Nonlocal eigenvalue problems with variable exponent

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ABSTRACT. We consider the nonlocal eigenvalue problem of the following form

$$(\mathcal{P}_K) \begin{cases} \mathcal{L}_K^{p(x)} u(x) + |u(x)|^{\bar{p}(x)-2} u(x) = \lambda |u(x)|^{r(x)-2} u(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a smooth open and bounded set in $\mathbb{R}^N$ ($N \geq 3$), $\lambda > 0$ is a real number, K is a suitable kernel and p, r are two bounded continuous functions on $\bar{\Omega}$. The main result of this paper establishes that any $\lambda > 0$ sufficiently small is an eigenvalue of the above nonhomogeneous nonlocal problem. The proof relies on some variational arguments based on Ekeland's variational principle.

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1. Introduction

We begin this work by remembering the definition of the fractional Sobolev spaces with variable exponent, to this end we follow the work of Pezzo and Rossi in [6].

Let Ω be a smooth bounded open set in $\mathbb{R}^N$. We start by fixing $s \in (0, 1)$ and let $p : \bar{\Omega} \times \bar{\Omega} \rightarrow (1, +\infty)$ be a continuous function. We assume that

$$1 < p^- = \min_{(x,y) \in \bar{\Omega} \times \bar{\Omega}} p(x,y) \leq p(x,y) \leq p^+ = \max_{(x,y) \in \bar{\Omega} \times \bar{\Omega}} p(x,y) < +\infty \quad (1.1)$$

and

$$p \text{ is symmetric, that is, } p(x,y) = p(y,x) \text{ for all } (x,y) \in \bar{\Omega} \times \bar{\Omega} \quad (1.2)$$

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Let denote by :

$$\bar{p}(x) = p(x, x) \text{ for all } x \in \bar{\Omega}.$$

We define the fractional Sobolev space with variable exponent via the Gagliardo approach as follows,

$$W = W^{s,p(x,y)}(\Omega) = \left\{ u \in L^{\bar{p}(x)}(\Omega) : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{sp(x,y)+N}} dx dy < +\infty, \text{ for some } \lambda > 0 \right\},$$

where $L^{\bar{p}(x)}(\Omega)$ is the Lebesgue space with variable exponent, (see section 2).

The space $W^{s,p(x,y)}(\Omega)$ is a Banach space (see [13]) if it is endowed with the norm,

$$\|u\|_{W^{s,p(x,y)}(\Omega)} = \|u\|_W = \|u\|_{L^{\bar{p}(x)}(\Omega)} + [u]_{s,p(x,y)},$$

where $[\cdot]_{s,p(x,y)}$ is a Gagliardo seminorm with variable exponent, which is defined by

$$[u]_{s,p(x,y)} = \inf \left\{ \lambda > 0 : \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)} |x - y|^{sp(x,y)+N}} dx dy \leq 1 \right\}.$$

The space $(W, \|\cdot\|_W)$ is a separable and reflexive, see [2, Lemma 3.1].

Let us consider the fractional $p(x)$ -Laplacian operator given by

$$(-\Delta_{p(x)})^s u(x) = p.v. \int_{\Omega} \frac{|u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y))}{|x - y|^{n+sp(x,y)}} dy \text{ for all } x \in \Omega,$$

where $p.v.$ is a commonly used abbreviation in the principal value sense.

Note that the operator $(-\Delta_{p(x)})^s$ is the fractional version of well known $p(x)$ -Laplacian operator $\Delta_{p(x)} u(x) = \operatorname{div}(|\nabla u(x)|^{p(x)-2} u(x))$. On the other hand, we remark that in the constant exponent case it is known as the fractional p -Laplacian operator $(-\Delta_p)^s$. This nonlocal nonlinear operator is consistent, up to some normalization constant depending upon N and s , with the linear fractional Laplacian $(-\Delta)^s$ in the case $p = 2$. The interest for this last operator and more generally pseudo-differential operators, has constantly increased over the last few years, although such operators have been a classical topic of functional analysis since long ago. Nonlocal operators such as $(-\Delta)^s$ and its generalisation $\mathcal{L}_K$ (see for instance [12, 14, 20, 21, 22]) naturally arise in continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical outcome of stochastic stabilization of Lévy processes, see e.g. [4, 16, 17]. We refer the reader to [3, 7] and to the references included for a self-contained overview of the basic properties of fractional Sobolev spaces and fractional Laplacian operator.

Now, we introduce the nonlocal integro-differential operator of elliptic type $\mathcal{L}_K^{p(x)}$ which generalizes $(-\Delta_{p(x)})^s$, for any fixed $s \in (0, 1)$, as follows:

$$\begin{aligned} \mathcal{L}_K^{p(x)}(u(x)) &= p.v. \int_{\mathbb{R}^N} |u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) K(x, y) dy \text{ for all } x \in \mathbb{R}^N \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} |u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) K(x, y) dy \text{ for all } x \in \mathbb{R}^N, \end{aligned}$$

where $p.v.$ is a commonly used abbreviation in the principal value sense, $p : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty)$ be a continuous bounded function satisfying (1.1), (1.2) on $\mathbb{R}^N \times \mathbb{R}^N$, and

$$p((x, y) - (z, z)) = p(x, y) \text{ for all } (x, y), (z, z) \in \mathbb{R}^N \times \mathbb{R}^N. \quad (1.3)$$

The kernel $K : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (0, +\infty)$ is a measurable function with the following properties

$$K(x, y) = K(y, x) \text{ for any } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N, \quad (1.4)$$

there exists a positive constant $k_0 > 0$ such that

$$K(x, y) \geq k_0 |x - y|^{-(N+sp(x,y))} \text{ for all } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N \text{ and } x \neq y, \quad (1.5)$$

$$mK \in L^1(\mathbb{R}^N \times \mathbb{R}^N), \text{ where } m(x, y) = \min \{1, |x - y|^{p(x, y)}\}. \quad (1.6)$$

A typical example for K is given by the singular kernel $K(x, y) = |x - y|^{-(N+sp(x, y))}$. In this case $\mathcal{L}_K^{p(x)} = (-\Delta_{p(x)})^s$.

Another example for K is given by the kernel

$$K_1(x, y) = |x - y|^{-(N+sp(x, y))} a(x - y),$$

where $a : \mathbb{R}^N \rightarrow [1, +\infty)$ is a bounded function, that is, $a \in L^\infty(\mathbb{R}^N)$. It is easy to see that K_1 satisfying (1.4), (1.5) and (1.6).

Due to the non-localness of the operator $\mathcal{L}_K^{p(x)}$, we introduce the general fractional Sobolev space with variable exponent as follows

$$E = W^{K, p(x, y)}(\Omega) = \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \text{ measurable, such that } u|_\Omega \in L^{\bar{p}(x)}(\Omega) \text{ with } \int_Q \frac{|u(x) - u(y)|^{p(x, y)}}{\lambda^{p(x, y)}} K(x, y) dx dy < +\infty, \text{ for some } \lambda > 0 \right\},$$

where Ω be an open bounded subset of $\mathbb{R}^N$ and Q defined by

$$Q := \mathbb{R}^N \times \mathbb{R}^N \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), \quad \text{with } \mathcal{C}\Omega = \mathbb{R}^N \setminus \Omega.$$

The space $W^{K, p(x, y)}(\Omega)$ is a Banach space (see [1]) if it is endowed with the norm

$$\|u\|_{W^{K, p(x, y)}(\Omega)} = \|u\|_{K, p(x, y)} = \|u\|_{L^{\bar{p}(x)}(\Omega)} + [u]_{K, p(x, y)},$$

where, $[u]_{K, p(x, y)} = \inf \left\{ \lambda > 0 : \int_Q \frac{|u(x) - u(y)|^{p(x, y)}}{\lambda^{p(x, y)}} K(x, y) dx dy \leq 1 \right\}$.

