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Notes on Knaster-Tarski Theorem versus Monotone Nonexpansive Mappings

Dedicated to Ibn al-Banna' al-Marrakushi (c. 1256 c. 1321)

Mohamed Amine Khamsi[1,a]

Abstract. *The purpose of this note is to discuss the recent paper of Espínola and Wiśnicki about the fixed point theory of monotone nonexpansive mappings. In their work, it is claimed that most of the fixed point results of this class of mappings boil down to the classical Knaster-Tarski fixed point theorem. We will show that their approach is very restrictive and fail to have any meaningful usefulness in applications.*

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1. Introduction

The fixed point theory is considered as one of the most powerful tools of modern mathematics. It allowed for example the recent explosion of nonlinear functional analysis. Fixed point theory finds its roots in the earlier works of Lefschetz-Hopf, Leray-Schauder and Poincaré. For example, the existence of solutions in many problems are usually translated into a fixed point problem, like the existence of closed periodic orbits in dynamical systems and answer sets in logic programming.

Metric fixed point theory is a sub-branch of the general fixed point theory. It usually deals with mappings defined within metric spaces and satisfy metric conditions. The most famous result in this sub-branch is known as the Banach Contraction Principle [9]. This theorem finds its roots in the early works of Cauchy, Fredholm, Liouville, Lipschitz, Peano and Picard. The most interesting part of this sub-branch is the possibility to study successive approximations of the fixed points even when such fixed point is proved to exist using different tools. This point is very important for our work at hand.

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[1] *Department of Mathematical Sciences, The University of Texas at El Paso El Paso, TX 79968, U.S.A. and Department of Mathematics and Statistics, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia.*
[a] *e-mail: mohamed@utep.edu, mkhamsi@kfupm.edu.sa .*

Nonexpansive mappings are those maps, which have Lipschitz constant equal to one. These mappings can obviously be viewed as a natural extension of contraction mappings (for which the Lipschitz constant is less than one). However the fixed point problem for nonexpansive mappings differ sharply from that of the contraction mappings in the sense that additional structure of the domain set is needed to insure the existence of fixed points. It took almost four decades to see the first fixed points results for nonexpansive mappings in Banach spaces following the publication in 1965 of the work of Browder [14], Göhde [20], and Kirk [26]. It is worth mentioning that Kirk's fixed point theorem is more general than that of Browder and Göhde. But Kirk's theorem does not allow for approximations of the fixed point since it is deeply based on the use of Zorn's lemma.

Following the publication of Ran and Reurings' fixed point theorem [32], in which the authors investigated the extension of the Banach Contraction Principle to metric sets endowed with a partial order for monotone or order preserving mappings, many mathematicians got interested into the investigation of the fixed point problem for monotone mappings. In [23], Jachymski was the first to give a general unified version of these extensions by considering oriented graphs. In this regard, we would like to point to Ben-El-Mechaiekh's work [10] where he gives an extension of Ran and Reurings' fixed point theorem in a metric space endowed with a binary relation.
Since in this work, we mostly discuss the fixed point theory for monotone mappings or order preserving mappings, we refer the readers to the books by Khamsi and Kirk [25], and by Zeidler [36].

2. Banach Contraction Principle for Monotone Mappings

The *Banach's Contraction Principle* was first published in 1922 by S. Banach [9]. It is stated for Lipschitz mappings.

Definition 2.1. *Let (M,d) be a metric space. The mapping $T : M \to M$ is said to be* **Lipschitzian** *if there exists a constant $k > 0$ (called Lipschitz constant) such that*

$$d\Big(T(x), T(y)\Big) \leq k\ d(x,y)$$

for all $x, y \in M$. A Lipschitzian mapping with a Lipschitz constant $k < 1$ (resp. $k = 1$), is called contraction (resp. nonexpansive). A point $x \in M$ is called a fixed point of T whenever $T(x) = x$. The set of fixed points of T will be denoted $Fix(T)$.

Theorem 2.1. [9] *(***Banach Contraction Principle***) Let (M,d) be a complete metric space and let $T : M \to M$ be a contraction mapping, with Lipschitz constant $k < 1$. Then T has a unique fixed point ω in M. Moreover, we have*

$$d\Big(T^n(x), \omega\Big) \leq \frac{k^n}{1-k}\ d\Big(T(x), x\Big),$$

for any $x \in M$ and $n \geq 1$, which implies $\lim\limits_{n\to\infty} T^n(x) = \omega$.

