$

where $K_s(y)$ is the K-Bessel function, defined for $\Re x > 0$ by $K_s(x) = \int_0^\infty e^{x\cosh(t)} \cosh(st)\, dt$. The Prime Number Theorem is equivalent to the non-vanishing of the Riemann zeta-function on the line $\Re(s) = 1$ (Hadamard et de la Vallée -Poussin). Eisenstein series can be used to show this fact. Indeed the n-th Fourier coefficient of $E(z,s)$ is

$$\int_0^1 E(z,s)e^{2i\pi nx}dx = \frac{2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)}{\pi^{-s}\Gamma(s)\zeta(2s)}.$$

Now the holomorphy of $E(z,s)$ on $\Re(s) = 1$ implies the non-vanishing of $\zeta(s)$ on $\Re(s) = 1$. Namely, the information of Eisenstein series give rise to the information on ζ function. This is the beginning observation of Langlands-Shahidi method.

5.8. Robin's theorem, Lagarias theorem. The behaviour of the sigma function $\sigma_1(n)$ is irregular. The asymptotic growth rate of the sigma function can be expressed by:

$$\overline{\lim_{n\to\infty}} \frac{\sigma_1(n)}{\log\log n} = e^\gamma.$$

This result is Grnwall's theorem, published in 1913 (Grönwall 1913). His proof uses Mertens' 3rd theorem, which says that

$$\lim_{n\to\infty} \frac{1}{n} \prod_{p\le n,\, p\,\text{prime}} \frac{p}{p-1} = e^\gamma.$$

In 1915, Ramanujan proved that under the assumption of the Riemann hypothesis, the inequality:

$$\sigma_1(n) \le e^\gamma n \log\log n$$

holds for all sufficiently large n (Ramanujan 1917). The largest known value that violates the inequality is $n = 7! = 5040$. In 1984 Guy Robin proves [50]

Theorem 5.7. *The previous inequality is true for all $n > 5040$ if and only if the Riemann hypothesis is true.*

Moreover Robin also proved, unconditionally, that the inequality

$$\sigma_1(n) \le e^\gamma n \log\log n + \frac{0.6483n}{\log\log n}$$

holds for all $n \ge 3$. Finally Jeffrey Lagarias showed in 2002 that the Riemann hypothesis is equivalent to the statement that

$$\sigma_1(n) < H_n + \ln(H_n)e^{H_n}$$

for every natural number $n > 1$, where H_n is the n-th harmonic number.

5.9. The Euler constant γ and the theta function. There are so many properties in [30] on the Euler constant. We would like to add some others. The function $\chi(z) = \sum_{n=0} z^{2^n}$ is related to the paper folding problem [51], [52] and [53], where a relation to Jacobi theta function is given.

Proposition 5.1. *We have the equality*

$$\gamma = 1 + \log 2 - \int_0^1 \frac{\chi(x)}{1+x}\, dx$$

To see this we first repeat Vacca formula(1910) which says that:

$$\begin{aligned}\gamma &= \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)\\ &= \frac{1}{2} - \frac{1}{3} + 2\left(\frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7}\right) + 3\left(\frac{1}{8} - \frac{1}{9} + \cdots - \frac{1}{15}\right) + \cdots\end{aligned}$$

In fact we know that for the Mercator's series

$$\frac{\pi}{4} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

and we define

$$\sigma_n = 1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2^n - 1} - \log 2.$$

Then

$$\sigma_1 + \sigma_2 + \cdots + \sigma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n - 1} - n \log 2$$

and therefore

$$\gamma = \lim_{n \to \infty} (\sigma_1 + \sigma_2 + \cdots + \sigma_n)$$

To finish the proof of Vacca's formula we observe that

$$\sigma_n = \tau_n + \tau_{n+1} + \cdots$$

where

$$\tau_n = \frac{1}{2^n} - \frac{1}{2^n + 1} + \frac{1}{2^n + 2} - \cdots - \frac{1}{2^{n+} - 1}.$$

Let $F(x) = \chi(x) - 1$, since Vacca's formula may be written in the form

$$\gamma = \int_0^1 \left(\frac{x - x^3}{1 + x} + 2\frac{x^3 - x^7}{1 + x} + 3\frac{x^7 - x^{15}}{1 + x} + \cdots \right) dx = \int_0^1 \frac{F(x)}{x(1 + x)} \, dx$$

and since

$$\int_0^1 \frac{F(x)}{x} \, dx = \sum_{n \geq 1} \frac{1}{2^n} = 1$$

we obtain the desired result.

Another arithmetical function also related to the Euler gamma constant is the Euler totient function

$$\Phi(n) = \sharp\{j \in \mathbb{Z}_+, \, \gcd(j, n) = 1\} = n \prod_{p|n} (1 - \frac{1}{p})$$

for which we have

$$n = \sum_{d|n} \Phi(d).$$

Landau(1903) proved that

$$\underline{\lim}_{n \to \infty} \Phi(n) \frac{\log \log n}{n} = e^{-\gamma}$$

6. Almost periodicity

We refer to [34] Let $w_j(n)$ be 1 or 0 according as j is or is not a divisor of n, and let $\gamma(n)$ be defined by:

$$2^{\gamma(n)} \mid n, \quad 2^{\gamma(n)+1} \nmid n.$$

Let $r_4(n)$ be the number of different representations of n as a sum of 4 squares. In other words

$$1 + \sum_{n=1}^{\infty} r_4(n) z^n = \left(1 + \sum_{n=1}^{\infty} z^{n^2}\right)^4, \quad |z| < 1.$$

Jacobi's well known theorem concerning the representation of $r_4(n)$ in terms of $\sigma_1(n)$ may be written as follows:

$$\frac{r_4(n)}{n} = 8 \frac{1 + 2\omega_2(n)}{2^{\gamma(n)+1} - 1} \frac{\sigma_1(n)}{n} = 8 \frac{1 + 2\omega_2(n)}{2^{\gamma(n)+1} - 1} \sum_{j=1}^{\infty} \frac{\omega_j(n)}{j}$$

If $f(n)$ is a function defined for $n = 0, 1, 2, \cdots$ we define the average of f by

$$M(f) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f(j).$$

Then we have [34]

$$M\left(\frac{\sigma_1(n)}{n} - \sum_{j=1}^{N} \frac{\omega_j(n)}{j}\right)^2 \longrightarrow 0, \quad N \to \infty.$$

Since the finite sums $\displaystyle\sum_{j=1}^{N} \frac{\omega_j(n)}{j}$, $N \geq 1$ are periodic, we then obtain:

Theorem 6.1. *The arithmetical function* $\dfrac{\sigma_1(n)}{n}$ *is almost periodic* $(\mathbf{B^2})$*, that is almost periodic is in the sense of Besicovitch.*