The space $(E, \|\cdot\|_E)$ is separable and uniformly convex reflexive, see ([1]).

We remark that even in the model case in which $K = |x - y|^{-(N+sp(x, y))}$, the norms $\|\cdot\|_{K, p(x, y)}$ and $\|\cdot\|_{s, p(x, y)}$ are not the same because $\Omega \times \Omega$ is strictly contained in Q this makes the fractional Sobolev space with variable exponent approach $(W^{s, p(x, y)}(\Omega))$ not sufficient for studying the nonlocal problem. For this we use the generalized fractional Sobolev space with variable exponent approach $(W^{K, p(x, y)}(\Omega))$ have been considered in [1], in order to correctly encode the Dirichlet boundary datum in the variational formulation.

In this paper, we are concerned with the study of the following nonlocal eigenvalue problem

$$(\mathcal{P}_K) \quad \begin{cases} \mathcal{L}_K^{p(x)} u(x) + |u(x)|^{\bar{p}(x)-2} u(x) &= \lambda |u(x)|^{r(x)-2} u(x) & \text{in } \Omega, \\ u &= 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where Ω is a smooth open and bounded set in $\mathbb{R}^N$ ($N \geq 3$), $\lambda > 0$ is a real number, $p : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty)$ is a continuous function satisfying (1.1) and (1.2). And $r : \bar{\Omega} \rightarrow (1, +\infty)$ is a bounded continuous function such that

$$1 < r^- = \min_{x \in \bar{\Omega}} r(x) \leq r(x) \leq r^+ = \max_{x \in \bar{\Omega}} r(x) < p^- \quad \text{for all } x \in \bar{\Omega}. \quad (1.7)$$

Our purpose is to establish the existence of a continuous family of eigenvalues for problem $(\mathcal{P}_K)$ in a neighborhood of the origin. More precisely, we show that there exists λ^* such that any $\lambda \in (0, \lambda^*)$ is an eigenvalue of the above problem.

In the context of eigenvalue problems involving variable exponent represent a starting point in analyzing more complicated equations. A first contribution in this sense is the paper of X. L. Fan, Q. H. Zhang and D. Zhao [11] where the following eigenvalue problem has been considered,

$$(\mathcal{P}_1) \begin{cases} -\operatorname{div}(|\nabla u(x)|^{p(x)-2} \nabla u(x)) &= \lambda |u(x)|^{p(x)-2} u(x) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded domain with smooth boundary $\partial\Omega$, the function $p : \overline{\Omega} \rightarrow (1, +\infty)$ is continuous and $\lambda > 0$ is a real number. The result obtained in [11] establishes the existence of infinitely many eigenvalues for problem $(\mathcal{P}_1)$ by using an argument based on the Ljusternik-Schnirelmann critical point theory. Denoting by Λ the set of all nonnegative eigenvalues, the authors showed that $\sup \Lambda = +\infty$ and they pointed out that only under special conditions, which are somehow connected with a kind of monotony of the function $p(x)$, we have $\inf \Lambda > 0$ (this is in contrast with the case when $p(x)$ is a constant; then, we always have $\inf \Lambda > 0$).

Going further, another eigenvalue problem involving variable exponent growth conditions intensively studied is the following

$$(\mathcal{P}_2) \begin{cases} -\operatorname{div}(|\nabla u(x)|^{p(x)-2} \nabla u(x)) &= \lambda |u(x)|^{q(x)-2} u(x) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

where $p, q : \overline{\Omega} \rightarrow (1, +\infty)$ are two continuous functions and $\lambda > 0$ is a real number.

Note that when $p(x) \neq q(x)$, the competition between the growth rates involved in problem $(\mathcal{P}_2)$ is essential in describing the set of eigenvalues of this problem and we cite the following:

- In the case when $\min_{x \in \overline{\Omega}} q(x) < \min_{x \in \overline{\Omega}} p(x)$ and $q(x)$ has a subcritical growth, M. Mihăilescu and V. Rădulescu [19] used *Ekeland's variational principle* in order to prove the existence of a continuous family of eigenvalues which lies in a neighborhood of the origin. This result was later extended by X. Fan in [8].
- In the case when $\max_{x \in \overline{\Omega}} p(x) < \min_{x \in \overline{\Omega}} q(x)$ and $q(x)$ has a subcritical growth, a mountain pass argument, similar with that used by Fan and Zhang [9], can be applied in order to show that any $\lambda > 0$ is an eigenvalue of problem $(\mathcal{P}_2)$.
- Finally, in the case when $\max_{x \in \overline{\Omega}} q(x) < \min_{x \in \overline{\Omega}} p(x)$ it can be proved that the energetic functional which can be associated with the eigenvalue problem has a nontrivial minimum for any positive λ large enough (see, [9]). Clearly, in this case, the result of M. Mihăilescu and V. Rădulescu [19] can be also applied. Consequently, in this situation there exist two positive constants λ^* and λ^{**} such that any $\lambda \in (0, \lambda^*) \cup (\lambda^{**}, +\infty)$ is an eigenvalue of the problem.

In an appropriate context we also point out the study of the eigenvalue problem,

$$(\mathcal{P}_3) \begin{cases} -\operatorname{div}((|\nabla u(x)|^{p_1(x)-2} + |\nabla u(x)|^{p_2(x)-2}) \nabla u(x)) &= \lambda |u(x)|^{q(x)-2} u(x) & \text{in } \Omega, \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

where $p_1, p_2, q : \overline{\Omega} \rightarrow (1, +\infty)$ are continuous functions satisfying

$$1 < p_2(x) < \min_{x \in \overline{\Omega}} q(x) \leq \max_{x \in \overline{\Omega}} q(x) < p_1(x) < N \quad \text{for all } x \in \overline{\Omega}$$

and

$$\max_{x \in \overline{\Omega}} q(x) < \frac{N p_2(x)}{N - p_2(x)} \quad \text{for all } x \in \overline{\Omega}.$$

For this problem M. Mihăilescu and V. Rădulescu [18] have proved the existence of two positive constants λ_0 and λ_1 with $\lambda_0 \leq \lambda_1$ such that any $\lambda \in (\lambda_1, +\infty)$ is an eigenvalue of problem $(\mathcal{P}_3)$ while any $\lambda \in (0, \lambda_0)$ is not an eigenvalue of problem $(\mathcal{P}_3)$.

One typical feature of problem $(\mathcal{P}_K)$ is the nonlocality, in the sense that the value of $\mathcal{L}_K^{p(x)}(u(x))$ at any point $x \in \Omega$ depends not only on the values of u on Ω , but actually on the entire space $\mathbb{R}^N$.

