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Inequalities of Hermite-Hadamard Type for HA-Convex Functions

S. S. Dragomir[1,2]

Abstract. Some new inequalities of Hermite-Hadamard type for HA-convex functions defined on positive intervals are given.

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1. Introduction

Following [4] (see also [40]) we say that the function $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is *HA-convex* or *harmonically convex* if

$$f\left(\frac{xy}{tx+(1-t)\,y}\right) \leq (1-t)\,f(x) + tf(y) \tag{1}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1) is reversed, then f is said to be *HA-concave* or *harmonically concave.*

In order to avoid any confusion with the class of *AH-convex* functions, namely the functions satisfying the condition

$$f((1-t)\,x + ty) \leq \frac{f(x)\,f(y)}{(1-t)\,f(y) + tf(x)}, \tag{2}$$

we call the class of functions satisfying (1) as *HA-convex functions.*

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[1] *Mathematics, College of Engineering & Science Victoria University, PO Box 14428 Melbourne City, MC 8001, Australia.*
[2] School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa
e-mail:sever.dragomir@vu.edu.au

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If $I\subset(0,\infty)$ and f is convex and nondecreasing function then f is HA-convex and if f is HA-convex and nonincreasing function then f is convex.

If $[a,b]\subset I\subset(0,\infty)$ and if we consider the function $g:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$, defined by $g(t)=f\left(\frac{1}{t}\right)$, then f is HA-convex on $[a,b]$ if and only if g is convex in the usual sense on $\left[\frac{1}{b},\frac{1}{a}\right]$. Therefore, as examples of HA-convex functions we can take $f(t)=g\left(\frac{1}{t}\right)$, where g is any convex function on $\left[\frac{1}{b},\frac{1}{a}\right]$.

For a convex function $h:[c,d]\to\mathbb{R}$, the following inequality is well known in the literature as the *Hermite-Hadamard inequality*

$$h\left(\frac{c+d}{2}\right)\leq\frac{1}{d-c}\int_c^d h(t)\,dt\leq\frac{h(c)+h(d)}{2}. \tag{3}$$

For related results, see [1]-[20], [23]-[26], [27]-[36] and [37]-[48].

If we write the Hermite-Hadamard inequality for the convex function $g(t)=f\left(\frac{1}{t}\right)$ on the closed interval $\left[\frac{1}{b},\frac{1}{a}\right]$, then we have

$$f\left(\frac{1}{\frac{\frac{1}{a}+\frac{1}{b}}{2}}\right)\leq\frac{1}{\frac{1}{a}-\frac{1}{b}}\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right)dt\leq\frac{f\left(\frac{1}{\frac{1}{b}}\right)+f\left(\frac{1}{\frac{1}{a}}\right)}{2}$$

that is equivalent to

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right)dt\leq\frac{f(b)+f(a)}{2}. \tag{4}$$

Using the change of variable $s=\frac{1}{t}$, then

$$\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{t}\right)dt=\int_a^b\frac{f(s)}{s^2}ds$$

and by (4) we get

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int_a^b\frac{f(s)}{s^2}ds\leq\frac{f(b)+f(a)}{2}. \tag{5}$$

The inequality (5) has been obtained in a different manner in [40] by I. Işcan.

Motivated by the above results, we establish in this paper some new inequalities of Hermite-Hadamard type for HA-convex functions.

2. A Refinement

We have the following representation result, see [25]. For the sake of completeness we give here a simple proof.

Lemma 2.1. *Let $g:[x,y]\subset\mathbb{R}\to\mathbb{C}$ be a Lebesgue integrable function on $[x,y]$. Then for any $\lambda\in[0,1]$ we have the representation*

$$\begin{aligned}\int_0^1 g[(1-t)x+ty]\,dt&=(1-\lambda)\int_0^1 g[(1-t)((1-\lambda)x+\lambda y)+ty]\,dt \\ &\quad+\lambda\int_0^1 g[(1-t)x+t((1-\lambda)x+\lambda y)]\,dt.\end{aligned} \tag{6}$$

Proof. For $\lambda = 0$ and $\lambda = 1$ the equality (6) is obvious.

Let $\lambda \in (0,1)$. Observe that

$$\int_0^1 g\left[(1-t)(\lambda y + (1-\lambda)x) + ty\right] dt = \int_0^1 g\left[((1-t)\lambda + t)y + (1-t)(1-\lambda)x\right] dt$$

and

$$\int_0^1 g\left[t(\lambda y + (1-\lambda)x) + (1-t)x\right] dt = \int_0^1 g\left[t\lambda y + (1-\lambda t)x\right] dt.$$

If we make the change of variable $u := (1-t)\lambda + t$ then we have $1 - u = (1-t)(1-\lambda)$ and $du = (1-\lambda)du$. Then

$$\int_0^1 g\left[((1-t)\lambda + t)y + (1-t)(1-\lambda)x\right] dt = \frac{1}{1-\lambda}\int_\lambda^1 g\left[uy + (1-u)x\right] du.$$

If we make the change of variable $u := \lambda t$ then we have $du = \lambda dt$ and

$$\int_0^1 g\left[t\lambda y + (1-\lambda t)x\right] dt = \frac{1}{\lambda}\int_0^\lambda g\left[uy + (1-u)x\right] du.$$

Therefore

$$\begin{aligned}
&(1-\lambda)\int_0^1 g\left[(1-t)(\lambda y + (1-\lambda)x) + ty\right] dt \\
&+ \lambda\int_0^1 g\left[t(\lambda y + (1-\lambda)x) + (1-t)x\right] dt \\
&= \int_\lambda^1 g\left[uy + (1-u)x\right] du + \int_0^\lambda g\left[uy + (1-u)x\right] du \\
&= \int_0^1 g\left[uy + (1-u)x\right] du
\end{aligned}$$

and the identity (6) is proved.

Corollary 2.1. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{C}$ be a Lebesgue integrable function on $[a,b]$ and $\lambda \in [0,1]$, then we have the representation*

$$\begin{aligned}
\int_0^1 f\left(\frac{ab}{(1-t)a + tb}\right) dt &= (1-\lambda)\int_0^1 f\left(\frac{ab}{(1-t)((1-\lambda)a + \lambda b) + tb}\right) dt \\
&\quad + \lambda\int_0^1 f\left(\frac{ab}{(1-t)a + t((1-\lambda)a + \lambda b)}\right) dt.
\end{aligned} \tag{7}$$

Proof. Consider the function $g : \left[\frac{1}{b}, \frac{1}{a}\right]$, $g(s) = f\left(\frac{1}{s}\right)$, $s \in \left[\frac{1}{b}, \frac{1}{a}\right]$.