Dirichlet showed that the average order of the divisor function $\sigma_0(n) = d(n)$ satisfies the following equality, valid for all $x \geq 1$:

$$\sum_{n\leq x} d(n) = x\log x + (2\gamma - 1)x + \mathrm{O}(\sqrt{x})$$

For prime divisors we have a nice probabilistic result:

Theorem 6.2. (Erdös-Kac Theorem). *For any positive integer $n \geq 1$, let $\omega(n)$ denote the number of prime divisors of n, counted without multiplicity. Then for any real numbers $a < b$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \left| \left\{ 1 \leq n \leq N,\ a \leq \frac{\omega(n) - \log\log N}{\sqrt{\log\log N}} \leq b \right\} \right| = \frac{1}{2\pi} \int_a^b e^{-x^2/2} dx$$

7. Motivation: Classical Polylogarithm function

The present section was the main origin of these notes. In [6], Besicovich solved the following problem: does there exist a continuous function $f(x)$ in the interval $0 \leq x \leq 1$ such that:

(1) $f(x) + f(1-x) \not\equiv 0$,

(2) For each integer $n > 0$, $\displaystyle\sum_{1\leq m\leq n} f(\frac{m}{n}) = 0$.

Following an idea of Chowla and Bateman, we give an example of a non-trivial real-valued continuous function on the real line which has period unity, is not odd, and has the property

$$\sum_{0<h\leq k} f\left(\frac{h}{k}\right) = 0.$$

It turns out that this example is really a particular case of a more general theorem that exhibits a new function, similar to the polylogarithm function in its form, but whose analytic properties depend heavily on the non trivial zeros of the Riemann zeta function. We recall first some well known facts concerning the polylogarithm and some related functions [19], [39]. The function

$$\Phi(z, s, v) = \sum_{n=0}^{\infty} (v+n)^{-s} z^n, \quad |z| < 1, \quad v \neq 0, -1, -2, \cdots. \tag{7.1}$$

is well known and has so many important properties. It reduces to Hurwitz ζ-function for $z = 1$

$$\zeta(s, v) = \sum_{n=0}^{\infty} (v+n)^{-s} \quad \Re s > 1, \quad v \neq 0, -1, -2, \cdots. \tag{7.2}$$

and to classical polylogarithm function with complex argument for $v = 1$

$$\mathcal{L}i_s(z) = \sum_{n=1}^{\infty} n^{-s} z^n. \tag{7.3}$$

We also introduce

$$\mathfrak{l}i_s(z) = \sum_{n=1}^{\infty} n^{-s} e^{2i\pi n\theta},\ \theta \in \mathbb{R}. \tag{7.4}$$

The function (7.1) has the remarquable Lindelöf decomposition:

$$\begin{aligned} \Phi(z,s,v) &= \frac{\Gamma(1-s)}{z^v}\left(\log\frac{1}{z}\right)^{s-1} + z^{-v}\sum_{r=0}^{\infty}\zeta(s-r,v)\frac{(\log z)^r}{r!}, \\ |\log z| &< 2\pi, \quad s \neq 1,2,3\cdots, v \neq 0,-1,-2,\cdots. \end{aligned} \tag{7.5}$$

This expansion is a consequence of the Hurwitz' formula [39]:

$$\zeta(s,v) = 2(2\pi)^{-(s-1)}\Gamma(1-s)\sum_{n=1}^{\infty} n^{s-1}\sin(2\pi n v + \frac{1}{2}\pi s),\ \Re s < 0,\ 0 < v \le 1. \tag{7.6}$$

When v is a positive integer, the equality (7.1) is a kind of decomposition of singularities, $\log z$ being a local coordinate at $z = 1$ and $\frac{\Gamma(1-s)}{z^v}\left(\log\frac{1}{z}\right)^{s-1}$ is the source of the multivaludeness of the function $\zeta(s,v)$. Our aim is to prove a similar formula for the function

$$a\Psi(z,s,v) = \sum_{n=0}^{\infty}\mu(n)(v+n)^{-s}z^n, \quad |z| < 1, \quad v \neq 0,-1,-2,\cdots. \tag{7.7}$$

where μ is the previously defined Möbius function

$$\mu(n) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n \text{ has one or more repeated prime factors} \\ (-1)^k \text{ if } n \text{ is the product of } k \text{ prime factors.} \end{cases}$$

This function is fundamental in number theory. Its absolute value is the characteristic function of squarefree integers.

If x denotes a positive real, then the Mertens function $M(x)$ is defined by

$$M(x) = \sum_{n \le x}\mu(n) \tag{7.8}$$

By partial summation

$$\begin{aligned} \frac{1}{\zeta(s)} &= \sum_{n=1}^{\infty}\frac{\mu(n)}{n^s} = \sum_{n=1}^{\infty}\frac{M(n)-M(n-1)}{n^s} \\ &= \sum_{n=1}^{\infty} M(n)\left\{\frac{1}{n^s} - \frac{1}{(n+1)^s}\right\} = \sum_{n=1}^{\infty} M(n)\int_n^{n+1}\frac{s}{x^{s+1}}dx \\ &= s\sum_{n=1}^{\infty}\int_n^{n+1}\frac{M(x)}{x^{s+1}}dx = s\int_1^{\infty}\frac{M(x)}{x^{s+1}}dx. \end{aligned}$$

The asymptotics

$$\sum_{n=1}^{N}\mu(n) = o(N), \quad N \to \infty$$

is equivalent to the prime number theorem, and the the more quantitative statement that for every $\epsilon > 0$

$$\sum_{n=1}^{N} \mu(n) = O_\epsilon(N^{\frac{1}{2}+\epsilon})$$

is equivalent to the Riemann hypothesis. Riele and Odlyzko [43], disproving the Mertens conjecture, showed the following:

Theorem 7.1. *There are explicit constants $C_1 > 1$ and $C_2 < -1$ such that*

$$\limsup_{x\to+\infty} \frac{M(x)}{\sqrt{x}} \geq C_1, \quad \liminf_{x\to+\infty} \frac{M(x)}{\sqrt{x}} \leq C_2.$$

This means that each of the inequalities $-\sqrt{x} \leq M(x)$ and $M(x) \leq \sqrt{x}$ fails for infinitely many values of x.