This paper is organized as follows. In section 2, we briefly review the definitions and collect some preliminary results for the Lebesgue spaces with variable exponent $L^{q(x)}$ and the generalized fractional Sobolev spaces with

variable exponent $W^{K,p(x,y)}$. In section 3, we introduce some important lemmas which show that the functional $\mathcal{J}_\lambda$ (see section 3) satisfies the geometrical conditions of the mountain pass theorem. Finally, using Ekeland's variational principle we prove that the problem $(\mathcal{P}_K)$ has a continuous spectrum which concentrates around the origin.

2. Some preliminary results

In this section, we recall some necessary properties of variable exponent spaces. For more details we refer the reader to [5, 10, 15], and the references therein.

Consider the set

$$C_+(\overline{\Omega}) = \{q \in C(\overline{\Omega}) : q(x) > 1, \text{ for all } x \in \overline{\Omega}\}.$$

For all $q \in C_+(\overline{\Omega})$, we set

$$q^+ = \sup_{x \in \overline{\Omega}} q(x) \text{ and } q^- = \inf_{x \in \overline{\Omega}} q(x).$$

Such that

$$1 < q^- \leq q(x) \leq q^+ < +\infty$$

For any $q \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space as

$$L^{q(x)}(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R} \text{ measurable} : \int_{\Omega} |u(x)|^{q(x)} dx < +\infty \right\}.$$

This vector space endowed with the *Luxemburg norm*, which is defined by

$$\|u\|_{L^{q(x)}(\Omega)} = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{q(x)} dx \leq 1 \right\}$$

is a separable reflexive Banach space.

Let $\hat{q} \in C_+(\overline{\Omega})$ be the conjugate exponent of q , that is, $\frac{1}{q(x)} + \frac{1}{\hat{q}(x)} = 1$. Then we have the following Hölder-type inequality

Lemma 2.1. (Hölder inequality). *If $u \in L^{q(x)}(\Omega)$ and $v \in L^{\hat{q}(x)}(\Omega)$; then*

$$\left| \int_{\Omega} uv dx \right| \leq \left(\frac{1}{q^-} + \frac{1}{\hat{q}^-} \right) \|u\|_{L^{q(x)}(\Omega)} \|v\|_{L^{\hat{q}(x)}(\Omega)} \leq 2 \|u\|_{L^{q(x)}(\Omega)} \|v\|_{L^{\hat{q}(x)}(\Omega)}$$

A very important role in manipulating the generalized Lebesgue spaces with variable exponent is played by the modular of the $L^{q(x)}(\Omega)$ space, which defined by

$$\begin{aligned} \rho_{q(\cdot)} : L^{q(x)}(\Omega) &\longrightarrow \mathbb{R} \\ u &\longrightarrow \rho_{q(\cdot)}(u) = \int_{\Omega} |u(x)|^{q(x)} dx \end{aligned}$$

Proposition 2.1. *Let $u \in L^{q(x)}(\Omega)$, then we have,*

- (1) $\|u\|_{L^{q(x)}(\Omega)} < 1$ (resp. $= 1, > 1$) $\Leftrightarrow \rho_{q(\cdot)}(u) < 1$ (resp. $= 1, > 1$),
- (2) $\|u\|_{L^{q(x)}(\Omega)} < 1 \Rightarrow \|u\|_{L^{q(x)}(\Omega)}^{q^+} \leq \rho_{q(\cdot)}(u) \leq \|u\|_{L^{q(x)}(\Omega)}^{q^-}$,
- (3) $\|u\|_{L^{q(x)}(\Omega)} > 1 \Rightarrow \|u\|_{L^{q(x)}(\Omega)}^{q^-} \leq \rho_{q(\cdot)}(u) \leq \|u\|_{L^{q(x)}(\Omega)}^{q^+}$.

Proposition 2.2. *If $u, u_k \in L^{q(x)}(\Omega)$ and $k \in \mathbb{N}$, then the following assertions are equivalent:*

- (1) $\lim_{k \rightarrow +\infty} \|u_k - u\|_{L^{q(x)}(\Omega)} = 0$,
- (2) $\lim_{k \rightarrow +\infty} \rho_{q(\cdot)}(u_k - u) = 0$.

In [1], the authors have proved the following basic theorem.

Theorem 2.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and let $s \in (0, 1)$. Let $p : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (1, +\infty)$ be a continuous variable exponent with $sp(x, y) < N$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$. Let (1.1) and (1.2) be satisfied. Let $r : \overline{\Omega} \longrightarrow (1, +\infty)$ be a continuous bounded variable exponent such that,*

$$1 < r(x) < p_s^*(x) = \frac{N\bar{p}(x)}{N - s\bar{p}(x)} \quad \text{for all } x \in \overline{\Omega}.$$

Suppose that $K : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0, +\infty)$ is a measurable function satisfying (1.4), (1.5) and (1.6). Then

(1) *There exists a positive constant $C = C(N, p, r, s, \Omega) > 0$, such that for any $u \in W^{K,p(x,y)}(\Omega)$, one has*

$$\|u\|_{L^{r(x)}(\Omega)} \leq C\|u\|_{W^{s,p(x,y)}(\Omega)} \leq C \max\{1, \tilde{k}_0\} \|u\|_{K,p(x,y)} = \tilde{C}\|u\|_{K,p(x,y)},$$

(2) *there exists a positive constant $C_0 = C_0(N, p, s, \tilde{k}_0, \Omega) > 0$, such that*

$$[u]_{K,p(x,y)} \leq \|u\|_{K,p(x,y)} \leq C_0[u]_{K,p(x,y)}.$$

For any $u \in W^{K,p(x,y)}(\Omega)$, we define the functional

$$\rho_{K,p(\cdot,\cdot)}(u) = \int_Q |u(x) - u(y)|^{p(x,y)} K(x, y) \, dx dy + \int_{\Omega} |u(x)|^{\bar{p}(x)} \, dx$$

It is easy to see that $\rho_{K,p(\cdot,\cdot)}$ is a *convex modular* on $W^{K,p(x,y)}(\Omega)$. The norm associated with $\rho_{K,p(\cdot,\cdot)}$ is given by

$$\|u\|_{\rho_{K,p(\cdot,\cdot)}} = \inf \left\{ \lambda > 0 : \rho_{K,p(\cdot,\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}.$$

Remark 2.1. $\rho_{K,p(\cdot,\cdot)}$ also check the results of Propositions 2.1 and 2.2.

Using the same argument as in [5, Theorem 2.17], we prove that $\|\cdot\|_{\rho_{K,p(\cdot,\cdot)}}$ is a norm on $W^{K,p(x,y)}(\Omega)$, which is equivalent to the norm $\|\cdot\|_{K,p(x,y)}$.

We also define the closed linear subspace of $W^{K,p(x,y)}(\Omega)$ by

$$E_0 = W_0^{K,p(x,y)}(\Omega) = \left\{ u \in W^{K,p(x,y)}(\Omega) : u(x) = 0 \text{ a.e. } \in \mathbb{R}^N \setminus \Omega \right\}.$$

Remark 2.2.