We have by (6) for g and $x = \frac{1}{b}$, $y = \frac{1}{a}$ that

$$\begin{aligned}
&\int_0^1 f\left(\frac{ab}{(1-t)a + tb}\right) dt \\
&= \int_0^1 f\left(\frac{1}{(1-t)\frac{1}{b} + t\frac{1}{a}}\right) dt
\end{aligned} \tag{8}$$

$$= \int_0^1 g\left((1-t)\frac{1}{b}+t\frac{1}{a}\right)dt$$
$$= (1-\lambda)\int_0^1 g\left[(1-t)\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)+t\frac{1}{a}\right]dt$$
$$+\lambda\int_0^1 g\left[(1-t)\frac{1}{b}+t\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)\right]dt$$

$$= (1-\lambda)\int_0^1 f\left(\frac{1}{(1-t)\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)+t\frac{1}{a}}\right)dt$$
$$+\lambda\int_0^1 f\left(\frac{1}{(1-t)\frac{1}{b}+t\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)}\right)dt$$
$$= (1-\lambda)\int_0^1 f\left(\frac{ab}{(1-t)\left((1-\lambda)a+\lambda b\right)+tb}\right)dt$$
$$+\lambda\int_0^1 f\left(\frac{ab}{(1-t)a+t\left((1-\lambda)a+\lambda b\right)}\right)dt.$$

The following result holds.

Theorem 2.1. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then for any $\lambda\in[0,1]$ we have the inequalities*

$$f\left(\frac{2ab}{a+b}\right)\le(1-\lambda)f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)+\lambda f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right) \tag{9}$$
$$\le\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt$$
$$\le\frac{1}{2}\left[f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+(1-\lambda)f(a)+\lambda f(b)\right]$$
$$\le\frac{f(a)+f(b)}{2}.$$

Proof. Consider the function $g:\left[\frac{1}{b},\frac{1}{a}\right]$, $g(s)=f\left(\frac{1}{s}\right)$, $s\in\left[\frac{1}{b},\frac{1}{a}\right]$.

Since g is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$, then by Hermite-Hadamard inequality for convex functions we have for $\lambda\in[0,1]$

$$g\left(\frac{(1-\lambda)a+(\lambda+1)b}{2ab}\right)=g\left(\frac{(1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}+\frac{1}{a}}{2}\right) \tag{10}$$
$$\le\int_0^1 g\left((1-t)\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)+t\frac{1}{a}\right)dt$$
$$\le\frac{g\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)+g\left(\frac{1}{a}\right)}{2}$$
$$=\frac{g\left(\frac{(1-\lambda)a+\lambda b}{ab}\right)+g\left(\frac{1}{a}\right)}{2}$$

and

$$g\left(\frac{(2-\lambda)a+\lambda b}{2ab}\right)=g\left(\frac{\frac{1}{b}+(1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}}{2}\right) \tag{11}$$
$$\leq \int_0^1 g\left((1-t)\frac{1}{b}+t\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)\right)dt$$
$$\leq \frac{g\left(\frac{1}{b}\right)+g\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)}{2}$$
$$=\frac{g\left(\frac{1}{b}\right)+g\left(\frac{(1-\lambda)a+\lambda b}{ab}\right)}{2}.$$

If we multiply (10) by $(1-\lambda)$ and 11 by λ, add the obtained inequalities and use the first part of the equality (8) we get

$$(1-\lambda)f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)+\lambda f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right) \tag{12}$$
$$\leq \int_0^1 f\left(\frac{ab}{(1-t)a+tb}\right)dt$$
$$\leq (1-\lambda)\frac{f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+f(a)}{2}+\lambda\frac{f(b)+f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)}{2}$$
$$=\frac{1}{2}\left[f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+(1-\lambda)f(a)+\lambda f(b)\right].$$

By the convexity of g we have

$$(1-\lambda)f\left(\frac{2ab}{(1-\lambda)a+(\lambda+1)b}\right)+\lambda f\left(\frac{2ab}{(2-\lambda)a+\lambda b}\right)$$
$$=(1-\lambda)g\left(\frac{(1-\lambda)a+(\lambda+1)b}{2ab}\right)+\lambda g\left(\frac{(2-\lambda)a+\lambda b}{2ab}\right)$$
$$\geq g\left(\frac{(1-\lambda)[(1-\lambda)a+(\lambda+1)b]}{2ab}+\frac{\lambda[(2-\lambda)a+\lambda b]}{2ab}\right)$$
$$=g\left(\frac{a+b}{2ab}\right)=f\left(\frac{2ab}{a+b}\right)$$

and

$$f\left(\frac{ab}{(1-\lambda)a+\lambda b}\right)+(1-\lambda)f(a)+\lambda f(b)$$
$$=g\left((1-\lambda)\frac{1}{b}+\lambda\frac{1}{a}\right)+(1-\lambda)f(a)+\lambda f(b)$$
$$\leq (1-\lambda)f(b)+\lambda f(a)+(1-\lambda)f(a)+\lambda f(b)$$
$$=f(a)+f(b)$$

and the desired inequality (9) is proved.

Corollary 2.2. *With the assumptions of Theorem 2.1 we have*

$$f\left(\frac{2ab}{a+b}\right)\leq \frac{1}{2}\left[f\left(\frac{4ab}{a+3b}\right)+f\left(\frac{4ab}{3a+b}\right)\right]\leq \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2}dt \tag{13}$$

$$\leq \frac{1}{2}\left[f\left(\frac{2ab}{a+b}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}.$$

3. New Results

We recall some facts on the lateral derivatives of a convex function.

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_-(x) \leq f'_+(x) \leq f'_-(y) \leq f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \geq f(a) + (x-a)\varphi(a) \text{ for any } x, a \in I. \tag{14}$$

It is also well known that if f is convex on I, then ∂f is nonempty, f'_-, $f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \leq \varphi(x) \leq f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

The *identric mean* $I(a,b)$ is defined by

$$I(a,b) := \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$$

while the *logarithmic mean* is defined by

$$L(a,b) := \frac{b-a}{\ln b - \ln a}.$$

Theorem 3.1. *Let $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then*

$$f(L(a,b)) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{(L(a,b)-a)\,bf(b) + (b-L(a,b))\,af(a)}{(b-a)\,L(a,b)}. \tag{15}$$

Proof. Since $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is an HA-convex function on the interval $[a,b]$, then the function $g : \left[\frac{1}{b}, \frac{1}{a}\right]$, $g(s) = f\left(\frac{1}{s}\right)$, is convex on $\left[\frac{1}{b}, \frac{1}{a}\right]$. Therefore f has partial derivatives in each point of (a,b) and by the gradient inequality for g we have for any $x, y \in (a,b)$ that

$$\begin{aligned} f(x) - f(y) &= g\left(\frac{1}{x}\right) - g\left(\frac{1}{y}\right) \geq g'_+\left(\frac{1}{y}\right)\left(\frac{1}{x}-\frac{1}{y}\right) \\ &= g'_+\left(\frac{1}{y}\right)\frac{y-x}{xy}. \end{aligned} \tag{16}$$

Since

$$g'_+(s) = f'_-\left(\frac{1}{s}\right)\left(-\frac{1}{s^2}\right), \; s \in \left(\frac{1}{b}, \frac{1}{a}\right)$$

then

$$g'_+\left(\frac{1}{y}\right) = f'_-(y)\left(-y^2\right)$$

and by (16) we have

$$f(x)-f(y)\geq f'_-(y)\frac{y-x}{xy}\left(-y^2\right)=f'_-(y)\,y\left(1-\frac{y}{x}\right).$$

Therefore we have

$$f(x)-f(y)\geq f'_-(y)\,y\left(1-\frac{y}{x}\right) \tag{17}$$

for any $x,y\in(a,b)$.