Let (M,d) be metric space partially ordered by $\preceq$. Recall that $x, y \in M$ are said to be comparable if and only if $x \preceq y$ or $y \preceq x$.

Definition 2.2. *Let $(M, d, \preceq)$ be a metric space endowed with a partial order. Let $T : M \to M$ be a map.*

(i) *T is said to be monotone or order-preserving if $T(x) \preceq T(y)$ whenever $x \preceq y$, for any $x, y \in M$.*
(ii) *T is said to be monotone Lipschitzian mapping if T is monotone and there exists $k \geq 0$ such that*

$$d(T(x), T(y)) \leq k\ d(x,y),$$

for any $x, y \in M$ such that x and y are comparable. If $k < 1$, then we say that T is a monotone contraction mapping. And if $k = 1$, T is called a monotone nonexpansive mapping.

Note that monotone Lipschitzian mappings are not necessarily continuous. Next we state the theorem of Ran and Reurings.

Theorem 2.2. [32] *Let $(M, d, \preceq)$ be a complete metric space endowed with a partial order. Let $T : M \to M$ be a continuous monotone contraction mapping. Assume that there exists $x_0 \in M$ such that x_0 and $T(x_0)$ are comparable. Then the sequence $\{T^n(x_0)\}$ converges to a fixed point ω of T. Moreover if $x \in M$ is comparable to x_0, then we have $\lim\limits_{n\to+\infty} T^n(x) = \omega$.*

In fact under suitable conditions, the fixed point may be unique. Indeed, if we assume that every pair $x, y \in M$ has an upper bound or a lower bound in M, then T has a unique fixed point ω and $\lim\limits_{n\to+\infty} T^n(x) = \omega$, for any $x \in M$.

Remark 2.1. *The continuity assumption in Theorem 2.2 may be relaxed. This point was noted by Nieto and Rodríguez-López* [29]*. Indeed, if we assume that $\preceq$ satisfies the following property*

(NRL) *for any $\{x_n\}$ in M such that $x_n \preceq x_{n+1}$ (resp. $x_{n+1} \preceq x_n$), for any $n \geq 1$, and $\lim\limits_{n\to+\infty} x_n = x$, then $x_n \preceq x$ (resp. $x \preceq x_n$),*

then the conclusion of Theorem 2.2 still holds. If we assume that order intervals are closed, then $\preceq$ satisfies the property (NRL). Recall that an order interval is any of the subsets

$$[a, \to) = \{x \in M; a \preceq x\},\ (\leftarrow, b] = \{x \in M; x \preceq b\},\ [a, b] = [a, \to) \cap (\leftarrow, b],$$

for any $a, b \in M$.

Before we conclude this section, let us state Jachymski's formulation of Theorem 2.2 to metric spaces endowed with an oriented graph [23]. Recall that an oriented or digraph G consists of a set of vertices $V(G)$ and a set of edges $E(G) \subset V(G) \times V(G)$. Throughout, we assume that G is reflexive, i.e., $(x, x) \in E(G)$ for any $x \in V(G)$ and that G has no parallel edges (arcs) and so we can identify G with the pair $(V(G), E(G))$. G will be said to be transitive whenever $(x, z) \in E(G)$ holds provided $(x, y) \in E(G)$ and $(y, z) \in E(G)$, for any $x, y, z \in V(G)$. The graph theory notations and terminology are standard and can be found in all graph theory books, like [16, 24]. Clearly a partial order generate easily a digraph but not any digraph is generated by a partial order. Let (M, d) be a metric space and $G = (V(G), E(G))$ be a digraph such that $V(G) = M$. The analogue to Definition 2.2 in terms of graphs is the following:

Definition 2.3. *Let (M, d) be a metric space endowed with a digraph G such that $V(G) = M$. Let $T : M \to M$ be a map.*

(i) *T is said to be G-monotone or edge-preserving if $(T(x), T(y)) \in E(G)$ whenever $(x, y) \in E(G)$, for any $x, y \in M$.*
(ii) *T is said to be monotone G-Lipschitzian mapping if T is G-monotone and there exists $k \geq 0$ such that*

$$d(T(x), T(y)) \leq k\ d(x, y),$$

for any $x, y \in M$ such that $(x, y) \in E(G)$ or $(y, x) \in E(G)$. If $k < 1$, then we say that T is a monotone G-contraction mapping. And if $k = 1$, T is called a monotone G-nonexpansive mapping.