- (1) *The assertion (i) in Theorem 2.1 implies that $W^{K,p(x,y)}(\Omega)$ and $W_0^{K,p(x,y)}(\Omega)$ are continuously embedded in $L^{r(x)}(\Omega)$, where $1 < r(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$.*
- (2) *As a consequence of in Theorem 2.1-(ii), $[\cdot]_{K,p(x,y)}$ is an equivalent norm of $\|u\|_{K,p(x,y)}$ on $W_0^{K,p(x,y)}(\Omega)$.*
- (3) *$(W_0^{K,p(x,y)}(\Omega), [\cdot]_{K,p(x,y)})$ is a separable, reflexive, and uniformly convex Banach space (see [1, Lemma 3.5]).*

On the other hand, for any $u \in W_0^{K,p(x,y)}(\Omega)$, we define the functional

$$\rho_{K,p(\cdot,\cdot)}^o(u) = \int_Q |u(x) - u(y)|^{p(x,y)} K(x, y) \, dx dy.$$

$\rho_{K,p(\cdot,\cdot)}^o$ is a *convex modular* on $W_0^{K,p(x,y)}(\Omega)$. The norm associated with $\rho_{K,p(\cdot,\cdot)}^o$ is given by

$$\|u\|_{\rho_{K,p(\cdot,\cdot)}^o} = [u]_{K,p(x,y)} = \inf \left\{ \lambda > 0 : \rho_{K,p(\cdot,\cdot)}^o\left(\frac{u}{\lambda}\right) \leq 1 \right\} = \|u\|_{E_0}.$$

Remark 2.3.

- (1) $\rho_{K,p(\cdot,\cdot)}^o$ also check the results of Proposition 2.2.

(2) The modular $\rho_{K,p(\cdot,\cdot)}^o$ does not satisfy the triangle inequality, that is,

$$\rho_{K,p(\cdot,\cdot)}^o(u+v) \leq \rho_{K,p(\cdot,\cdot)}^o(u) + \rho_{K,p(\cdot,\cdot)}^o(v)$$

However, there is a substitute that is sometimes useful.

$$\rho_{K,p(\cdot,\cdot)}^o(u+v) \leq 2^{p^+-1} (\rho_{K,p(\cdot,\cdot)}^o(u) + \rho_{K,p(\cdot,\cdot)}^o(v)).$$

We will refer to this as the modular triangle inequality.

We could also get the following properties:

Lemma 2.2. ([1]). Let $p : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty)$, be a continuous variable exponent and let $K : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (0, +\infty)$ is a measurable function satisfy (1.4) and (1.5). Then For any $u \in W_0^{K,p(x,y)}$, we have

- (1) $1 \leq [u]_{K,p(x,y)} \Rightarrow [u]_{K,p(x,y)}^{p^-} \leq \rho_{K,p(\cdot,\cdot)}^o(u) \leq [u]_{K,p(x,y)}^{p^+}$,
- (2) $[u]_{K,p(x,y)} \leq 1 \Rightarrow [u]_{K,p(x,y)}^{p^+} \leq \rho_{K,p(\cdot,\cdot)}^o(u) \leq [u]_{K,p(x,y)}^{p^-}$.

Lemma 2.3. ([1]) Let (1.1), (1.2) and (1.3) be satisfied. Then the space $C_0^\infty(\mathbb{R}^N)$ of smooth functions with compact support is dense in $W^{K,p(x,y)}(\Omega)$.

Let (1.1) and (1.2), be satisfied and let $K : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (0, +\infty)$ is a measurable function satisfy (1.4), (1.5) and (1.6). Then

$$\begin{aligned} \mathcal{L}_K^{p(x)} : E_0 &\longrightarrow E_0^* \\ u &\longmapsto \mathcal{L}_K^{p(x)}(u) : E_0 \longrightarrow \mathbb{R} \\ \varphi &\longmapsto \langle \mathcal{L}_K^{p(x)}(u), \varphi \rangle \end{aligned}$$

such that

$$\langle \mathcal{L}_K^{p(x)}(u), \varphi \rangle = \int_{\mathbb{R}^{2N}} |u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) (\varphi(x) - \varphi(y)) K(x, y) \, dx dy.$$

Where $E_0^* = \left(W_0^{K,p(x,y)}(\Omega) \right)^*$ is the dual space of $E = W_0^{K,p(x,y)}(\Omega)$.

In the following Lemma, we introduce fundamental properties of the operator $\mathcal{L}_K^{p(x)}$.

Lemma 2.4. ([1]) Suppose that (1.1) and (1.2) be satisfied and let $K : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (0, +\infty)$ be a measurable function which satisfies (1.4), (1.5) and (1.6). Then, The following assertions hold:

- (1) $\mathcal{L}_K^{p(x)}$ is well defined and bounded,
- (2) $\mathcal{L}_K^{p(x)}$ is a strictly monotone operator,
- (3) $\mathcal{L}_K^{p(x)}$ is a mapping of type (S_+) , that is, if $u_k \rightharpoonup u$ in E_0 and $\limsup_{k \rightarrow +\infty} \langle \mathcal{L}_K^{p(x)}(u)_k - \mathcal{L}_K^{p(x)}(u), u_k - u \rangle \leq 0$,

then $u_k \rightarrow u$ in $W_0^{K,p(x,y)}$,

- (4) $\mathcal{L}_K^{p(x)}$ is a homeomorphism,
- (5) $\mathcal{L}_K^{p(x)}$ is coercive.

3. Main results

Definition 3.1. We say that $u \in E_0$ is a weak solution of problem $(\mathcal{P}_K)$, if for all $\varphi \in E_0$, we have

$$\begin{aligned} &\int_Q |u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) (\varphi(x) - \varphi(y)) K(x, y) \, dx dy \\ &+ \int_\Omega |u(x)|^{\bar{p}(x)-2} u(x) \varphi(x) dx - \lambda \int_\Omega |u(x)|^{r(x)-2} u(x) \varphi(x) dx = 0. \end{aligned} \tag{3.1}$$

Moreover, we say that λ is an eigenvalue of problem $(\mathcal{P}_K)$, if there exists $u \in E_0 \setminus \{0\}$ which satisfies (3.1), i.e. u is the corresponding eigenfunction to λ .

Let us consider the energy functional $\mathcal{J}_\lambda$ corresponding to problem $(\mathcal{P}_K)$, defined by, $\mathcal{J}_\lambda : W_0^{K,p(x,y)}(\Omega) \longrightarrow \mathbb{R}$

$$\mathcal{J}_\lambda(u) = \int_Q \frac{1}{p(x,y)} |u(x) - u(y)|^{p(x,y)} K(x,y) \, dx dy + \int_\Omega \frac{1}{\bar{p}(x)} |u(x)|^{\bar{p}(x)} dx - \lambda \int_\Omega \frac{1}{r(x)} |u(x)|^{r(x)} dx,$$

for any $\lambda > 0$.

Now, we introduce some important lemmas that show that the functional $\mathcal{J}_\lambda$ satisfies the geometrical conditions of the mountain pass theorem which are necessary to establish the proof of the existence result.

Lemma 3.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and let $s \in (0, 1)$. Let $p : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (1, +\infty)$ be a continuous variable exponent with $sp(x, y) < N$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$. Let (1.1) and (1.2) be satisfied. Let $r : \bar{\Omega} \longrightarrow (1, +\infty)$ be a continuous bounded variable exponent satisfy (1.7). Suppose that $K : \mathbb{R}^N \times \mathbb{R}^N \longrightarrow (0, +\infty)$ is a measurable function satisfying (1.4), (1.5) and (1.6). Then:*

(1) $\mathcal{J}_\lambda$ is well defined.