An explicit formula for the Mertens function was first published in 1951 by Titchmarsh on the assumption of the Riemann hypothesis ([55], 14.27). More precisely let $\rho = \frac{1}{2} + i\gamma$ with $\gamma \in \mathbb{R}$. Specifically, on Riemann hypothesis and the simplicity of the non-trivial zeros, there exists a sequence (T_ν), $\nu \leq T_\nu \leq \nu + 1$ such that

$$M(x) = -2 + \lim_{\nu\to+\infty} \sum_{|\gamma|<T_\nu} \frac{x^\rho}{\rho\zeta'(\rho)} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2\pi)^n}{(2n)!n\zeta(2n+1)} x^{-n} \tag{7.9}$$

if x is not an integer. If x is an integer, $M(x)$ is to be replaced by $M(x) - \frac{1}{2}\mu(x)$. The crucial property of the Riemann zeta function needed in this result is the following estimate of minimum modulus type: Each interval $(T, T+1)$ contains a t such that

$$\zeta(\sigma + it) > \exp\left(-A\frac{\log t \log\log\log t}{\log\log t}\right), \quad \frac{1}{2} \leq \sigma \leq 2.$$

As consequence we obtain the following result, due to Littlewood:

Proposition 7.1. *If ϵ is any positive number, each $(T, T+1)$ contains a t such that:*

$$\frac{1}{\zeta(s)} = \mathrm{O}(t^\epsilon). \tag{7.10}$$

Let us recall that the Prime Number Theorem states that

$$\pi(x) \approx \frac{x}{\log x},\ x \to \infty.$$

where $\pi(x) = \sharp\{p \in P | p \leq x\}$ and P is the set of prime numbers. If P is replaced by by the larger set

$$Q = \{2, 3, 4, 5, 7, 9, 11, 13, 16, 17, \cdots\}$$

of prime powers and denote by

$$\Pi(x) = \sharp\{q \in Q, q \leq x\}$$

suitably normalized at integer values, in such a way that

$$\Pi(x) = \pi(x) + \frac{1}{2}\pi(x^{\frac{1}{2}}) + \frac{1}{3}\pi(x^{\frac{1}{3}}) + \cdots.$$

The link between the function $\Pi(x)$ and complex zeros of the ζ-function is given by Riemann as follows

$$\Pi(x) = \mathrm{Li}(x) - \Sigma'_\rho \mathrm{Li}(x^\rho) + \int_x^\infty \frac{1}{x^2-1}\frac{dx}{x\log x} - \log 2$$

where $\mathrm{Li}(x^s) = \displaystyle\int_0^x \frac{t^{s-1}dt}{\log t}$. In summary, under the Riemann hypothesis, Riemann connecting formula between primes and complex zeros of the ζ-function is

$$\frac{\displaystyle\int_2^x \frac{dt}{\log t} - \sharp\{\text{primes } \leq x\}}{\sqrt{x}/x} \sim 1 + 2 \sum_{\gamma>0,\, \zeta(\frac{1}{2}+\gamma)=0} \frac{\sin(\gamma \log x)}{\gamma}, \quad x \to \infty.$$

The Möbius function has some interesting functional identities

$$F(x) = \sum_{n=1}^{\infty} f(nx) \quad \Leftrightarrow \quad f(x) = \sum_{n=1}^{\infty} \mu(n) F(nx)$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1-x^n} &= x \\
\sum_{n=1}^{\infty} \mu(n) \frac{x^n}{1+x^n} &= x - 2x^2 \\
\sum_{n=1}^{\infty} \mu(n) x^n &= \sum_{n=1}^{\infty} \frac{x^n}{1-x^n}
\end{aligned}$$

Another essential point where the Möbius function occurs is that, according to de la Vallée Poussin for example, a zero-free region for the ζ-function is

$$\left\{ z = x + iy \in \mathbb{C},\, 1 - x \leq \frac{c}{\log |y|} \right\}$$

where c is some positive constant. Hence $\dfrac{1}{\zeta(z)} = \displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n^z}$ is holomorphic in some disc $D(\frac{1}{2}, r)$, $r > 0$.

Let us return to the lindelöf expansion (7.5). We first observe that for $v = 1$, $z = 1$ the function $\Psi(z, s, v)$ reduces to $\dfrac{1}{\zeta(s)}$ for $\Re s > 1$ and the series $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ converges for $\Re s > \frac{1}{2}$ under the Riemann hypothesis. We introduce, similarly to (7.3), the series

$$\mathcal{M}i_s(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} z^n. \tag{7.11}$$

It converges for any complex number s if $|z| < 1$. For $z = 1$ this series converges for $\Re s > 1$ and also for $\Re s > \frac{1}{2}$ if and only if the Riemann hypothesis is true. One way to predict formally the decomposition (7.5) when $v = 0$, that is

$$\mathcal{L}i_s(z) = \Gamma(1-s) \left(\log \frac{1}{z} \right)^{s-1} + \sum_{r=0}^{\infty} \zeta(s-r) \frac{(\log z)^r}{r!} \tag{7.12}$$

is to proceed as follows:

$$\begin{aligned}
\mathcal{L}i_s(z) &= \sum_{n=1}^{\infty} n^{-s} z^n = \sum_{n=1}^{\infty} n^{-s} e^{n \log z} \\
&= \sum_{n=1}^{\infty} n^{-s} \sum_{m=0}^{\infty} \frac{(n \log z)^m}{m!}.
\end{aligned}$$

by formally interverting the two summations. In this way we obtain the infinite sum in the decomposition formula (7.5) of $\mathcal{L}i_s(z)$. The factor $\Gamma(1-s) \left(\log \dfrac{1}{z} \right)^{s-1}$ is present to make this inversion of sums rigorous. This suggests that for $\mathcal{M}i_s(z)$, the sum $\displaystyle\sum_{r=0}^{\infty} \frac{1}{\zeta(s-r)} \frac{(\log z)^r}{r!}$ should appear.

The Mellin inversion formula shows that, for any complex number z and any $c > 0$, we have

$$e^{-nz} = \frac{1}{2i\pi} \int_{\Re u = c} \frac{\Gamma(u)}{(nz)^u} du.$$

If c is large enough and $\Im z > 0$, we obtain

$$\mathcal{M}i_s(e^{-z}) = \frac{1}{2i\pi} \int_{\Re u = c} \frac{\Gamma(u)}{z^u} \frac{du}{\zeta(s+u)}.$$

The poles of the integrated function are those of the Γ function $u = -m$, $m = 0, 1, 2, \cdots$ which are simple with residues $\frac{(-1)^m}{m!}$ and the zeros of the function $\zeta(s+u)$ which are at $u = -s - 2m$, $m = 1, 2, \cdots$ and $u = -s + \rho, \rho$ being the non-trivial zeros of the Riemann ζ-function. It is known that

$$\zeta'(-2m) = \frac{(-1)^m \zeta(2m+1)(2m)!}{2^{2m+1}\pi^{2m}}$$

and by shifting the line of integration to the left, we obtain the decomposition formula, under the assumption that the non trivial zeros of the Riemann ζ-function are simple.