The graph version of Theorem 2.2 may be stated as

Theorem 2.3. [23] *Let (M, d) be a complete metric space endowed with a digraph G such that $V(G) = M$. Let $T : M \to M$ be a continuous monotone G-contraction mapping. Assume that there exists $x_0 \in M$ such that $(x_0, T(x_0) \in E(G)$ or $(T(x_0, x_0) \in E(G)$. Then the sequence $\{T^n(x_0)\}$ converges to a fixed point ω of T. Moreover if $x \in M$ is such that $(x_0, x) \in E(G)$ or $(x, x_0) \in E(G)$, then we have $\lim\limits_{n\to+\infty} T^n(x) = \omega$.*

Jachymski also pointed out that the continuity assumption in Theorem 2.3 may be relaxed by introducing a similar property as (NRL):

(NRLJ) for any $\{x_n\}$ in M such that $(x_n, x_{n+1}) \in E(G)$ (resp. $(x_{n+1}, x_n) \in E(G)$), for any $n \in \mathbb{N}$, and $\lim\limits_{n\to+\infty} x_n = x$, then $(x_n, x) \in E(G)$ (resp. $(x, x_n) \in E(G)$).

A sequence $\{x_n\}$ such that $(x_n, x_{n+1}) \in E(G)$ (resp. $(x_{n+1}, x_n) \in E(G)$), for any $n \in \mathbb{N}$, is known as G-monotone increasing (resp. decreasing) sequence. If we assume that G-intervals are closed, then $(NRLJ)$ holds. Recall that a G-interval is any of the subsets

$$[a, \rightarrow) = \{x \in M; (a, x) \in E(G)\},\ (\leftarrow, b] = \{x \in M; (x, b) \in E(G)\},$$

and $[a, b] = [a, \rightarrow) \cap (\leftarrow, b]$, for any $a, b \in M$.

Remark 2.2. *In fact, a serious look at Jachymski's ideas, one will notice that we do not need $V(G) = M$ the entire metric space. Therefore, we just need to associate to an edge a number that behave like a distance. This is known in graph theory as the weighted graphs. We refer to* [1, 2, 5, 13] *for some examples and applications of the above ideas to weighted graphs.*

3. Monotone Nonexpansive Mappings in Banach Spaces

Fixed point theory for nonexpansie mappings finds its roots in the papers of Browder [14], Göhde [20] and Kirk [26] published in 1965. It took little over forty years to extend the contractive condition to the case of nonexpansive mappings. Early on, it was clear that nonexpansive mappings have a very different behavior than contraction mappings. For example, the first positive results were all discovered when the domain is a convex subset of a Banach space. The extension to nonlinear sets took few decades more. Similarly it was natural to try to extend these fixed point results to the case of monotone nonexpansive mappings. This was done in [7].

Throughout $(X, \|.\|)$ stands for a Banach space and $G = (V(G), E(G))$ is a digraph such that $V(G) = X$. We will assume that G satisfies the following convexity property:

$$(x, z) \in E(G) \textit{ and } (y, w) \in E(G) \Longrightarrow (\alpha x + (1-\alpha)y, \alpha z + (1-\alpha)w) \in E(G),$$

$x, y, z, w \in X$ and $\alpha \in [0, 1]$. This property will force G-intervals to be convex. We will also assume that G-intervals are closed. Very early on, our approach to study the fixed point problem for monotone G-nonexpansive mappings was based on extending known results for nonexpansive mappings. Clearly any result that uses set theoretical techniques is hard to prove in this setting. Because monotone G-nonexpansive mappings do not enjoy a global good behavior. Our initial approach looked at results obtained from iterative techniques and successive approximations. The following result played a key role in the study of monotone G-nonexpansive mappings:

Proposition 3.1. [6, 19] *Let $(X, \|.\|)$ be a Banach space endowed with a digraph G such that $V(G) = X$. Assume that G is reflexive and transitive and G-intervals are convex and closed. Let C be a nonempty convex subset of X. Let $T : C \to C$ be a monotone G-nonexpansive mapping. Fix $\lambda \in (0, 1)$ and $x_0 \in C$ such that $(x_0, T(x_0)) \in E(G)$ or $(T(x_0), x_0) \in E(G)$. Consider the sequence $\{x_n\}$ in C defined by*

$$x_{n+1} = (1-\lambda)\ x_n + \lambda\ T(x_n),$$

for any $n \in \mathbb{N}$. Then

$$(1 + n\ \lambda)\ \|T(x_i) - x_i\| \leq \|T(x_{i+n}) - x_i\| + (1-\lambda)^n \Big(\|T(x_i) - x_i\| - \|T(x_{i+n}) - x_{i+n}\| \Big),$$

for any $i, n \in \mathbb{N}$. If C is bounded, then we have $\lim\limits_{n \to +\infty} \|x_n - T(x_n)\| = 0$, i.e., $\{x_n\}$ is an approximate fixed point sequence of T.

This technical result allows the extension of the Browder [14], Göhde [20] fixed point theorem for monotone G-nonexpansive mappings.

Theorem 3.1. [6] *Let $(X, \|.\|)$ be a uniformly convex Banach space endowed with a digraph G such that $V(G) = X$. Assume that G is reflexive and transitive and G-intervals are convex and closed. Let C be a nonempty bounded and convex subset of X. Let $T : C \to C$ be a monotone G-nonexpansive mapping. Assume there exists $x_0 \in C$ such that $(x_0, T(x_0)) \in E(G)$ or $(T(x_0), x_0) \in E(G)$. Then T has a fixed point.*

In this result and many others related to monotone G-nonexpansive mappings, the iterative sequence considered in Proposition 3.1 is heavily used. This iterative approach is closely related to what is known as Ishikawa, Mann and other type of iterations [22, 19, 28, 31, 33]. All these iterations played a major role in proving the existence of the fixed points of monotone G-nonexpansive mappings [4, 11, 12, 8]. Another beautiful aspect of these iterations is that they provide an algorithm for approximating the fixed points.

4. The case of Monotone Asymptotically Nonexpansive Mappings

The fixed point problem for asymptotic nonexpansie mappings finds its root in the work of Goebel and Kirk [18]. These type of mappings were introduced to extend the family of nonexpansive mappings. Here we extend this class to the case of monotone mappings.

Definition 4.1. [3, 18] *Let (M, d) be a metric space endowed with a digraph G such that $V(G) = M$. Let $T : M \to M$ be a map. T is said to be monotone G-asymptotically nonexpansive mapping if T is G-monotone and there exists $\{k_n\}$, a sequence of positive numbers, such that $\lim\limits_{n\to+\infty} k_n = 1$ and*

$$d(T^n(x), T^n(y)) \leq k_n \ d(x, y),$$

for any $x, y \in M$ such that $(x, y) \in E(G)$, and any $n \in \mathbb{N}$.

Following many successful results on monotone G-nonexpansive mappings, it was natural to try to extend Goebel and Kirk's fixed point theorem for asymptotically nonexpansive mappings [18] to the case of monotone G-asymptotically nonexpansive mappings.

Theorem 4.1. [3] *Let $(X, \|.\|)$ be a uniformly convex Banach space endowed with a digraph G such that $V(G) = X$. Assume that G is reflexive and transitive and G-intervals are convex and closed. Let C be a nonempty bounded and convex subset of X and $T : C \to C$ be a continuous monotone G-asymptotically nonexpansive mapping. Assume there exists $x_0 \in C$ such that $(x_0, T(x_0) \in E(G)$ or $(T(x_0, x_0) \in E(G)$. Then T has a fixed point.*

The continuity assumption on T may be relaxed if we use the concept of monotone-Opial conditions.

Definition 4.2. *Let $(X, \|.\|)$ be a uniformly convex Banach space endowed with a digraph G such that $V(G) = X$.*

(i) [30] *X is said to satisfy the weak-Opial condition if whenever any sequence $\{x_n\}$ in X which weakly converges to x, we have*

$$\limsup_{n\to+\infty} \|x_n - x\| < \limsup_{n\to+\infty} \|x_n - y\|,$$

for any $y \in X$ such that $x \neq y$.

(ii) *X is said to satisfy the G-monotone weak-Opial condition if whenever any G-monotone increasing (resp. decreasing) sequence $\{x_n\}$ in X which weakly converges to x, we have*

$$\limsup_{n\to+\infty} \|x_n - x\| \leq \limsup_{n\to+\infty} \|x_n - y\|,$$

for any $y \in X$ such that $(x, y) \in E(G)$ (resp. $(y, x) \in E(G)$).