If we take the integral mean over x in (17), then we have

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx-f(y) &\geq \left(1-y\frac{1}{b-a}\int_a^b\frac{1}{x}dx\right)f'_-(y)\,y \\ &= \left(1-\frac{y}{L(a,b)}\right)f'_-(y)\,y\end{aligned} \tag{18}$$

for any $y\in(a,b)$.

Now, if we take $y=L(a,b)$ in (18), then we get the first inequality in (15).

Observe that for any $x\in[a,b]$ we have

$$\frac{1}{x}=\frac{\left(\frac{1}{a}-\frac{1}{x}\right)\frac{1}{b}+\left(\frac{1}{x}-\frac{1}{b}\right)\frac{1}{a}}{\frac{1}{a}-\frac{1}{b}}.$$

By the convexity of g on $\left[\frac{1}{b},\frac{1}{a}\right]$ we then have

$$\begin{aligned}f(x)=g\left(\frac{1}{x}\right)&=g\left(\frac{\left(\frac{1}{a}-\frac{1}{x}\right)\frac{1}{b}+\left(\frac{1}{x}-\frac{1}{b}\right)\frac{1}{a}}{\frac{1}{a}-\frac{1}{b}}\right)\\ &\leq\frac{\left(\frac{1}{a}-\frac{1}{x}\right)g\left(\frac{1}{b}\right)+\left(\frac{1}{x}-\frac{1}{b}\right)g\left(\frac{1}{a}\right)}{\frac{1}{a}-\frac{1}{b}}\\ &=\frac{\left(\frac{1}{a}-\frac{1}{x}\right)f(b)+\left(\frac{1}{x}-\frac{1}{b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}}\end{aligned} \tag{19}$$

for any $x\in[a,b]$.

Taking the integral mean in (19) we get

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)\,dx &\leq\frac{\left(\frac{1}{a}-\frac{1}{b-a}\int_a^b\frac{1}{x}dx\right)f(b)+\left(\frac{1}{b-a}\int_a^b\frac{1}{x}dx-\frac{1}{b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}}\\ &=\frac{\left(\frac{1}{a}-\frac{1}{L(a,b)}\right)f(b)+\left(\frac{1}{L(a,b)}-\frac{1}{b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}}\\ &=\frac{\frac{L(a,b)-a}{aL(a,b)}f(b)+\frac{b-L(a,b)}{L(a,b)b}f(a)}{\frac{b-a}{ab}}\end{aligned}$$

and the second inequality in (15) is also proved.

Remark 3.1. *If $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is a differentiable HA-convex function on the interval (a,b), then from (18) we have the following inequality*

$$\frac{1}{b-a}\int_a^b f(x)\,dx-f(y)\geq\left(1-\frac{y}{L(a,b)}\right)f'(y)\,y \tag{20}$$

for any $y \in (a,b)$.

We have

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f(A(a,b)) \geq \left(1-\frac{A(a,b)}{L(a,b)}\right) f'(A(a,b))\,A(a,b) \tag{21}$$

and if $f'(A(a,b)) \leq 0$, *then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geq f(A(a,b)). \tag{22}$$

We have

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f(I(a,b)) \geq \left(1-\frac{I(a,b)}{L(a,b)}\right) f'(I(a,b))\,I(a,b) \tag{23}$$

and if $f'(I(a,b)) \leq 0$, *then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geq f(I(a,b)). \tag{24}$$

We have

$$\frac{1}{b-a}\int_a^b f(x)\,dx - f(G(a,b)) \geq \left(1-\frac{G(a,b)}{L(a,b)}\right) f'(G(a,b))\,G(a,b) \tag{25}$$

and if $f'(G(a,b)) \geq 0$, *then*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \geq f(G(a,b)). \tag{26}$$

We have:

Theorem 3.2. *Let* $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be a HA-convex function on the interval* $[a,b]$. *Then*

$$f\left(\frac{a+b}{2}\right)\frac{a+b}{2} \leq \frac{1}{b-a}\int_a^b xf(x)\,dx \leq \frac{bf(b)+af(a)}{2}. \tag{27}$$

Proof. From the inequality (17), by multiplying with $x>0$ we have

$$xf(x) - xf(y) \geq f'_-(y)\,y\,(x-y) \tag{28}$$

for any $x,y \in (a,b)$.

Taking the integral mean over $x \in [a,b]$ we have

$$\frac{1}{b-a}\int_a^b xf(x)\,dx - f(y)\frac{1}{b-a}\int_a^b x dx \geq \left(\frac{1}{b-a}\int_a^b x dx - y\right) f'_-(y)\,y,$$

that is equivalent to

$$\frac{1}{b-a}\int_a^b xf(x)\,dx - f(y)\frac{a+b}{2} \geq \left(\frac{a+b}{2} - y\right) f'_-(y)\,y, \tag{29}$$

for any $y \in (a,b)$.

If we take in (29) $y=\frac{a+b}{2}$, then we get the first inequality in (27).

From the inequality (19) we also have

$$xf(x) \leq \frac{\left(\frac{x}{a}-1\right)f(b) + \left(1-\frac{x}{b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}} \tag{30}$$

for any $x \in [a,b]$.

Taking the integral mean on (30) we get

$$\frac{1}{b-a}\int_a^b xf(x)\,dx \le \frac{\left(\frac{a+b}{2a}-1\right)f(b)+\left(1-\frac{a+b}{2b}\right)f(a)}{\frac{1}{a}-\frac{1}{b}} \tag{31}$$
$$= \frac{\frac{b-a}{2a}f(b)+\frac{b-a}{2b}f(a)}{\frac{b-a}{ab}} = \frac{bf(b)+af(a)}{2}$$

and the second inequality in (27) is proved.

Remark 3.2. *If $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is a differentiable HA-convex function on the interval (a,b), then from (29) we have*

$$f(y)A(a,b)-\frac{1}{b-a}\int_a^b xf(x)\,dx \le (y-A(a,b))f'(y)y, \tag{32}$$

for any $y\in(a,b)$.