Theorem 7.2.

$$\begin{aligned} \mathcal{M}i_s(e^{-z}) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\zeta(s-k)} z^k + \left(\sum_{n=1}^{\infty} \frac{c_{2n}\Gamma(-2n-s)}{\zeta(2n+1)} z^{2n} \right) z^s \\ &+ \left(\sum_{\rho} \frac{\Gamma(\rho - s)}{\zeta'(\rho)} z^{-\rho} \right) z^s. \end{aligned} \tag{7.13}$$

The third sum in the right hand side of (7.13) *is over all the nontrivial zeros of the ζ-function. The constants c_{2n} in the second sum are given by $c_{2n} = \frac{(-1)^n (2n)!}{2^{2n+1}\pi^{2n}}$.*

The assumption that the zeros are simple is connected with some conjectures, also related to Riemann hypothesis which is equivalent to the fact that $\frac{1}{\zeta(s)} = \sum_{1}^{\infty} \mu(n) n^{-s}$ is convergent for $\sigma = \Re s > \frac{1}{2}$. By taking Mellin's transform we obtain another equivalent formulation

$$M(x) := \sum_{n \leq x} \mu(n) \ll x^{\frac{1}{2}+\epsilon}.$$

A weaker statement is

$$\int_1^x \left(\frac{M(x)}{x} \right)^2 dx \ll \log x. \tag{7.14}$$

This hypothesis seems more tractable and has the following consequence [55]:

Theorem 7.3. *If* (7.14) *is valid then all the zeros of $\zeta(s)$ are simple and*

$$\sum_{\rho} |\rho \zeta'(\rho)|^{-2} < \infty \tag{7.15}$$

$$|\zeta'(\frac{1}{2} + it)| \gg exp \left(\frac{A \log^2 t}{\log \log t} \right), \quad t \gg 3, \tag{7.16}$$

If $\rho = \frac{1}{2} + i\gamma$ and $\rho' = \frac{1}{2} + i\gamma'$ are two different zeros of $\zeta(s)$, then:

$$|\rho - \rho'| \gg \gamma^{-1} exp \left(\frac{-A \log \gamma}{\log \log \gamma} \right). \tag{7.17}$$

The previous computations could be justified by using a deep theorem of Davenport [16] that we use many times in this work.

Theorem 7.4. *If A is a given positive constant, then for real x greater than unity we have*

$$\sum_{n \leq x} \mu(n) e^{2i\pi n\theta} = \mathrm{O}\left(x \log^{-A}(x+1) \right)$$

uniformly in θ.

The series appearing previously in theorem (7.2) has some similarities with well known series of M. Riesz [49] on one hand and G.H.Hardy and J.E.Littlewood [29] on the other. M.Riesz proved that the Riemann Hypothesis is equivalent to the following statement: For each $\epsilon > 0$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{(k-1)!\zeta(2k)} << x^{\frac{1}{2}+\epsilon}.$$

Hardy-Littlewood proved that the Riemann Hypothesis is equivalent to

$$\sum_{k=1}^{\infty} \frac{(-x)^k}{k!\zeta(2k+1)} << x^{-\frac{1}{4}}, \qquad x \to \infty.$$

Remark 7.1. *Concerning the series of Riesz and Hardy-Littelwood, Titchmarsh wrote in* [55], *p.328 "These conditions have a superficial attractiveness since they depend explicitly only on values taken by $\zeta(s)$ at points in $\sigma > 1$, but actually no use has ever been made of them". Less ambitiously the meromorphic continuation of the Riemann zeta function can be obtained from an expansion of the same kind* [17]*(ch.2, ex.2.3):*

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} \frac{\Gamma(s+n)}{\Gamma(s) n!} \frac{\zeta(n+s)}{2^{n+s}} \tag{7.18}$$

The series $\mathcal{M}i_s(z)$ given in (7.2) is the Hadamard product of $\mathcal{L}i_s(z)$ given in (7.1) and $\varphi(z) = \sum_{n=1}^{\infty} \mu(n) z^n$. Little is known about the last series. The well-known theorem of Polya-Carlson asserts that a complex power series with integer coefficients and radius of convergence 1 is either a rational function or has the unit circle as its natural boundary. It follows from this fact and from the theorem (7.4) of Davenport that the series $\varphi(z)$ is holomorphic in the unit disk $\{|z| < 1\}$, continuous on $\{|z| \leq 1\}$ and has the unit circle as a natural boundary. Hence

Corollary 7.5. *For each positive integer k, the power series $\mathcal{M}i_k(z)$ has the unit circle as natural boundary.*

To formulate the main theorem, we introduce the last arithmetical function:

Definition 7.6. *The Liouville function λ is defined by $\lambda(1) = 1$ and $\lambda(n) = (-1)^j$ if n is the product of j (not necessarily distinct) prime numbers. The Liouville function is closely related to the Möbius μ function for they are equal on square-free integers*

We can similarly to (7.3) and (7.11) define

$$\mathcal{N}i_s(z) = \sum_{n \geq 1} \frac{\lambda(n)}{n^s} z^n.$$

Our main result is

Theorem 7.7. *Let:*

$$\mathfrak{m}i_s(\theta) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} e^{2i\pi n\theta}, \qquad \mathfrak{n}i_s(\theta) = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s} e^{2i\pi n\theta}; \qquad \theta \in \mathbb{R}$$

Then for every positive k we have

$$\sum_{h=1}^{k} \mathfrak{m}i_s(h/k) = \frac{\mu(k)}{k^{s-1}} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}, \qquad \sum_{h=1}^{k} \mathfrak{n}i_s(h/k) = \frac{\lambda(k)}{k^{s-1}} \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}$$

Proof. We give just an idea of the proof:

$$\sum_{h=1}^{k} \mathfrak{m}i_s(h/k) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \sum_{h=1}^{k} e^{2iphn/k} = \frac{\mu(k)}{k^{s-1}} \sum_{n=1,(n,k)=1}^{\infty} \frac{\mu(n)}{n^s}.$$

We obtain the conclusion by using (4.17) and the following remark:

$$\sum_{n=1,(n,k)=1}^{\infty} \frac{\mu(n)}{n} = \lim_{s \to 1^+} \sum_{n=1,(n,k)=1}^{\infty} \frac{\mu(n)}{n^s}$$

$$= \lim_{s\to 1^+} \prod_{p\nmid k}(1-p^{-s}) = \lim_{s\to 1^+} \left\{\zeta(s)\prod_{p|k}(1-p^{-s})\right\}^{-1} = 0.$$

The same proof goes for $\mathfrak{n}i_s$ by using that for $x > 0$ be a large real number

$$L(x) = \sum_{n\le x}\lambda(n) = \mathrm{O}(\sqrt{x})$$

and for $\Re s > 1$

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)}$$

hence

$$\sum_{n=1}^{\infty}\frac{\lambda(n)}{n} = \lim_{s\to 1, s>1}\frac{\zeta(2s)}{\zeta(s)} = 0.$$