(2) $\mathcal{J}_\lambda \in C^1(W_0^{K,p(x,y)}(\Omega), \mathbb{R})$ and for all $u, \varphi \in W_0^{K,p(x,y)}(\Omega)$, its Gâteaux derivative is given by:

$$\begin{aligned} \langle \mathcal{J}'_\lambda(u), \varphi \rangle &= \int_Q |u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) (\varphi(x) - \varphi(y)) K(x, y) \, dx dy \\ &\quad + \int_\Omega |u(x)|^{\bar{p}(x)-2} u(x) \varphi(x) dx - \lambda \int_\Omega |u(x)|^{r(x)-2} u(x) \varphi(x) dx \end{aligned}$$

Proof. (i)- Let $u \in W_0^{K,p(x,y)}(\Omega)$; then

$$\begin{aligned} \mathcal{J}_\lambda(u) &= \int_Q \frac{1}{p(x,y)} |u(x) - u(y)|^{p(x,y)} K(x, y) \, dx dy + \int_\Omega \frac{1}{\bar{p}(x)} |u(x)|^{\bar{p}(x)} dx \\ &\quad - \lambda \int_\Omega \frac{1}{r(x)} |u(x)|^{r(x)} dx \\ &\leq \frac{1}{p^-} \left[\int_Q |u(x) - u(y)|^{p(x,y)} K(x, y) \, dx dy + \int_\Omega |u(x)|^{\bar{p}(x)} dx \right] \\ &\quad - \frac{\lambda}{r^+} \int_\Omega |u(x)|^{r(x)} dx \\ &= \frac{1}{p^-} \rho_{K,p(\cdot,\cdot)}(u) - \frac{\lambda}{r^+} \rho_{r(\cdot)}(u) \end{aligned}$$

By Proposition 2.1 and Remark 2.1, we get

$$\mathcal{J}_\lambda(u) \leq \frac{1}{p^-} \left[\|u\|_{E_0}^{p^+} + \|u\|_{E_0}^{p^-} \right] - \frac{\lambda}{r^+} \left[\|u\|_{L^{r(x)}(\Omega)}^{r^+} + \|u\|_{L^{r(x)}(\Omega)}^{r^-} \right]$$

Using the Remark 2.2-(1), we obtain

$$\begin{aligned} \mathcal{J}_\lambda(u) &\leq \frac{1}{p^-} \left[\|u\|_{E_0}^{p^+} + \|u\|_{E_0}^{p^-} \right] - \frac{\lambda}{r^+} \left[\tilde{C}^{r^+} \|u\|_{E_0}^{r^+} + \tilde{C}^{r^-} \|u\|_{E_0}^{r^-} \right] \\ &\leq \frac{1}{p^-} \left[\|u\|_{E_0}^{p^+} + \|u\|_{E_0}^{p^-} \right] - \frac{\lambda}{r^+} \max \{ \tilde{C}^{r^+}, \tilde{C}^{r^-} \} \left[\|u\|_{E_0}^{r^+} + \|u\|_{E_0}^{r^-} \right] \\ &\leq \left(\frac{1}{p^-} - \frac{\lambda}{r^+} \max \{ \tilde{C}^{r^+}, \tilde{C}^{r^-} \} \right) \left[\|u\|_{E_0}^{p^+} + \|u\|_{E_0}^{r^+} \right] < +\infty. \end{aligned}$$

(ii)- **Existence of the Gâteaux derivative.** We define

$$\Psi(u) = \int_Q \frac{1}{p(x,y)} |u(x) - u(y)|^{p(x,y)} K(x, y) \, dx dy, \quad \Phi(u) = \int_\Omega \frac{1}{\bar{p}(x)} |u(x)|^{\bar{p}(x)} dx$$

and

$$\Phi_\lambda(u) = \lambda \int_{\Omega} \frac{1}{r(x)} |u(x)|^{r(x)} dx.$$

Then

$$\mathcal{J}_\lambda(u) = \Psi(u) + \Phi(u) - \Phi_\lambda(u) \text{ and } \mathcal{J}'_\lambda(u) = \Psi'(u) + \Phi'(u) - \Phi'_\lambda(u). \quad (3.2)$$

• For any $u, \varphi \in E_0$, we have

$$\langle \Psi'(u), \varphi \rangle = \int_Q |u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) (\varphi(x) - \varphi(y)) K(x, y) dx dy. \quad (3.3)$$

Indeed,

$$\begin{aligned} \langle \Psi'(u), \varphi \rangle &= \lim_{t \rightarrow 0} \frac{\Psi(u+t\varphi) - \Psi(u)}{t} \\ &= \lim_{t \rightarrow 0} \left\{ \int_Q \frac{|(u(x) + t\varphi(x)) - (u(y) + t\varphi(y))|^{p(x,y)} - |u(x) - u(y)|^{p(x,y)}}{tp(x, y)} K(x, y) dx dy \right\}. \end{aligned} \quad (3.4)$$

Let us consider $M : [0, 1] \rightarrow \mathbb{R}$

$$\alpha \mapsto \frac{|(u(x) - u(y)) + \alpha t(\varphi(x) - \varphi(y))|^{p(x,y)}}{tp(x, y)} K(x, y).$$

M is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Then by the mean value theorem, there exists $\theta \in (0, 1)$ such that

$$M'(\alpha)(\theta) = M(1) - M(0)$$

Then

$$\begin{aligned} & |(u(x) - u(y)) + \theta t(\varphi(x) - \varphi(y))|^{p(x,y)-2} [(u(x) - u(y)) + t\theta(\varphi(x) - \varphi(y))] (\varphi(x) - \varphi(y)) K(x, y) \\ &= S_t(u, \varphi) = \frac{|(u(x) - u(y)) + t(\varphi(x) - \varphi(y))|^{p(x,y)} - |u(x) - u(y)|^{p(x,y)}}{tp(x, y)} K(x, y) \end{aligned} \quad (3.5)$$

Combining (3.4) and (3.5), we get,

$$\langle \Psi'(u), \varphi \rangle = \lim_{t \rightarrow 0} \int_Q S_t(u, \varphi) dx dy.$$

Since $t, \theta \in [0, 1]$, so $t\theta \leq 1$, which implies

$$S_t(u, \varphi) \leq |(u(x) - u(y)) + (\varphi(x) - \varphi(y))|^{p(x,y)-2} [(u(x) - u(y)) + (\varphi(x) - \varphi(y))] (\varphi(x) - \varphi(y)) K(x, y).$$

On the other hand

$$S_t(u, \varphi) \xrightarrow{t \rightarrow 0} |u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) (\varphi(x) - \varphi(y)) K(x, y).$$

Hence, by the dominated convergence theorem, we obtain (3.3).

By the same argument, we have

$$\langle \Phi'(u), \varphi \rangle = \int_{\Omega} |u(x)|^{\bar{p}(x)-2} u(x) \varphi(x) dx \text{ and } \langle \Phi'_\lambda(u), \varphi \rangle = \lambda \int_{\Omega} |u(x)|^{r(x)-2} u(x) \varphi(x) dx.$$

Then by relation (3.2), the result holds.