If we take in (32) $y=I(a,b)$, then we get

$$f(I(a,b))A(a,b)-\frac{1}{b-a}\int_a^b xf(x)\,dx \tag{33}$$
$$\le (I(a,b)-A(a,b))f'(I(a,b))I(a,b).$$

If $f'(I(a,b))\ge 0$, then

$$f(I(a,b))A(a,b)\le\frac{1}{b-a}\int_a^b xf(x)\,dx. \tag{34}$$

If we take in (32) $y=L(a,b)$, then we get

$$f(L(a,b))A(a,b)-\frac{1}{b-a}\int_a^b xf(x)\,dx \tag{35}$$
$$\le (L(a,b)-A(a,b))f'(L(a,b))L(a,b).$$

If $f'(L(a,b))\ge 0$, then

$$f(L(a,b))A(a,b)\le\frac{1}{b-a}\int_a^b xf(x)\,dx. \tag{36}$$

If we take in (32) $y=G(a,b)$, then we get

$$f(G(a,b))A(a,b)-\frac{1}{b-a}\int_a^b xf(x)\,dx \tag{37}$$
$$\le (G(a,b)-A(a,b))f'(G(a,b))G(a,b).$$

If $f'(G(a,b))\ge 0$, then

$$f(G(a,b))A(a,b)\le\frac{1}{b-a}\int_a^b xf(x)\,dx. \tag{38}$$

We use the following results obtained by the author in [21] and [22]

Lemma 3.1. *Let $h:[\alpha,\beta]\to\mathbb{R}$ be a convex function on $[\alpha,\beta]$. Then we have the inequalities*

$$\frac{1}{8}\left[h'_+\left(\frac{\alpha+\beta}{2}\right)-h'_-\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha) \tag{39}$$

$$\le \frac{h(\alpha)+h(\beta)}{2}-\frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt$$
$$\le \frac{1}{8}\left[h'_-(\beta)-h'_+(\alpha)\right](\beta-\alpha)$$

and

$$\frac{1}{8}\left[h'_+\left(\frac{\alpha+\beta}{2}\right)-h'_-\left(\frac{\alpha+\beta}{2}\right)\right](\beta-\alpha) \tag{40}$$
$$\le \frac{1}{\beta-\alpha}\int_\alpha^\beta h(t)\,dt-h\left(\frac{\alpha+\beta}{2}\right)$$
$$\le \frac{1}{8}\left[h'_-(\beta)-h'_+(\alpha)\right](\beta-\alpha).$$

The constant $\frac{1}{8}$ *is best possible in (39) and (40).*

We have:

Theorem 3.3. *Let* $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ *be a HA-convex function on the interval* $[a,b]$. *Then*

$$\frac{1}{2}\left[f'_+\left(\frac{2ab}{a+b}\right)-f'_-\left(\frac{2ab}{a+b}\right)\right]\frac{ab}{(a+b)^2}(b-a) \tag{41}$$
$$\le \frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt$$
$$\le \frac{1}{8}\left[\frac{f'_-(b)\,b^2-f'_+(a)\,a^2}{ab}\right](b-a)$$

and

$$\frac{1}{2}\left[f'_+\left(\frac{2ab}{a+b}\right)-f'_-\left(\frac{2ab}{a+b}\right)\right]\frac{ab}{(a+b)^2}(b-a) \tag{42}$$
$$\le \frac{ab}{b-a}\int_a^b\frac{f(t)}{t^2}dt-f\left(\frac{2ab}{a+b}\right)$$
$$\le \frac{1}{8}\left[\frac{f'_-(b)\,b^2-f'_+(a)\,a^2}{ab}\right](b-a).$$

Proof. Since $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ is an *HA*-convex function on the interval $[a,b]$, then the function $g:\left[\frac{1}{b},\frac{1}{a}\right]$, $g(s)=f\left(\frac{1}{s}\right)$, is convex on $\left[\frac{1}{b},\frac{1}{a}\right]$.

We know that

$$g'_\pm(s)=f'_\mp\left(\frac{1}{s}\right)\left(-\frac{1}{s^2}\right),\ s\in\left(\frac{1}{b},\frac{1}{a}\right).$$

If we use the inequality (39) for the convex function $g:\left[\frac{1}{b},\frac{1}{a}\right]\to\mathbb{R}$, then we have

$$\frac{1}{8}\left[f'_-\left(\frac{1}{\frac{\frac{1}{b}+\frac{1}{a}}{2}}\right)\left(-\frac{1}{\left(\frac{\frac{1}{b}+\frac{1}{a}}{2}\right)^2}\right)-f'_+\left(\frac{1}{\frac{\frac{1}{b}+\frac{1}{a}}{2}}\right)\left(-\frac{1}{\left(\frac{\frac{1}{b}+\frac{1}{a}}{2}\right)^2}\right)\right]$$
$$\times\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$\leq \frac{f(a)+f(b)}{2} - \frac{1}{\frac{1}{a}-\frac{1}{b}} \int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{s}\right) ds$$
$$\leq \frac{1}{8}\left[f'_{+}\left(\frac{1}{\frac{1}{a}}\right)\left(-\frac{1}{\left(\frac{1}{a}\right)^2}\right) - f'_{-}\left(\frac{1}{\frac{1}{b}}\right)\left(-\frac{1}{\left(\frac{1}{b}\right)^2}\right)\right]\left(\frac{1}{a}-\frac{1}{b}\right)$$

that is equivalent to

$$\frac{1}{8}\left[f'_{+}\left(\frac{1}{\frac{\frac{1}{b}+\frac{1}{a}}{2}}\right)\left(\frac{1}{\left(\frac{\frac{1}{b}+\frac{1}{a}}{2}\right)^2}\right) - f'_{-}\left(\frac{1}{\frac{\frac{1}{b}+\frac{1}{a}}{2}}\right)\left(\frac{1}{\left(\frac{\frac{1}{b}+\frac{1}{a}}{2}\right)^2}\right)\right]$$
$$\times\left(\frac{1}{a}-\frac{1}{b}\right)$$
$$\leq \frac{f(a)+f(b)}{2} - \frac{1}{\frac{1}{a}-\frac{1}{b}} \int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{s}\right) ds$$
$$\leq \frac{1}{8}\left[f'_{-}\left(\frac{1}{\frac{1}{b}}\right)\left(\frac{1}{\left(\frac{1}{b}\right)^2}\right) - f'_{+}\left(\frac{1}{\frac{1}{a}}\right)\left(\frac{1}{\left(\frac{1}{a}\right)^2}\right)\right]\left(\frac{1}{a}-\frac{1}{b}\right),$$

namely

$$\frac{1}{8}\left[f'_{+}\left(\frac{2ab}{a+b}\right) - f'_{-}\left(\frac{2ab}{a+b}\right)\right]\frac{4a^2b^2}{(a+b)^2}\left(\frac{b-a}{ab}\right)$$
$$\leq \frac{f(a)+f(b)}{2} - \frac{1}{\frac{1}{a}-\frac{1}{b}} \int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{s}\right) ds$$
$$\leq \frac{1}{8}\left[f'_{-}(b)\,b^2 - f'_{+}(a)\,a^2\right]\left(\frac{b-a}{ab}\right).$$

Observe that

$$\int_{\frac{1}{b}}^{\frac{1}{a}} f\left(\frac{1}{s}\right) ds = \int_a^b \frac{f(t)}{t^2} dt$$

and the inequality (41) is proved.