Corollary 7.8. *Les fonctions $\Re mi_1(\theta)$, $\Re ni_1(\theta)$ are non-trivial real-valued continuous functions f on the real line which have period unity, are even, and have the property $\sum_{h=1}^{n} f(h/k) = 0$ for every positive integer k.*

It is very likely that these functions are nowhere differentiable. This has not been proved. Moreover, one can prove directly that $\sum(-1)^n\lambda(n)/n = 0$. Indeed:

$$\sum(-1)^n\lambda(n)/n^s + \sum\lambda(n)/n^s = 2\sum\lambda(2n)/(2n)^s.$$

By using $\lambda(2) = -1$ the fact that λ is multiplicative we obtain

$$\lambda(2n)/(2n)^s = -2/2^s\lambda(n)/n^s$$

and hence

$$(-1)^n\lambda(n)/n^s = (-1-2/2^s)\lambda(n)/n^s$$

$$(-1)^{(n+1)}\lambda(n)/n^s = (1+2/2^s)\zeta(2s)/\zeta(s).$$

It is generally believed that the values of the Liouville function enjoy various randomness properties and one manifestation of this principle is an old conjecture of Chowla [11] which asserts that for all $l \in \mathbb{N}$ and all distinct $n_1, n_2, \cdots, n_l \in \mathbb{N}$ we have

$$\lim_{M\to\infty}\sum_{m=1}^{M}\lambda(m+n_1)\cdots\lambda(m+n_l) = 0.$$

8. Leroy-Lindelöf theory

We complete the remark (4.1). Any power series with positive radius r of convergence $g(z) = \sum a_n z^n$ can be written

$$\sum a_n z^n = \sum f(n) z^n$$

where $f(z)$ is an entire function of exponential type and order one. From Cauchy formula for derivatives of analytic functions we have for the sum $f(z) = \sum a_n z^n$

$$a_n = \frac{g^{(n)}(0)}{n!} = \frac{1}{2i\pi}\int_\gamma \frac{g(z)}{z^{n+1}}dz,$$

γ is small circle, centered at 0 and of radius smaller than r. Now if we set $z = e^{-u}$ we obtain

$$a_n = f(n), \quad f(\zeta) = \frac{1}{2i\pi}\int_\Gamma g(e^{-u})e^{\zeta u}du.$$

Here Γ is a path of ends z_1 and z_2 with

$$\Re z_1 = \Re z_2, \quad \Im z_1 - \Im z_2 = 2\pi.$$

The function $f(z)$ is not unique but, according to a result of Carlson it is unique if we require that the exponential type is less than π. By the remark (4.1) the Borel transform of the entire function $f(z)$ is the Cauchy transform of $G(u) = g(e^{-u})$.

Leroy-Lindelöf theory, suitably extended to power series $\sum_{n=0}^{\infty} f(n)z^n$ where $f(z)$ is a an holomorphic function of exponential type only on right half plane [39] (chapter V), enables us to give the precise asymptotic expansions. In particular this applies to many power series having coefficients depending on Riemann ζ-function. The most known series of this type are certainly the polygamma function. We first recall that the function $\frac{1}{\Gamma}$ is entire with the following Weierstrass factorization in infinite product (γ is Euler's constant)

$$\frac{1}{\Gamma(z)} = ze^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}.$$

We define

$$\Psi^{(0)}(z) = \Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}, \quad \Psi^{(m)}(z) = \frac{d^m}{dz^m}\Psi(z).$$

The Ψ-function has the following integral and series representations

$$\Psi(z) = -\gamma + \int_0^{\infty} \frac{e^{-t} - e^{-zt}}{1 - e^{-t}} dt = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(z+n)}, \quad \Re z > 0,$$

so that

$$\Psi^{(m)}(z) = (-1)^{m+1} m! \sum_{k=0}^{\infty} \frac{1}{(z+k)^{m+1}}.$$

In particular we have, for $|z| < 1$, the Taylor expansion having the zeta values $\zeta(n)$ as coefficients

$$\Psi(1+z) = -\gamma + \zeta(2)z - \zeta(3)z^2 + \zeta(4)z^3 - \cdots.$$

We recover and complete many known results. Let

$$G_a(x) = \sum_{n > a+1} \zeta(n-a) \frac{(-x)^n}{n!}.$$

Then we have the following expansions:

(1) $G_0(x) = x\log x + (2\gamma - 1)x + \frac{1}{2} + O(e^{-c\sqrt{x}}), \quad x \to \infty$ (Chowla-Hawkins),
(2) $-G_{-1}(x) = \log x + 2\gamma + O(x^{-1})$ (Verma),
(3) $G_{-2}(x) = x^{-1} + o(x^{-k})$ for all $k > 1$ (Tennenbaum),
(4) $G_1(x) = -\frac{x^2}{2}\log x - (\gamma - \frac{3}{4})x^2 - \frac{x}{2} + \frac{1}{12} + O(x^{-1})$ (Verma and Prasad).

8.1. Lacunarity of $\mathfrak{m}i_s(\theta) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} e^{2i\pi n\theta}$. The Fourier coefficient in $\mathfrak{m}i_s(\theta)$ does not vanish when n is square free. The function $|\mu(n)| = \mu^2(n)$ is the characteristic function of the square free integers, with

$$\sum_{n=1}^{\infty} \frac{|\mu(n)|}{n^s} = \frac{\zeta(s)}{\zeta(2s)}, \quad \Re s > 1. \tag{8.1}$$

Let $Q(x)$ be the number of positive squarefree numbers less than x, then

$$Q(x) = \sum_{n=1}^{x} |\mu(n)| = \frac{6x}{\pi^2} + O(\sqrt{x}).$$

Let $Q_2(n)$ be the number of consecutive numbers $(k, k+1)$ with $k \leq n$ such that k and $k+1$ are both squarefree. A result of Carlitz [10] asserts that if p_n is the n^{th} prime, then $\dfrac{Q_2(n)}{n}$ is, asymptotically,

$$\prod_{k=1}^{\infty}\left(1-\frac{2}{p_k^2}\right) = 2F-1 = 0.3226340989\cdots$$

and F is the Feller-Tornier constant. If q_k is the k-th squarefree number, then

$$q_{k+1}-q_k = \mathrm{O}(\sqrt{n})$$

Though the proof of this result (and of much more general character) may be found in many standard texts on number theory we give an idea of the proof. The function μ is multiplicative, then

$$\mu^2(n) = \sum_{d^2|n} \mu(d).$$

It follows that

$$\begin{aligned}\sum_{n\leq x} |\mu(n)| &= \sum_{n\leq x} \mu^2(n) = \sum_{n\leq x}\sum_{d^2|n} \mu(d)\\ &= \sum_{d\leq\sqrt{x}} \mu(d)\left(\frac{x}{d^2}+\mathrm{O}(1)\right) = x\sum_{d\leq\sqrt{x}}\frac{\mu(d)}{d^2}+\mathrm{O}(\sqrt{x})\\ &= x\sum_{d\geq 1}\frac{\mu(d)}{d^2}+\mathrm{O}\left(\sqrt{x}+\sum_{d>\sqrt{x}}\frac{1}{d^2}\right) = \frac{x}{\zeta(2)}+\mathrm{O}(\sqrt{x})\end{aligned}$$