Note that the classical Banach spaces $L^p([0, 1])$ (for $p \geq 1$), satisfy the G-monotone weak-Opial condition [3], where G is generated by the natural partial order of $L^p([0, 1])$ and fails the weak-Opial condition [30].

Theorem 4.2. [3] *Let $(X, \|.\|)$ be a uniformly convex Banach space endowed with a digraph G such that $V(G) = X$. Assume that G is reflexive and transitive and G-intervals are convex and closed. Assume that X satisfies the G-monotone weak-Opial condition. Let C be a nonempty bounded and convex subset of X and $T : C \to C$ be a monotone G-asymptotically nonexpansive mapping. Assume there exists $x_0 \in C$ such that $(x_0, T(x_0) \in E(G)$ or $(T(x_0, x_0) \in E(G)$. Then T has a fixed point.*

Despite the fact that asymptotic nonexpansive mappings were proven to have a fixed point [18], it was interesting to find out if an algorithm may be found to approximate such fixed points. This was done by the wonderful work of Schu [34]. His approach was based on a modified Mann iteration defined by:

$$x_{n+1} = \lambda_n\ T^n(x_n) + (1 - \lambda_n)\ x_n, \tag{SMI}$$

for $\lambda_n \in [0,1]$ and $n \in \mathbb{N}$, where T is an asymptotically nonexpansive mapping and x_0 is a fixed element in the domain of T. In the graphical case, if $(x_0, T(x_0) \in E(G)$ (resp. $(T(x_0, x_0) \in E(G))$, it is still not known if the above sequence $\{x_n\}$ is G-monotone increasing (resp. decreasing). This forced the authors of [4] to modify the iteration (SMI) by using the Fibonacci integer sequence $\{f(n)\}$ defined by

$$f(0) = f(1) = 1, \ \ and \ \ f(n+1) = f(n) + f(n-1),$$

for any $n \geq 1$. The new iteration scheme, which they called Fibonacci-Mann iteration, is defined by

$$x_{n+1} = \lambda_n\ T^{f(n)}(x_n) + (1 - \lambda_n)\ x_n, \tag{FMI}$$

for $t_n \in [0,1]$ and $n \in \mathbb{N}$. This new iteration allowed them to capture the evasive monotonicity.

Lemma 4.1. [4] *Let $(X, \|.\|)$ be a Banach space endowed with a digraph G such that $V(G) = X$. Assume that G is reflexive and transitive and G-intervals are convex and closed. Let C be a nonempty bounded and convex subset of X and $T : C \to C$ be a G-monotone mapping. Let $x_0 \in C$ be such that $(x_0, T(x_0) \in E(G)$ (resp. $(T(x_0, x_0) \in E(G))$. Let $\{\lambda_n\} \subset [0,1]$. Consider the (FMI) sequence $\{x_n\}$ generated by x_0 and $\{\lambda_n\}$. Let z be a fixed point of T such that $(x_0, z) \in E(G)$ (resp. $(z, x_0) \in E(G)$). Then*

(i) *$(T^n(x_0), T^{n+1}(x_0)) \in E(G)$ (resp. $(T^{n+1}(x_0), T^n(x_0)) \in E(G)$),*
(ii) *$(x_0, x_n) \in E(G)$ and $(x_n, z) \in E(G)$ (resp. $(z, x_n) \in E(G)$ and $(x_n, x_0) \in E(G)$),*
(iii) *$(T^{f(n)}(x_0), T^{f(n)}(x_n)) \in E(G)$ and $(T^{f(n)}(x_n), z) \in E(G)$ (resp. $(z, T^{f(n)}(x_n)) \in E(G)$ and $(T^{f(n)}(x_n), T^{f(n)}(x_0)) \in E(G)$),*
(iv) *$(x_n, x_{n+1}) \in E(G)$ and $(x_{n+1}, T^{f(n)}(x_n)) \in E(G)$ (resp. $(T^{f(n)}(x_n), x_{n+1}) \in E(G)$ and $(x_{n+1}, x_n) \in E(G)$),*

for any $n \in \mathbb{N}$.

Using this technical Lemma, the authors were able to prove the analogs results to Schu [34]. The first of which consists of the strong convergence of the iterative sequence generated by (FMI). Recall that the map T is said to be compact if it maps bounded sets into relatively compact ones.