The inequality (42) follows by (40).

Corollary 3.1. *If $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is a differentiable HA-convex function on the interval (a,b), then*

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2} dt \leq \frac{1}{8}\left[\frac{f'_{-}(b)\,b^2 - f'_{+}(a)\,a^2}{ab}\right](b-a) \tag{43}$$

and

$$0 \leq \frac{ab}{b-a}\int_a^b \frac{f(t)}{t^2} dt - f\left(\frac{2ab}{a+b}\right) \leq \frac{1}{8}\left[\frac{f'_{-}(b)\,b^2 - f'_{+}(a)\,a^2}{ab}\right](b-a). \tag{44}$$

4. Related Results

We have the following result:

Theorem 4.1. *Let $f:[a,b]\subset(0,\infty)\to\mathbb{R}$ be an HA-convex function on the interval $[a,b]$. Then*

$$\frac{1}{2}\left[xf(x)+\frac{(b-x)\,bf(b)+(x-a)\,af(a)}{b-a}\right]\geq\frac{1}{b-a}\int_a^b yf(y)\,dy \tag{45}$$

for any $x\in[a,b]$.

Proof. From (17) we have

$$f(x)-f(y)\geq f'_-(y)\,y\left(1-\frac{y}{x}\right) \tag{46}$$

for any $x,y\in(a,b)$.

If we take the integral mean over y in (46), then we have

$$f(x)-\frac{1}{b-a}\int_a^b f(y)\,dy\geq\frac{1}{b-a}\int_a^b f'_-(y)\,ydy-\frac{1}{x}\frac{1}{b-a}\int_a^b f'_-(y)\,y^2dy \tag{47}$$

for any $x\in(a,b)$.

Integrating by parts in the Lebesgue integral, we have

$$\int_a^b f'_-(y)\,ydy=bf(b)-af(a)-\int_a^b f(y)\,dy$$

and

$$\int_a^b f'_-(y)\,y^2dy=b^2f(b)-a^2f(a)-2\int_a^b yf(y)\,dy.$$

Utilising (47) we obtain

$$\begin{aligned} & f(x)-\frac{1}{b-a}\int_a^b f(y)\,dy \\ & \geq\frac{1}{b-a}\left(bf(b)-af(a)-\int_a^b f(y)\,dy\right) \\ & -\frac{1}{x}\frac{1}{b-a}\left(b^2f(b)-a^2f(a)-2\int_a^b yf(y)\,dy\right) \\ & =\frac{bf(b)-af(a)}{b-a}-\frac{1}{b-a}\int_a^b f(y)\,dy \\ & -\frac{1}{x}\frac{b^2f(b)-a^2f(a)}{b-a}+\frac{2}{x}\frac{1}{b-a}\int_a^b yf(y)\,dy \end{aligned} \tag{48}$$

that is equivalent to

$$f(x)+\frac{1}{x}\frac{b^2f(b)-a^2f(a)}{b-a}-\frac{bf(b)-af(a)}{b-a}\geq\frac{2}{x}\frac{1}{b-a}\int_a^b yf(y)\,dy.$$

If we multiply this inequality by $\frac{x}{2}$, then we get

$$\frac{1}{2}\left[xf(x)+\frac{b^2f(b)-a^2f(a)-xbf(b)+xaf(a)}{b-a}\right]\geq\frac{1}{b-a}\int_a^b yf(y)\,dy,$$

and the inequality (45) is proved.

Remark 4.1. *If we take in (45)* $x = \frac{a+b}{2}$, *then we get*

$$\frac{1}{2}\left[\frac{a+b}{2}f\left(\frac{a+b}{2}\right)+\frac{bf(b)+af(a)}{2}\right]\geq\frac{1}{b-a}\int_a^b yf(y)\,dy. \tag{49}$$

If we take in (45) $x = \frac{2ab}{a+b}$, *then we get*

$$\frac{1}{2}\left[\frac{2ab}{a+b}f\left(\frac{2ab}{a+b}\right)+\frac{b^2f(b)+a^2f(a)}{a+b}\right]\geq\frac{1}{b-a}\int_a^b yf(y)\,dy. \tag{50}$$

We have:

Theorem 4.2. *Let* $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ *be a HA-convex function on the interval* $[a,b]$. *Then*

$$\begin{aligned}&\frac{1}{x}\left[\frac{(b-x)f(b)+(x-a)f(a)}{b-a}-\frac{1}{b-a}\int_a^b f(y)\,dy\right] \\ &\geq\frac{1}{L(a,b)}\left[\frac{1}{\ln b-\ln a}\int_a^b\frac{f(y)}{y}dy-f(x)\right]\end{aligned} \tag{51}$$

for any $x \in [a,b]$.

Proof. By dividing with $y > 0$ in (17) we have

$$\frac{1}{y}f(x)-\frac{f(y)}{y}\geq f'_-(y)\left(1-\frac{y}{x}\right) \tag{52}$$

for any $x, y \in (a,b)$.

By taking the integral mean over y in (52) we obtain

$$\begin{aligned}&\frac{\ln b-\ln a}{b-a}f(x)-\frac{1}{b-a}\int_a^b\frac{f(y)}{y}dy \\ &\geq\frac{1}{b-a}\int_a^b f'_-(y)\,dy-\frac{1}{x}\frac{1}{b-a}\int_a^b yf'_-(y)\,dy \\ &=\frac{f(b)-f(a)}{b-a}-\frac{1}{x}\frac{bf(b)-af(a)-\int_a^b f(y)\,dy}{b-a} \\ &=\frac{f(b)-f(a)}{b-a}-\frac{1}{x}\frac{bf(b)-af(a)\,y}{b-a}+\frac{1}{x}\frac{1}{b-a}\int_a^b f(y)\,dy\end{aligned}$$

that is equivalent to

$$\begin{aligned}&\frac{1}{x}\frac{bf(b)-af(a)\,y}{b-a}-\frac{f(b)-f(a)}{b-a}-\frac{1}{b-a}\int_a^b\frac{f(y)}{y}dy \\ &\geq\frac{1}{x}\frac{1}{b-a}\int_a^b f(y)\,dy-\frac{\ln b-\ln a}{b-a}f(x)\end{aligned}$$

or, to

$$\begin{aligned}&\frac{1}{b-a}\left[\frac{b-x}{x}f(b)+\frac{x-a}{x}f(a)\right]-\frac{1}{b-a}\int_a^b\frac{f(y)}{y}dy \\ &\geq\frac{1}{x}\frac{1}{b-a}\int_a^b f(y)\,dy-\frac{\ln b-\ln a}{b-a}f(x)\end{aligned} \tag{53}$$

for any $x \in (a,b)$.