On the other hand, at the same level of depth as the prime number theorem, we have the sharper result

$$Q(x) = \sum_{n=1}^{x} |\mu(n)| = \frac{6x}{\pi^2}+\mathrm{o}(\sqrt{x})$$

and then

$$q_{n+1}-q_n = \mathrm{o}(\sqrt{n}).$$

Actually this problem is also linked to the Riemann hypothesis. To give some details we start from the Perron integral:

$$\frac{1}{2i\pi}\int_{\Re s=2}\frac{y^s}{s}ds = \begin{cases} 0 & if \;\; y<1\\ \frac{1}{2} & if \;\; y=1\\ 1 & if \;\; y>1\end{cases}$$

where, in the second case, the principal value of the integral is taken. This gives for non-integers x, using (8.1):

$$\sum_{n\leq x}\mu^2(n) = \frac{1}{2i\pi}\int_{\Re s=2}\frac{\zeta(s)}{\zeta(2s)}\frac{y^s}{s}ds.$$

By shifting the path of integration to the left we obtain the expansion

$$\sum_{n\leq x}\mu^2(n) = \frac{x}{\zeta(2)}+\sum_{\rho,\,\zeta(\rho)=0} a_\rho x^{\frac{\rho}{2}} \tag{8.2}$$

where the coefficients a_ρ on the the residues of $\dfrac{1}{\zeta(s)}$ ar ρ. Assuming the Riemann hypothesis we can write $\rho = \dfrac{1}{2}+it$, $t\in\mathbb{R}$. The relation (8.2) becomes

$$\sum_{n\leq x}\mu^2(n) = \frac{x}{\zeta(2)}+x^{\frac{1}{4}}\sum_{\rho,\,\zeta(\rho)=0} a_\rho x^{\frac{it}{2}}.$$

This argument suggests that the right error term in an asymptotic formula for the number of square free integers should be [44]:

$$\sum_{n\le x}\mu^2(n)=\frac{6x}{\pi^2}+\mathrm{O}_\epsilon(x^{\frac14+\epsilon}).$$

Polya-Cramer theorem states if the exponents of the Dirichlet series

$$f(s)=\sum_{j=1}^{\infty}a_je^{-\lambda_js}$$

satisfies the gap condition

$$\liminf(\lambda_{j+1}-\lambda_j)\ge\gamma>0,$$

then every closed segment of length $\dfrac{2\pi}{\gamma}$ on the line of convergence contains a singular point of $f(s)$.

9. The Lemniscate or from $\mathbb{Q}$ to $\mathbb{Q}(i)$ or from Bernoulli to Hurwitz

9.1. Kubert identities. The generating function for the Bernoulli polynomials $B_q(t)$ is given by

$$\frac{ze^{tz}}{e^z-1}=\sum_{q=0}^{\infty}\frac{B_q(t)z^q}{q!}.$$

We have the addition formula

$$B_q(kx)=k^{k-1}\sum_{j=0}^{k-1}B_q(x+\frac{j}{k}) \tag{9.1}$$

for $q=1,2,\cdots$. It is also true for $q=0$ if we put $B_0(x)=1$. Putting $k=2,\ x=0,\ q=2m$ we obtain

$$B_{2m}(\frac12)=\left(\frac{1}{2^{2m-1}-1}\right)B_{2m}(0)=\left(\frac{1}{2^{2m-1}-1}B_{2m}\right)$$

where $B_{2m}=B_{2m}(0)=B_{2m}(1)$ are the Bernoulli numbers. The relation (9.1) remains true with the stipulation $B_0(x)=B_0=1$. The functions $\psi_q(t)=B_q(t-[t]),\ q\ge1$ are periodic of period 1. With the exception of $\psi_1(t)=B_1(t-[t]),B_1(t)=t-\dfrac12$ which has jumps at all integers, they are continuous, since $B_q(0)=B_q(1)$ for $q\ge2$. They are all of bounded variation and thus have the Fourier expansion, $k\ge1$

$$\psi_{2k-1}(t)=B_{2k-1}(t-[t])=2(-1)^k(2k-1)!\ \sum_{n=1}^{\infty}\frac{\sin 2\pi nt}{(2\pi n)^{2k-1}}, \tag{9.2}$$

$$\psi_{2k}(t)=B_{2k}(t-[t])=2(-1)^{k-1}(2k)!\ \sum_{n=1}^{\infty}\frac{\cos 2\pi nt}{(2\pi n)^{2k}}. \tag{9.3}$$

For $\psi_1(t)$ integer values have to be excluded, because of the discontinuity at $t=0$. Indeed

$$\frac12\left(\psi_1(-0)+\psi_1(+1)\right)=\frac12\left(\frac12-\frac12\right)=0.$$

In the same vein we have for the Γ-function the Gauss's multiplication formula

$$\Gamma(s)\Gamma(s+\frac1k)\cdots\Gamma(s+\frac{k-1}{k})=(2\pi)^{\frac{k-1}{2}}k^{\frac12-ks}\Gamma(ks). \tag{9.4}$$

The theorem (7.7) seems to extend to any arithmetical function η by considering series of the form

$$\sum_{n\ge1}\frac{\eta(n)}{n^s}z^n$$

and we can maybe consider Gauss or Dedekind instead of harmonic sums. The functions (7.1) and (7.2) verify both the Kubert identities, valid for every integer $m \in \mathbb{N}$ [42]:

$$\phi(z) = m^{s-1} \sum_{w^n=1} \phi(w). \tag{9.5}$$

9.2. From Bernoulli to Hurwitz. We first recall an important theorem:

Theorem 9.1 (Gauss-Wentzel's theorem)**.** *The division of the circle in n equal parts with straightedge and compass is possible if and only if*

$$n = 2^k p_1 \cdots p_t$$

where $p_1, \cdots, p_t$ are distinct Fermat primes.