Theorem 4.3. [4] *Let $(X, \|.\|)$ be a Banach space endowed with a digraph G such that $V(G) = X$. Assume that G is reflexive and transitive and G-intervals are convex and closed. Let C be a nonempty bounded and convex subset of X and $T : C \to C$ be a monotone G-asymptotically nonexpansive mapping with the Lipschitz constants $\{k_n\}$. Assume that $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$ and T^m is compact for some $m \geq 1$. Let $x_0 \in C$ be such that $(x_0, T(x_0) \in E(G)$ (resp. $(T(x_0, x_0) \in E(G))$. Let $\{\lambda_n\} \subset [0,1]$. Consider the (FMI) sequence $\{x_n\}$ generated by x_0 and $\{\lambda_n\}$. Then $\{x_n\}$ converges strongly to a fixed point of T such that $(x_0, z) \in E(G)$ (resp. $(z, x_0) \in E(G)$.*

If we relax the compactness assumption, we may get the weak convergence instead of the strong convergence but we need to assume the monotone Opial condition discussed before which holds in uniformly convex Banach spaces for which the norm is G-monotone, i.e. $\|x\| \leq \|y\|$ provided $(x, y) \in E(G)$.

Theorem 4.4. *Let $(X, \|.\|)$ be a uniformly convex Banach space endowed with a digraph G such that $V(G) = X$. Assume that G is reflexive and transitive and G-intervals are convex and closed and the norm $\|.\|$ is G-monotone. Let C be a nonempty bounded and convex subset of X and $T : C \to C$ be a monotone G-asymptotically nonexpansive mapping with the Lipschitz constants $\{k_n\}$. Assume that $\sum\limits_{n=1}^{\infty}(k_n - 1) < \infty$. Let $x_0 \in C$ be such that $(x_0, T(x_0) \in E(G)$ (resp. $(T(x_0, x_0) \in E(G))$. Let $\{\lambda_n\} \subset [0,1]$. Consider the (FMI) sequence $\{x_n\}$ generated by x_0 and $\{\lambda_n\}$. Then $\{x_n\}$ converges weakly to a fixed point of T such that $(x_0, z) \in E(G)$ (resp. $(z, x_0) \in E(G)$).*

In the next section we discuss the relationship between the Knaster-Tarski theorem versus the fixed point theorems discovered for monotone nonexpansive mappings.

5. Concluding remarks

As we said earlier of this work, the approach taken over the recent years dealing with the fixed point problem for monotone mappings relied heavily on iterative or successive approximations. Therefore, it was unknown whether an analogue to Kirk's fixed point theorem [26] for monotone nonexpansive mappings does exist. It was this problem that led Espínola and Wiśnicki [17] to discover the following results.

Lemma 5.1. [17] *Let X be a partially ordered set for which each family of order intervals of the form $[a, b]$, $[a, \rightarrow)$ with the finite intersection property has a nonempty intersection. Then every directed subset of X has a supremum.*

Remember that a subset J of a partially ordered set X is directed if each finite subset of J has an upper bound in J. If we combine Lemma 5.1 with the classical the Knaster-Tarski's fixed point theorem [27, 35], we get the following result:

Theorem 5.1. [17] *Let X be a topological space with a partial order $\preceq$ for which order intervals are compact and let $T : X \to X$ be monotone. If there exists $c \in X$ such that $c \preceq T(c)$, then the set of all fixed points of T is nonempty and has a maximal element.*

Using the theorem, Espínola and Wiśnicki concluded that most of the fixed point results related to monotone nonexpansive mappings are a particular case of Theorem 5.1. In other words, these fixed point theorems are reduced to the classical Knaster-Tarski theorem. First, this approach works only if we consider partially ordered sets. They fail to extend to the case of sets endowed with a digraph. This is why I presented most of the results here in terms of digraphs. Moreover, the Knaster-Tarski's fixed point theorem fails to provide an algorithm to approximate the fixed points. This is exactly the approach taken by Schu [34] under restrictive conditions of the Lipschitz constants of the map though the existence of the fixed point was known to Goebel and Kirk [18].
Before, we close this work, we recommend the books [15, 21] for very interesting examples where fixed point theorems for monotone mappings are used to solve differential equations or integral equations for which classical fixed point theorems fail.

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