Rearranging the terms in (53) produces the desired result (51).

Remark 4.2. *If we take $x = L(a,b)$ in (51), then we get*

$$\frac{(b - L(a,b)) f(b) + (L(a,b) - a) f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(y)\,dy \quad (54)$$
$$\geq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(y)}{y}dy - f(L(a,b)).$$

If we take $x = A(a,b)$ in (51), then we get

$$\frac{f(b)+f(a)}{2} - \frac{1}{b-a}\int_a^b f(y)\,dy \quad (55)$$
$$\geq \frac{A(a,b)}{L(a,b)}\left[\frac{1}{\ln b - \ln a}\int_a^b \frac{f(y)}{y}dy - f(A(a,b))\right].$$

If we take $x = H(a,b) := \frac{2ab}{a+b}$ in (51), then we get

$$\frac{bf(b)+af(a)}{b+a} - \frac{1}{b-a}\int_a^b f(y)\,dy \quad (56)$$
$$\geq \frac{H(a,b)}{L(a,b)}\left[\frac{1}{\ln b - \ln a}\int_a^b \frac{f(y)}{y}dy - f(H(a,b))\right].$$

If we take $x = G(a,b)$ in (51)), then we get

$$\frac{(b - G(a,b)) f(b) + (G(a,b) - a) f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(y)\,dy \quad (57)$$
$$\geq \frac{G(a,b)}{L(a,b)}\left[\frac{1}{\ln b - \ln a}\int_a^b \frac{f(y)}{y}dy - f(G(a,b))\right].$$

If the function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is convex, then by Jensen's inequality we have

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(y)}{y}dy = \frac{1}{\int_a^b \frac{dy}{y}}\int_a^b \frac{f(y)}{y}dy \geq f\left(\frac{\int_a^b y\frac{dy}{y}}{\int_a^b \frac{dy}{y}}\right)$$
$$= f\left(\frac{b-a}{\ln b - \ln a}\right) = f(L(a,b)).$$

Therefore, for any function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ that is *convex and HA-convex*, by (54) we have

$$\frac{(b - L(a,b)) f(b) + (L(a,b) - a) f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(y)\,dy \quad (58)$$
$$\geq \frac{1}{\ln b - \ln a}\int_a^b \frac{f(y)}{y}dy - f(L(a,b)) \geq 0.$$

It is known that, if a function $f : [a,b] \subset (0,\infty) \to \mathbb{R}$ is *GA-convex*, namely

$$f\left(x^{1-\lambda}y^{\lambda}\right) \leq (\geq)(1-\lambda) f(x) + \lambda f(y)$$

for all $x, y \in [a,b]$ and $\lambda \in [0,1]$, then [25]

$$\frac{1}{\ln b - \ln a}\int_a^b \frac{f(y)}{y}dy \geq f(G(a,b)). \quad (59)$$

Therefore, for any function $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ that is *GA-convex and HA-convex*, by (57) we have

$$\frac{(b - G(a,b)) f(b) + (G(a,b) - a) f(a)}{b-a} - \frac{1}{b-a}\int_a^b f(y)\,dy \tag{60}$$
$$\geq \frac{G(a,b)}{L(a,b)}\left[\frac{1}{\ln b - \ln a}\int_a^b \frac{f(y)}{y}dy - f(G(a,b))\right] \geq 0.$$

Theorem 4.3. *Let $f : [a, b] \subset (0, \infty) \to \mathbb{R}$ be a HA-convex function on the interval $[a, b]$. Then*

$$\frac{1}{2x}\left(\frac{f(b)\,a(b-x) + f(a)\,b(x-a)}{b-a} + xf(x)\right) \geq \frac{ab}{b-a}\int_a^b \frac{f(y)}{y^2}dy \tag{61}$$

for any $x \in [a, b]$.

Proof. From (17) we have, by division with $y^2 > 0$, that

$$\frac{1}{y^2}f(x) - \frac{1}{y^2}f(y) \geq \frac{f'_-(y)}{y}\left(1 - \frac{y}{x}\right)$$

for any $x, y \in (a, b)$.

Taking the integral mean over y we have

$$f(x)\frac{1}{b-a}\int_a^b \frac{1}{y^2}dy - \frac{1}{b-a}\int_a^b \frac{f(y)}{y^2}dy$$
$$\geq \frac{1}{b-a}\int_a^b \frac{f'_-(y)}{y}dy - \frac{1}{x}\frac{1}{b-a}\int_a^b f'_-(y)\,dy$$

that is equivalent to

$$\frac{f(x)}{ab} - \frac{1}{b-a}\int_a^b \frac{f(y)}{y^2}dy$$
$$\geq \frac{1}{b-a}\left[\frac{f(b)}{b} - \frac{f(a)}{a} + \int_a^b \frac{f(y)}{y^2}dy\right] - \frac{1}{x}\frac{f(b)-f(a)}{b-a}$$
$$= \frac{1}{b-a}\left(\frac{f(b)}{b} - \frac{f(a)}{a}\right) + \frac{1}{b-a}\int_a^b \frac{f(y)}{y^2}dy - \frac{1}{x}\frac{f(b)-f(a)}{b-a},$$

for any $x \in (a, b)$. This can be written as

$$\frac{1}{x}\frac{f(b)-f(a)}{b-a} - \frac{1}{b-a}\left(\frac{f(b)}{b} - \frac{f(a)}{a}\right) \geq \frac{2}{b-a}\int_a^b \frac{f(y)}{y^2}dy - \frac{f(x)}{ab}$$

or as

$$\frac{1}{2}\left(\frac{1}{b-a}\left[f(b)\frac{b-x}{xb} + f(a)\frac{x-a}{ax}\right] + \frac{f(x)}{ab}\right) \geq \frac{1}{b-a}\int_a^b \frac{f(y)}{y^2}dy.$$

This is equivalent to the desired result (61).