Continuity of the Gâteaux derivative of $\mathcal{J}_\lambda$. Assume that $u_k \rightarrow u$ in E_0 , and we will show that $\Psi'(u_k) \rightarrow \Psi'(u)$ in E_0^* . Indeed,

$$\begin{aligned} & \langle \Psi'(u_k) - \Psi'(u), \varphi \rangle = \\ & \int_Q \left[|u_k(x) - u_k(y)|^{p(x,y)-2} (u_k(x) - u_k(y)) - |u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) \right] K(x, y) \\ & \quad \times (\varphi(x) - \varphi(y)) \, dx dy \\ & = \int_Q \left[|u_k(x) - u_k(y)|^{p(x,y)-2} (u_k(x) - u_k(y)) K(x, y)^{\frac{p(x,y)-1}{p(x,y)}} \right. \\ & \quad \left. - |u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) K(x, y)^{\frac{p(x,y)-1}{p(x,y)}} \right] \times (\varphi(x) - \varphi(y)) K(x, y)^{\frac{1}{p(x,y)}} \, dx dy \end{aligned}$$

Let us set

$$\begin{aligned} F_k(x, y) &= |u_k(x) - u_k(y)|^{p(x,y)-2} (u_k(x) - u_k(y)) K(x, y)^{\frac{p(x,y)-1}{p(x,y)}} \in L^{\hat{p}(x,y)}(Q), \\ F(x, y) &= |u(x) - u(y)|^{p(x,y)-2} (u(x) - u(y)) K(x, y)^{\frac{p(x,y)-1}{p(x,y)}} \in L^{\hat{p}(x,y)}(Q), \\ \bar{\varphi}(x, y) &= (\varphi(x) - \varphi(y)) K(x, y)^{\frac{1}{p(x,y)}} \in L^{p(x,y)}(Q), \end{aligned}$$

where $\frac{1}{p(x,y)} + \frac{1}{\bar{p}(x,y)} = 1$.

Hence, by the Hölder inequality (see Lemma 2.1), we obtain

$$\langle \Psi'(u_k) - \Psi'(u), \varphi \rangle \leq 2 \|F_k - F\|_{L^{\hat{p}(x,y)}(Q)} \|\bar{\varphi}\|_{L^{p(x,y)}(Q)}.$$

Thus

$$\|\Psi'(u_k) - \Psi'(u)\|_{E_0^*} \leq 2 \|F_k - F\|_{L^{\hat{p}(x,y)}(Q)}.$$

Now, let

$$v_k(x, y) = \frac{u_k(x) - u_k(y)}{|x - y|^{\frac{N}{p(x,y)} + s}} \in L^{p(x,y)}(Q) \text{ and } v(x, y) = \frac{u(x) - u(y)}{|x - y|^{\frac{N}{p(x,y)} + s}} \in L^{p(x,y)}(Q).$$

Since $u_k \rightarrow u$ in E_0 . Then $v_k \rightarrow v$ in $L^{p(x,y)}(Q)$.

Hence, for a subsequence of $(v_k)_{k \geq 0}$, we get

$$v_k(x, y) \rightarrow v(x, y) \text{ a.e. in } Q \text{ and } \exists h \in L^{p(x,y)}(Q) \text{ such that } |v_k(x, y)| \leq h(x, y).$$

So we have

$$F_k(x, y) \rightarrow F(x, y) \text{ a.e. in } Q \text{ and } |F_k(x, y)| = |v_k(x, y)|^{p(x,y)-1} \leq |h(x, y)|^{p(x,y)-1}.$$

Then, by the dominated convergence theorem, we deduce that

$$F_k \rightarrow F \text{ in } L^{\hat{p}(x,y)}(Q).$$

Consequently

$$\Psi'(u_k) \rightarrow \Psi'(u) \text{ in } E_0^*.$$

By the same argument, we show that

$$\Phi'(u_k) \rightarrow \Phi'(u) \text{ in } (L^{\bar{p}(x)}(\Omega))^* \quad \text{and} \quad \Phi'_\lambda(u_k) \rightarrow \Phi'_\lambda(u) \text{ in } (L^{r(x)}(\Omega))^*$$

Then by relation (3.2), we deduce the continuity of $\mathcal{J}'_\lambda$.
The proof of Lemma 3.1 is complete. $\square$

The following result shows that the functional $\mathcal{J}_\lambda$ satisfies the first geometrical condition of the mountain pass theorem;

Lemma 3.2. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and let $s \in (0, 1)$. Let $p : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty)$ be a continuous variable exponent with $sp(x, y) < N$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$. Let (1.1) and (1.2) be satisfied. Let $r : \bar{\Omega} \rightarrow (1, +\infty)$ be a continuous bounded variable exponent satisfy (1.7). Suppose that $K : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (0, +\infty)$ is a measurable function satisfying (1.4), (1.5) and (1.6). Then, there exists $\lambda^* > 0$ such that for any $\lambda \in (0, \lambda^*)$, there exist $R, a > 0$ such that $\mathcal{J}_\lambda(u) \geq a > 0$ for any $u \in W_0$ with $\|u\|_{W_0} = R$.*

Proof. Since $r(x) < p_s^*(x)$ for all $x \in \bar{\Omega}$, so by Remark 2.2-(1), E_0 is continuously embedded in $L^{r(x)}(\Omega)$. Then there exists a positive constant c_1 such that

$$\|u\|_{L^{r(x)}(\Omega)} \leq c_1 \|u\|_{E_0}, \quad \text{for all } u \in E_0. \quad (3.6)$$

We fix $R \in (0, 1)$ such that $R < \frac{1}{c_1}$. Then relation (3.6) implies

$$\|u\|_{L^{r(x)}(\Omega)} < 1 \quad \text{for all } u \in E_0 \text{ with } R = \|u\|_{E_0}.$$

By Proposition 2.1-(ii), we get

$$\int_{\Omega} |u(x)|^{r(x)} dx \leq \|u\|_{L^{r(x)}(\Omega)}^{r^-}, \quad \text{for all } u \in E_0 \text{ with } R = \|u\|_{E_0}. \quad (3.7)$$

Combining (3.6) and (3.7), we get

$$\int_{\Omega} |u(x)|^{r(x)} dx \leq c_1^{r^-} \|u\|_{W_0}^{r^-}, \quad \text{for all } u \in E_0 \text{ with } R = \|u\|_{E_0}. \quad (3.8)$$

Using the fact that $\|u\|_{E_0} < 1$ and (3.4), we deduce that for any $u \in E_0$ with $R = \|u\|_{E_0}$ the following inequalities hold true

$$\begin{aligned} \mathcal{J}_\lambda(u) &\geq \frac{1}{p^+} \int_Q |u(x) - u(y)|^{p(x,y)} K(x, y) dx dy + \frac{1}{p^+} \int_{\Omega} |u(x)|^{\bar{p}(x)} dx \\ &\quad - \frac{\lambda}{r^-} \int_{\Omega} |u(x)|^{r(x)} dx \\ &\geq \frac{1}{p^+} \left[\int_Q |u(x) - u(y)|^{p(x,y)} K(x, y) dx dy + \int_{\Omega} |u(x)|^{\bar{p}(x)} dx \right] \\ &\quad - \frac{\lambda}{r^-} \int_{\Omega} |u(x)|^{r(x)} dx \\ &\geq \frac{1}{p^+} \|u\|_{E_0}^{p^+} - \frac{\lambda}{r^-} c_1^{r^-} \|u\|_{E_0}^{r^-} \\ &\geq \frac{1}{p^+} R^{p^+} - \frac{\lambda}{r^-} c_1^{r^-} R^{r^-} \\ &\geq R^{r^-} \left(\frac{1}{p^+} R^{p^+ - r^-} - \frac{\lambda}{r^-} c_1^{r^-} \right). \end{aligned} \quad (3.9)$$