A Fermat prime is a prime of the form $2^{2^m} + 1$.
The lemniscate is the curve defined by the polar equation $r^2 = \cos(2\theta)$. In the first quadrant, the arc length t is related to the radial distance w by the elliptic integral

$$t = \int_0^w \frac{dz}{\sqrt{1-z^4}}.$$

We recall that the circular sine is the function $w = \sin t$, the function inverse to the integral

$$t = \int_0^w \frac{dz}{\sqrt{1-z^2}}.$$

The hyperbolic sine $w = \sinh t$ is the function inverse to the integral

$$t = \int_0^w \frac{dz}{\sqrt{1+z^2}}.$$

Usually the lemniscatic sine is denoted $w = \mathfrak{sl}\, t$

$$w = \mathfrak{sl}\, t \Longleftrightarrow t = \int_0^w \frac{dz}{\sqrt{1-z^4}}.$$

We define

$$\omega = 2\int_0^1 \frac{dz}{\sqrt{1-x^4}} = \frac{\Gamma(\frac{1}{4})^2}{2\sqrt{2\pi}} = 2.62205755\cdots$$

and the Weierstrass $\wp$-function associated to the elliptic curve (with complex multiplication by $\mathbb{Z}[i]$) $\mathbf{E} : y^2 = x^3 - x$, with periods ω and $i\omega$, defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\lambda \in \omega\mathbb{Z} + i\omega\mathbb{Z} \\ \lambda \neq 0}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right).$$

If $K = \mathbb{Q}(i)$ is the field of Gaussian rationals and n is a positive odd integer

$$Gal\left(K\left(\mathfrak{sl}\left(\frac{2w}{n}\right)\right)/K\right) \simeq (\mathbb{Z}[i]/n\mathbb{Z}[i])^*.$$

This Galois group is then abelian. We note that this is a complete analog of the following theorem for cyclotomic extensions:

Theorem 9.2. *if $\xi_n = e^{\frac{2i\pi}{n}}$, then*

$$Gal\left(\mathbb{Q}\left(\xi_n\right)/\mathbb{Q}\right) \simeq (\mathbb{Z}/n\mathbb{Z})^*.$$

The Weierstrass $\wp$-function also have complex multiplication by $\mathbb{Z}[i]$, that is

$$\wp(iz) = -\wp(z).$$

It satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - 4\wp(z).$$

The Laurent expansion at $z = 0$ is

$$\wp(z) = \frac{1}{z^2} + \sum_{n=2}^{\infty} \frac{2^n \mathbf{H_n}}{n} \frac{z^{n-2}}{(n-2)!}. \tag{9.6}$$

The coefficients $\mathbf{H_n}$ are called Hurwitz numbers. From $\wp(iz) = -\wp(z)$ we see that $\mathbf{H_n} = \mathbf{0}$ if n is not a multiple of 4. Similarly to (9.1) we have

$$\frac{1}{\sin^2(x)} = \frac{1}{x^2} + \sum_{n=2}^{\infty} \frac{(-1)^{\frac{n}{2}-1} 2^n B_n}{n} \frac{x^{n-2}}{(n-2)!}. \tag{9.7}$$

The equations (9.6) and (9.7) are analogous as was observed first by Hurwitz [31], [32] . One deduces, similarly to (4.1), that

$$\sum_{\substack{\lambda \in \omega\mathbb{Z} + i\mathbb{Z} \\ \lambda \neq 0}} \frac{1}{\lambda^2} = \frac{(2\omega)^{4n}}{(4n)!} \mathbf{H_{4n}}.$$

A very interesting property proved by Hurwitz for the $\mathbf{H_n}$ is the following analog of the Von Staudt-Clausen theorem (4.1). Let p be a prime of the form $4k+1$ so that $p = a^2 + b^2$. We take a odd and such that $a \equiv b + 1 (\bmod 4)$, then

$$\mathbf{H_n} = G_n + \frac{1}{2} + \sum \frac{(2a)^{\frac{4n}{p-1}}}{p}$$

where the summation is over all primes p of the form $4k+1$ such that $p-1|4n$ and G_n is an integer [1].
The Fagnano's addition theorem asserts that

$$2\int_0^a \frac{dx}{\sqrt{1-x^4}} = \int_0^{\frac{2a\sqrt{1-a^4}}{1+a^4}} \frac{dz}{\sqrt{1-z^4}}$$

hence the doubling of the length of a segment of the lemniscate is the length of an another segment of the lemniscate. Euler obtained a more general result by observing that the general solution of the differential equation

$$\frac{dx}{\sqrt{1-x^4}} = \frac{dz}{\sqrt{1-z^4}}$$

is

$$x^2 + z^2 + b^2x^2z^2 = b^2 + 2xz\sqrt{1-b^4}, \quad b \in \mathbb{R}.$$

It can be solved by

$$z = \frac{x\sqrt{1-b^4} + b\sqrt{1-x^4}}{1+b^2x^2}$$

so that

$$\int_0^a \frac{dx}{\sqrt{1-x^4}} = \int_b^{\frac{a\sqrt{1-b^4}+b\sqrt{1-a^4}}{1+a^2b^2}} \frac{dz}{\sqrt{1-z^4}}.$$

This gives the general addition theorem of Euler

$$\int_0^a \frac{dx}{\sqrt{1-x^4}} + \int_0^b \frac{dx}{\sqrt{1-x^4}} = \int_0^{\frac{a\sqrt{1-b^4}+b\sqrt{1-a^4}}{1+a^2b^2}} \frac{dx}{\sqrt{1-x^4}}.$$

We recover Fagnano's theorem for $a = b$.