Remark 4.3. *If we take in (61) $x = \frac{a+b}{2}$, then we get*

$$\frac{1}{2}\left(\frac{f(b)\,a + f(a)\,b}{a+b} + f\left(\frac{a+b}{2}\right)\right) \geq \frac{ab}{b-a}\int_a^b \frac{f(y)}{y^2}dy. \tag{62}$$

If we take in (32) $x=\frac{2ab}{a+b}$, *then we get*

$$\frac{1}{2}\left[\frac{f(b)+f(a)}{2}+f\left(\frac{2ab}{a+b}\right)\right]\geq\frac{ab}{b-a}\int_a^b\frac{f(y)}{y^2}dy. \tag{63}$$

5. Applications

We consider the *arithmetic mean* $A(a,b)=\frac{a+b}{2}$, the *geometric mean* $G(a,b)=\sqrt{ab}$ and *harmonic mean* $H(a,b)=\frac{2ab}{a+b}$ for the positive numbers $a,b>0$.

The following well know order between these means, including logarithmic and identric means defined above, holds

$$H(a,b)\leq G(a,b)\leq L(a,b)\leq I(a,b)\leq A(a,b). \tag{64}$$

If we consider the HA-convex function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t$ and we use the inequalities (9), then we have

$$\begin{aligned}H(a,b)&\leq(1-\lambda)\frac{2ab}{(1-\lambda)a+(\lambda+1)b}+\lambda\frac{2ab}{(2-\lambda)a+\lambda b} \\ &\leq\frac{G^2(a,b)}{L(a,b)}\leq\frac{1}{2}\left[\frac{ab}{(1-\lambda)a+\lambda b}+(1-\lambda)a+\lambda b\right]\leq A(a,b),\end{aligned} \tag{65}$$

for any $\lambda\in[0,1]$.

If we use the inequalities (43) and (44) we get

$$0\leq A(a,b)-\frac{G^2(a,b)}{L(a,b)}\leq\frac{1}{4}\frac{A(a,b)}{G^2(a,b)}(b-a)^2 \tag{66}$$

and

$$0\leq\frac{G^2(a,b)}{L(a,b)}-H(a,b)\leq\frac{1}{4}\frac{A(a,b)}{G^2(a,b)}(b-a)^2. \tag{67}$$

The first inequality in (66) also follows by (64).

Consider the function $f:(0,\infty)\to\mathbb{R}$, $f(t)=\frac{\ln t}{t}$. Observe that

$$g(t)=f\left(\frac{1}{t}\right)=-t\ln t,$$

which shows that f is HA-concave on $(0,\infty)$.

If we use the inequality (15) for HA-concave functions we have

$$\frac{\ln(L(a,b))}{L(a,b)}\geq\frac{1}{b-a}\int_a^b\frac{\ln t}{t}dt\geq\frac{(L(a,b)-a)\ln b+(b-L(a,b))\ln a}{(b-a)L(a,b)},$$

which is equivalent to

$$\frac{\ln(L(a,b))}{L(a,b)}\geq\frac{\ln G(a,b)}{L(a,b)}\geq\frac{(L(a,b)-a)\ln b+(b-L(a,b))\ln a}{(b-a)L(a,b)}. \tag{68}$$

The first inequality in (68) also follows (64).

From the second inequality we have

$$G(a,b)\geq b^{\frac{L(a,b)-a}{b-a}}a^{\frac{b-L(a,b)}{b-a}}. \tag{69}$$

If we write the inequality (63) for the HA-convex function $f:(0,\infty)\to\mathbb{R}$, $f(t)=t$, then we have

$$\frac{A(a,b)+H(a,b)}{2}\geq\frac{G^{2}(a,b)}{L(a,b)}. \tag{70}$$

If we write the inequalities (49) and (50) for the HA-concave function $f:(0,\infty)\to\mathbb{R}$, $f(t)=\frac{\ln t}{t}$, then we get

$$\sqrt{A(a,b)\,G(a,b)}\leq I(a,b) \tag{71}$$

and

$$\frac{1}{2}\left[\ln\left(\frac{2ab}{a+b}\right)+\frac{b\ln b+a\ln a}{a+b}\right]\leq\ln I(a,b). \tag{72}$$

References

[1] M. Alomari and M. Darus, The Hadamard's inequality for s-convex function. *Int. J. Math. Anal.* (Ruse) **2** (2008), no. 13-16, 639–646.

[2] M. Alomari and M. Darus, Hadamard-type inequalities for s-convex functions. *Int. Math. Forum* **3** (2008), no. 37-40, 1965–1975.

[3] G. A. Anastassiou, Univariate Ostrowski inequalities, revisited. *Monatsh. Math.,* **135** (2002), no. 3, 175–189.

[4] G. D. Anderson, M. K. Vamanamurthy and M. Vuorinen, Generalized convexity and inequalities, *J. Math. Anal. Appl.* **335** (2007) 1294–1308.

[5] N. S. Barnett, P. Cerone, S. S. Dragomir, M. R. Pinheiro,and A. Sofo, Ostrowski type inequalities for functions whose modulus of the derivatives are convex and applications. *Inequality Theory and Applications*, **Vol. 2** (Chinju/Masan, 2001), 19–32, Nova Sci. Publ., Hauppauge, NY, 2003. Preprint: *RGMIA Res. Rep. Coll.* **5** (2002), No. 2, Art. 1 `[Online http://rgmia.org/papers/v5n2/Paperwapp2q.pdf]`.

[6] E. F. Beckenbach, Convex functions, *Bull. Amer. Math. Soc.* **54**(1948), 439–460.

[7] M. Bombardelli and S. Varošanec, Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities. *Comput. Math. Appl.* **58** (2009), no. 9, 1869–1877.

[8] W. W. Breckner, Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen. (German) *Publ. Inst. Math.* (Beograd) (N.S.) **23(37)** (1978), 13–20.

[9] W. W. Breckner and G. Orbán, Continuity properties of rationally s-convex mappings with values in an ordered topological linear space. Universitatea "Babeş-Bolyai", Facultatea de Matematica, Cluj-Napoca, 1978. viii+92 pp.

[10] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view, Ed. G. A. Anastassiou, *Handbook of Analytic-Computational Methods in Applied Mathematics*, CRC Press, New York. 135-200.

[11] P. Cerone and S. S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, in *Advances in Statistics Combinatorics and Related Areas*, C. Gulati, et al. (Eds.), World Science Publishing, 2002, 53-62.

[12] P. Cerone, S. S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, *Demonstratio Mathematica*, **32**(2) (1999), 697—712.

[13] G. Cristescu, Hadamard type inequalities for convolution of h-convex functions. *Ann. Tiberiu Popoviciu Semin. Funct. Equ. Approx. Convexity* **8** (2010), 3–11.

[14] S. S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, *J. KSIAM,* **3**(1) (1999), 127-135.

[15] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, *Comp. Math. Appl.,* **38** (1999), 33-37.

[16] S. S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, *Korean J. Appl. Math.,* **7** (2000), 477-485.

[17] S. S. Dragomir, On the Ostrowski's inequality for mappings of bounded variation and applications, *Math. Ineq. & Appl.,* **4**(1) (2001), 33-40.