By the inequality (3.9), we can choose λ^* in order to

$$\frac{1}{p^+} R^{p^+ - r^-} - \frac{\lambda}{r^-} c_1^{r^-} > 0$$

Hence, if

$$\lambda^* = \frac{R^{p^+ - r^-}}{2p^+} \cdot \frac{r^-}{c_1^{r^-}}, \quad (3.10)$$

then, for any $\lambda \in (0, \lambda^*)$ and any $u \in W_0$ with $\|u\|_{W_0} = R$, there exists $a = \frac{R^{p^+}}{2p^+} > 0$ such that

$$\mathcal{J}_\lambda(u) \geq a > 0,$$

which completes the proof of Lemma 3.2. $\square$

The following result shows that the functional $\mathcal{J}_\lambda$ satisfies the second geometrical condition of mountain pass theorem;

Lemma 3.3. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and let $s \in (0, 1)$. Let $p : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty)$ be a continuous variable exponent with $sp(x, y) < N$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$. Let (1.1) and (1.2) be satisfied. Let $r : \bar{\Omega} \rightarrow (1, +\infty)$ be a continuous bounded variable exponent satisfy (1.7). Suppose that $K : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (0, +\infty)$ is a measurable function satisfying (1.3), (1.4) and (1.5). Then, there exists $\varphi \in W_0$ such that $\varphi \geq 0$, $\varphi \neq 0$ and $\mathcal{J}_\lambda(t\varphi) < 0$ for any t small enough.*

Proof. Assumption (1.6) implies that $r^- < p^-$. Let $\varepsilon > 0$ be such that $r^- + \varepsilon \leq p^-$. Since $r \in C(\bar{\Omega})$, then we can find an open set $\Omega_0 \subset \Omega$ such that

$$|r(x) - r^-| \leq \varepsilon \quad \text{for all } x \in \Omega_0.$$

Consequently

$$r(x) \leq r^- + \varepsilon \leq p^- \quad \text{for all } x \in \Omega_0.$$

Let $\varphi \in C_0^\infty(\Omega)$ be such that $\bar{\Omega}_0 \subset \text{supp } \varphi$, $\varphi(x) = 1$ for all $x \in \Omega_0$, and $0 \leq \varphi \leq 1$ in Ω . Then using the above information for any $t \in (0, 1)$ we have

$$\begin{aligned} \mathcal{J}_\lambda(t\varphi) &= \int_Q \frac{t^{p(x,y)}}{p(x,y)} |\varphi(x) - \varphi(y)|^{p(x,y)} K(x, y) \, dx dy + \int_\Omega \frac{t^{\bar{p}(x)}}{\bar{p}(x)} |\varphi(x)|^{\bar{p}(x)} \, dx \\ &\quad - \lambda \int_\Omega \frac{t^{r(x)}}{r(x)} |\varphi(x)|^{r(x)} \, dx \\ &\leq \frac{t^{p^-}}{p^-} \left[\int_Q |\varphi(x) - \varphi(y)|^{p(x,y)} K(x, y) \, dx dy + \int_\Omega |\varphi(x)|^{\bar{p}(x)} \, dx \right] \\ &\quad - \lambda \int_{\Omega_0} \frac{t^{r(x)}}{r(x)} |\varphi(x)|^{r(x)} \, dx \\ &\leq \frac{t^{p^-}}{p^-} \rho_{K,p(\cdot,\cdot)}(u) - \frac{\lambda}{r^+} t^{r^- + \varepsilon} \int_{\Omega_0} |\varphi(x)|^{r(x)} \, dx \\ &\leq t^{r^- + \varepsilon} \left[\frac{\rho_{K,p(\cdot,\cdot)}(\varphi)}{p^-} t^{p^- - r^- - \varepsilon} - \frac{\lambda}{r^+} \int_{\Omega_0} |\varphi(x)|^{r(x)} \, dx \right] \end{aligned}$$

Thus

$$\mathcal{J}_\lambda(t\varphi) < 0 \quad \text{for any } t < \xi^{\frac{1}{p^- - r^- - \varepsilon}}$$

Where

$$0 < \xi < \min \left\{ 1, \frac{\frac{\lambda p^-}{r^+} \int_{\Omega_0} |\varphi(x)|^{r(x)} \, dx}{\rho_{K,p(\cdot,\cdot)}(\varphi)} \right\}$$

Finally, we point out that $\rho_{p(\cdot,\cdot)}(\varphi) > 0$ (this fact implies that $\varphi \neq 0$).

Indeed, since $\Omega_0 \subset \text{supp } \varphi \subset \Omega$ and $0 \leq \varphi \leq 1$ in Ω , so we get

$$0 < \int_{\Omega_0} |\varphi(x)|^{r(x)} \, dx \leq \int_\Omega |\varphi(x)|^{r(x)} \, dx \leq \int_\Omega |\varphi(x)|^{r^-} \, dx \quad (3.11)$$

On the other hand, since $1 < r^- < p_s^*(x)$ for all $x \in \bar{\Omega}$, then W_0 is continuously embedded in $L^{r^-}(\Omega)$, so there exists $c_2 > 0$ such that

$$\|\varphi\|_{L^{r^-}(\Omega)} \leq c_2 \|\varphi\|_{E_0} \quad (3.12)$$

Combining (3.11) and (3.12), we get

$$0 < \frac{1}{c_2} \|\varphi\|_{L^{r^-}(\Omega)} \leq \|\varphi\|_{E_0}$$

This fact and Proposition 2.1 ((ii) or (iii)) imply that

$$\rho_{p(\cdot, \cdot)}(\varphi) > 0.$$

The lemma 3.3 is proved. $\square$

Our main result is given by the following theorem

Theorem 3.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and let $s \in (0, 1)$. Let $p : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (1, +\infty)$ be a continuous variable exponent with $sp(x, y) < N$ for all $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$. Let (1.1) and (1.2) be satisfied. Let $r : \bar{\Omega} \rightarrow (1, +\infty)$ be a continuous bounded variable exponent satisfy (1.7). Suppose that $K : \mathbb{R}^N \times \mathbb{R}^N \rightarrow (0, +\infty)$ is a measurable function satisfying (1.4), (1.5) and (1.6). Then there exists $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ is an eigenvalue of problem $(\mathcal{P}_K)$.*

The proof of theorem 3.1 is based on Ekeland's variational principle and the Mountain Pass theorem.

Proof. Let $\lambda^* > 0$ be defined as in (3.10) and let $\lambda \in (0, \lambda^*)$. By Lemma 3.2, it follows that

$$\inf_{\partial B_R(0)} \mathcal{J}_\lambda > 0, \quad (3.13)$$

where $\partial B_R(0) = \{u \in B_R(0) : \|u\|_{W_0} = R\}$ and $B_R(0)$ is the ball centered at the origin and of radius R in E_0 .