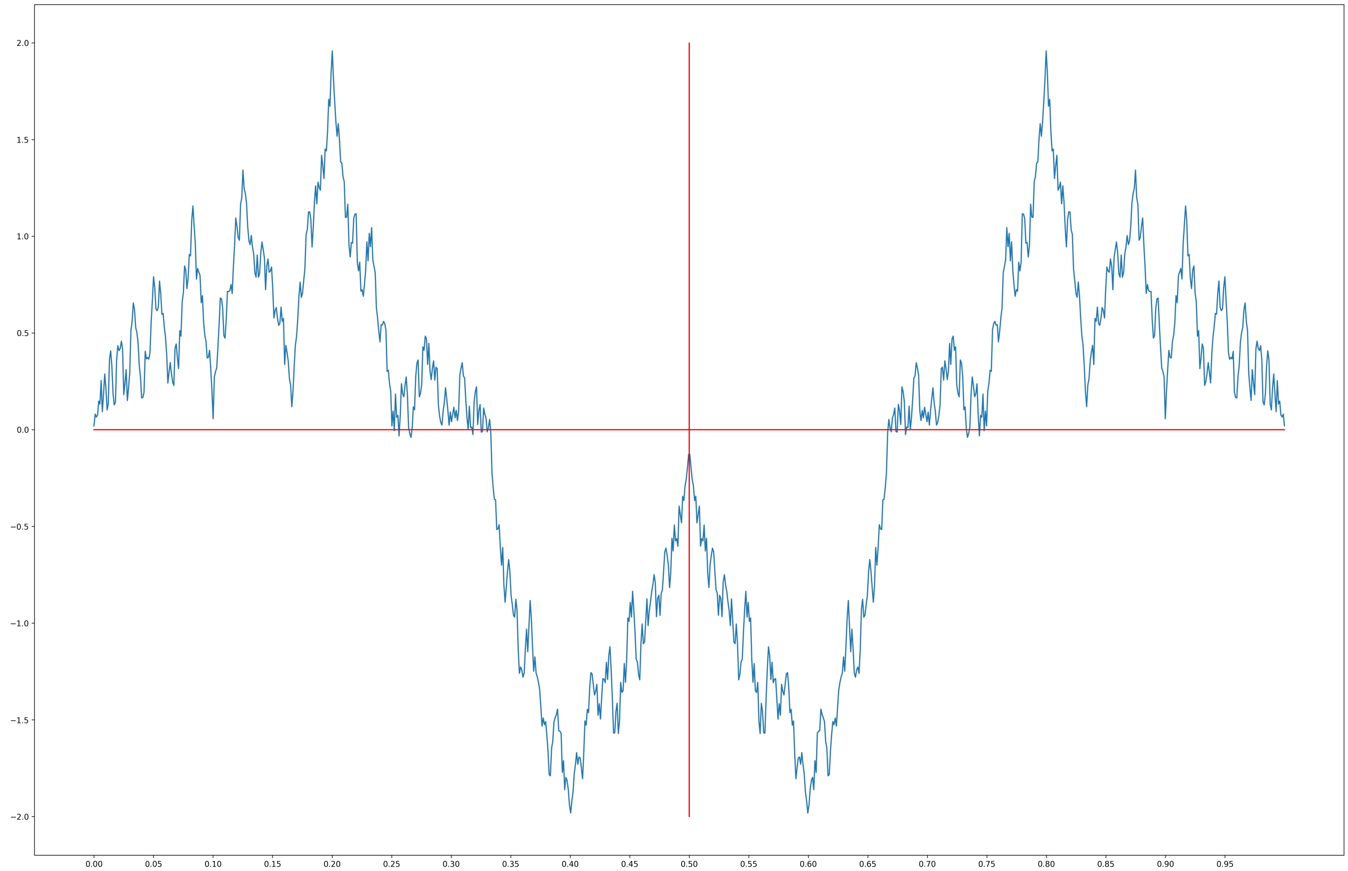

FIGURE 1. Graph of $\sum \frac{\mu(n)}{n} \cos(2\pi n t)$

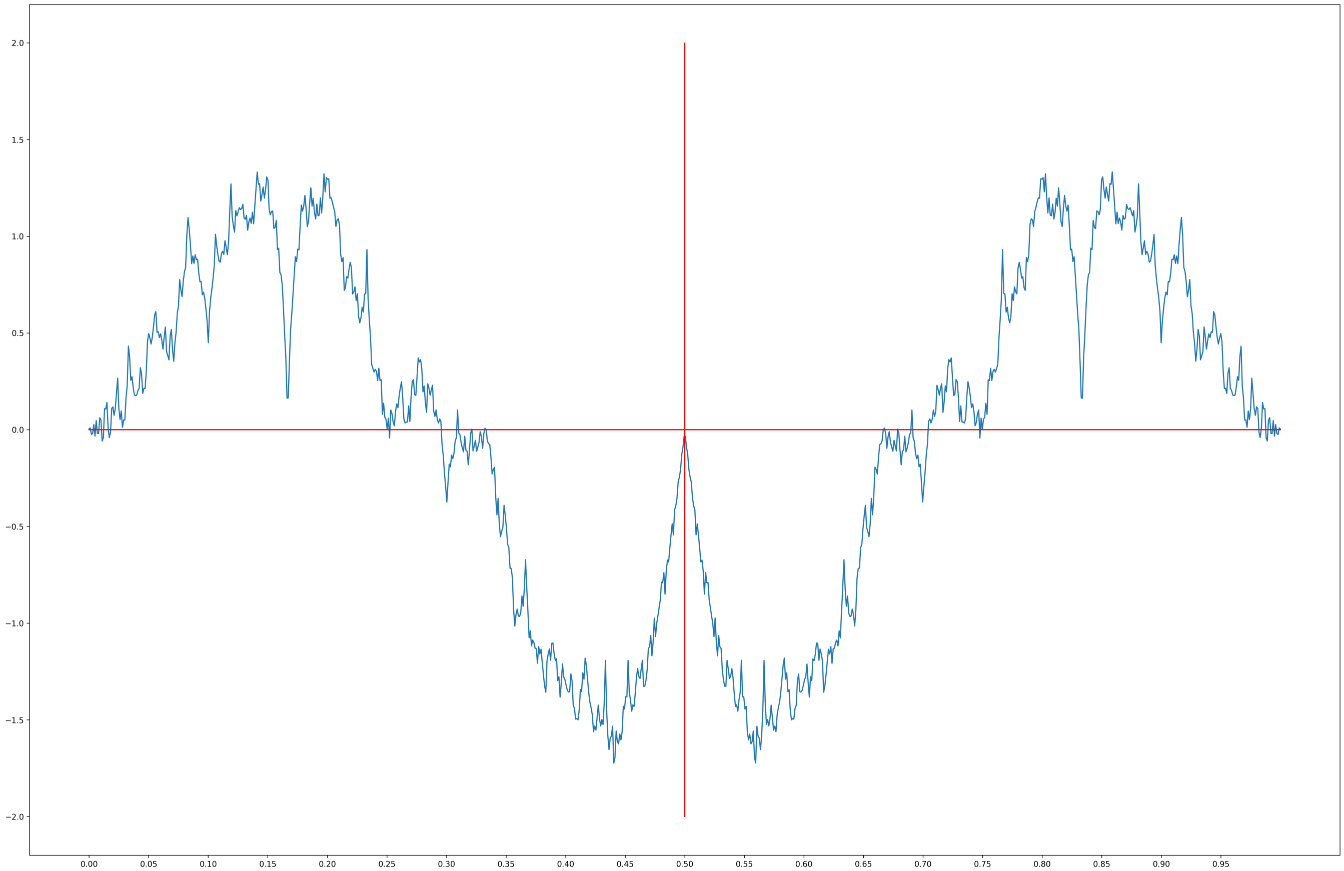

FIGURE 2. Graph of $\sum \frac{\lambda(n)}{n} \cos(2\pi n t)$

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