[18] S. S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation and applications, *J. KSIAM,* **5**(1) (2001), 35-45.

[19] S. S. Dragomir, Ostrowski type inequalities for isotonic linear functionals, *J. Inequal. Pure & Appl. Math.,* **3**(5) (2002), Art. 68.

[20] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. *J. Inequal. Pure Appl. Math.* **3** (2002), no. 2, Article 31, 8 pp.

[21] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No. 2, Article 31.

[22] S. S. Dragomir, An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products, *J. Inequal. Pure Appl. Math.* **3** (2002), No.3, Article 35.

[23] S. S. Dragomir, An Ostrowski like inequality for convex functions and applications, *Revista Math. Complutense*, **16**(2) (2003), 373-382.

[24] S. S. Dragomir, *Operator Inequalities of Ostrowski and Trapezoidal Type.* Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1

[25] S. S. Dragomir, Some new inequalities of Hermite-Hadamard type for GA-convex functions, Preprint *RGMIA Res. Rep. Coll.* 18 (2015), Art. 30. [http://rgmia.org/papers/v18/v18a30.pdf].

[26] S. S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, *Bull. Math. Soc. Sci. Math. Romanie,* **42(90)** (4) (1999), 301-314.

[27] S.S. Dragomir and S. Fitzpatrick, The Hadamard inequalities for s-convex functions in the second sense. *Demonstratio Math.* **32** (1999), no. 4, 687–696.

[28] S.S. Dragomir and S. Fitzpatrick,The Jensen inequality for s-Breckner convex functions in linear spaces. *Demonstratio Math.* **33** (2000), no. 1, 43–49.

[29] S. S. Dragomir and B. Mond, On Hadamard's inequality for a class of functions of Godunova and Levin. *Indian J. Math.* **39** (1997), no. 1, 1–9.

[30] S. S. Dragomir and C. E. M. Pearce, On Jensen's inequality for a class of functions of Godunova and Levin. *Period. Math. Hungar.* **33** (1996), no. 2, 93–100.

[31] S. S. Dragomir and C. E. M. Pearce, Quasi-convex functions and Hadamard's inequality, *Bull. Austral. Math. Soc.* **57** (1998), 377-385.

[32] S. S. Dragomir, J. Pečarić and L. Persson, Some inequalities of Hadamard type. *Soochow J. Math.* **21** (1995), no. 3, 335–341.

[33] S. S. Dragomir and Th. M. Rassias (Eds), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publisher, 2002.

[34] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, *Tamkang J. of Math.*, **28** (1997), 239-244.

[35] S. S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, *Appl. Math. Lett.*, **11** (1998), 105-109.

[36] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, *Indian J. of Math.,* **40**(3) (1998), 245-304.

[37] A. El Farissi, Simple proof and refeinment of Hermite-Hadamard inequality, *J. Math. Ineq.* **4** (2010), No. 3, 365–369.

[38] E. K. Godunova and V. I. Levin, Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. (Russian) *Numerical mathematics and mathematical physics* (Russian), 138–142, 166, Moskov. Gos. Ped. Inst., Moscow, 1985

[39] H. Hudzik and L. Maligranda, Some remarks on s-convex functions. *Aequationes Math.* **48** (1994), no. 1, 100–111.

[40] I. Işcan, Hermite-Hadamard type inequalities for harmonically convex functions, *Hacettepe Journal of Mathematics and Statistics*, **43** (6) (2014), 935 – 942.

[41] E. Kikianty and S. S. Dragomir, Hermite-Hadamard's inequality and the p-HH-norm on the Cartesian product of two copies of a normed space, *Math. Inequal. Appl.* (in press)

[42] U. S. Kirmaci, M. Klaričić Bakula, M. E Özdemir and J. Pečarić, Hadamard-type inequalities for s-convex functions. *Appl. Math. Comput.* **193** (2007), no. 1, 26–35.

[43] M. A. Latif, On some inequalities for h-convex functions. *Int. J. Math. Anal.* (Ruse) **4** (2010), no. 29-32, 1473–1482.

[44] D. S. Mitrinović and I. B. Lacković, Hermite and convexity, *Aequationes Math.* **28** (1985), 229–232.

[45] D. S. Mitrinović and J. E. Pečarić, Note on a class of functions of Godunova and Levin. *C. R. Math. Rep. Acad. Sci. Canada* **12** (1990), no. 1, 33–36.

[46] M. A. Noor, K. I. Noor and M. U. Awan, Some inequalities for geometrically-arithmetically h-convex functions, *Creat. Math. Inform.* **23** (2014), No. 1, 91 - 98.

[47] C. E. M. Pearce and A. M. Rubinov, P-functions, quasi-convex functions, and Hadamard-type inequalities. *J. Math. Anal. Appl.* **240** (1999), no. 1, 92–104.

[48] J. E. Pečarić and S. S. Dragomir, On an inequality of Godunova-Levin and some refinements of Jensen integral inequality. *Itinerant Seminar on Functional Equations, Approximation and Convexity* (Cluj-Napoca, 1989), 263–268, Preprint, 89-6, Univ. "Babeş-Bolyai", Cluj-Napoca, 1989.

[49] J. Pečarić and S. S. Dragomir, A generalization of Hadamard's inequality for isotonic linear functionals, *Radovi Mat.* (Sarajevo) **7** (1991), 103–107.

[50] M. Radulescu, S. Radulescu and P. Alexandrescu, On the Godunova-Levin-Schur class of functions. *Math. Inequal. Appl.* **12** (2009), no. 4, 853–862.

[51] M. Z. Sarikaya, A. Saglam, and H. Yildirim, On some Hadamard-type inequalities for h-convex functions. *J. Math. Inequal.* **2** (2008), no. 3, 335–341.

[52] E. Set, M. E. Özdemir and M. Z. Sarıkaya, New inequalities of Ostrowski's type for s-convex functions in the second sense with applications. *Facta Univ. Ser. Math. Inform.* **27** (2012), no. 1, 67–82.

[53] M. Z. Sarikaya, E. Set and M. E. Özdemir, On some new inequalities of Hadamard type involving h-convex functions. *Acta Math. Univ. Comenian.* (N.S.) **79** (2010), no. 2, 265–272.

[54] M. Tunç, Ostrowski-type inequalities via h-convex functions with applications to special means. *J. Inequal. Appl.* **2013**, 2013:326.

[55] S. Varošanec, On h-convexity. *J. Math. Anal. Appl.* **326** (2007), no. 1, 303–311.

[56] X.-M. Zhang, Y.-M. Chu and X.-H. Zhang, The Hermite-Hadamard type inequality of GA-convex functions and its application, *Journal of Inequalities and Applications*, Volume **2010**, Article ID 507560, 11 pages.