On the other hand, by Lemma 3.3, there exists $\varphi \in E_0$ such that $\mathcal{J}_\lambda(t\varphi) < 0$ for any t small enough. Moreover, by (3.5), for all $u \in B_R(0)$, we get

$$\mathcal{J}_\lambda(u) \geq \frac{1}{p^+} \|u\|_{E_0}^{p^+} - \frac{\lambda}{r^-} c_1^{r^-} \|u\|_{E_0}^{r^-}. \quad (3.14)$$

Then we have

$$-\infty < \bar{c} = \inf_{B_R(0)} \mathcal{J}_\lambda < 0. \quad (3.15)$$

Combining (3.13) and (3.11), then we can assume that

$$0 < \varepsilon < \inf_{\partial B_R(0)} \mathcal{J}_\lambda - \inf_{B_R(0)} \mathcal{J}_\lambda.$$

Applying Ekeland's variational principle to the functional

$\mathcal{J}_\lambda : \overline{B_R(0)} \rightarrow \mathbb{R}$, we find $u_\varepsilon \in \overline{B_R(0)}$ such that

$$\begin{cases} \mathcal{J}_\lambda(u_\varepsilon) < \inf_{B_R(0)} \mathcal{J}_\lambda + \varepsilon, \\ \mathcal{J}_\lambda(u_\varepsilon) < \mathcal{J}_\lambda(u) + \varepsilon \|u - u_\varepsilon\|_{E_0}, \quad \forall u \neq u_\varepsilon. \end{cases} \quad (3.16)$$

So

$$\mathcal{J}_\lambda(u_\varepsilon) \leq \inf_{B_R(0)} \mathcal{J}_\lambda + \varepsilon \leq \inf_{B_R(0)} \mathcal{J}_\lambda + \varepsilon < \inf_{\partial B_R(0)} \mathcal{J}_\lambda$$

It follows that $u_\varepsilon \in B_R(0)$.

Now, we consider $\mathcal{I}_\lambda^\varepsilon : \overline{B_R(0)} \rightarrow \mathbb{R}$

$$u \rightarrow \mathcal{J}_\lambda(u) + \varepsilon \|u - u_\varepsilon\|_{W_0}.$$

By (3.12), we get

$$\mathcal{I}_\lambda^\varepsilon(u_\varepsilon) = \mathcal{J}_\lambda(u) < \mathcal{I}_\lambda^\varepsilon(u) \quad \text{for all } u \neq u_\varepsilon.$$

Thus u_ε is a minimum point of $\mathcal{I}_\lambda^\varepsilon$ on $\overline{B_R(0)}$. It follows that for any $t > 0$ small enough and $v \in B_R(0)$,

$$\frac{\mathcal{I}_\lambda^\varepsilon(u_\varepsilon + tv) - \mathcal{I}_\lambda^\varepsilon(u_\varepsilon)}{t} \geq 0.$$

By this fact, we claim that

$$\frac{\mathcal{J}_\lambda(u_\varepsilon + tv) - \mathcal{J}_\lambda(u_\varepsilon)}{t} + \varepsilon \|v\|_{E_0} \geq 0.$$

When t tends to 0^+ , we have that

$$\langle \mathcal{J}'_\lambda(u_\varepsilon), v \rangle + \varepsilon \|v\|_{E_0} \geq 0.$$

And that gives

$$\|\mathcal{J}_\lambda(u_\varepsilon)\|_{E_0^*} \leq \varepsilon. \quad (3.17)$$

From (3.17), we deduce that there exists a sequence $\{w_k\} \subset B_r(0)$ such that

$$\mathcal{J}_\lambda(w_k) \longrightarrow \bar{c} \quad \text{and} \quad \mathcal{J}'_\lambda(w_k) \longrightarrow 0. \quad (3.18)$$

From (3.14) and (3.18), we have that $\{w_k\}$ is bounded in E_0 . Thus there exists $w \in E_0$ such that $w_k \rightharpoonup w$ in E_0 .

By (1.7), we have that $r(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$, so by Theorem 2.1 and Remark 2.2 – (1) we deduce that W_0 is compactly embedded in $L^{r(x)}(\Omega)$, then

$$w_k \longrightarrow w \text{ in } L^{r(x)}(\Omega). \quad (3.19)$$

Using Lemma 2.1, we have

$$\int_{\Omega} |w_k|^{r(x)-2} w_k (w_k - w) dx \leq 2 \|w_k\|_{L^{r(x)}(\Omega)} \|w_k - w\|_{L^{r(x)}(\Omega)}.$$

So, by (3.15), we get

$$\lim_{k \rightarrow +\infty} \int_{\Omega} |w_k|^{r(x)-2} w_k (w_k - w) dx = 0. \quad (3.20)$$

Since $\bar{p}(x) < p_s^*(x)$ for all $x \in \overline{\Omega}$, by the same argument, we have

$$\lim_{k \rightarrow +\infty} \int_{\Omega} |w_k|^{\bar{p}(x)-2} w_k (w_k - w) dx = 0. \quad (3.21)$$

On the other hand, from (3.18), we get

$$\lim_{k \rightarrow +\infty} \langle \mathcal{J}'_\lambda(w_k), w_k - w \rangle = 0.$$

Namely

$$\begin{aligned} \lim_{k \rightarrow +\infty} \left\{ \int_{\Omega \times \Omega} \frac{|w_k(x) - w_k(y)|^{p(x,y)-2} (w_k(x) - w_k(y)) ((w_k(x) - w_k(y)) - (w(x) - w(y)))}{|x - y|^{N+sp(x,y)}} dx dy \right. \\ \left. + \int_{\Omega} |w_k(x)|^{\bar{p}(x)-2} w_k(x) (w_k(x) - w(x)) dx - \lambda \int_{\Omega} |w_k(x)|^{r(x)-2} w_k(x) (w_k(x) - w(x)) dx \right\} = 0. \end{aligned}$$

Hence, relations (3.20) and (3.21) yield

$$\lim_{k \rightarrow +\infty} \int_{\Omega \times \Omega} \frac{|w_k(x) - w_k(y)|^{p(x,y)-2} (w_k(x) - w_k(y)) ((w_k(x) - w_k(y)) - (w(x) - w(y)))}{|x - y|^{N+sp(x,y)}} dx dy = 0.$$

Using the above information, Lemma 2.4 – (ii), and the fact that $w_k \rightarrow w$ in W_0 we get

$$\begin{cases} \limsup < \mathcal{L}(w_k), w_k - w > \leq 0, \\ w_k \rightarrow w \text{ in } W_0, \\ \mathcal{L} \text{ is a mapping of type } (S_+). \end{cases} \Rightarrow w_k \rightarrow w \text{ in } W_0.$$

Then by (3.18), we obtain

$$\mathcal{J}_\lambda(w) = \lim_{k \rightarrow +\infty} \mathcal{J}_\lambda(w_k) = \bar{c} < 0 \quad \text{and} \quad \mathcal{J}'_\lambda(w) = 0.$$

We conclude that w is a nontrivial critical point of $\mathcal{J}_\lambda$. Thus w is a nontrivial weak solution for problem $(\mathcal{P}_s)$. Finally any $\lambda \in (0, \lambda^*)$ is an eigenvalue of problem $(\mathcal{P}_s)$.

The proof of Theorem 3.1 is complete. $\square$

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