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Hermite-Hadamard type inequalities for p-convex functions via fractional integrals

Mehmet Kunt[a] and İmdat İşcan[b]

Abstract. In this paper, we present Hermite-Hadamard inequality for p-convex functions in fractional integral forms. we obtain an integral equality and some Hermite-Hadamard type integral inequalities for p-convex functions in fractional integral forms. We give some Hermite-Hadamard type inequalities for convex, harmonically convex and p-convex functions. Some results presented in this paper for p-convex functions, provide extensions of others given in earlier works for convex, harmonically convex and p-convex functions.

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1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

is well known in the literature as Hermite-Hadamard's inequality [3, 4].

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[a]*Department of Mathematics, Faculty of Sciences, Karadeniz Technical University, 61080, Trabzon, Turkey*
[b]*Department of Mathematics, Faculty of Sciences and Arts, Giresun University, 28200, Giresun, Turkey*
e-mail: mkunt@ktu.edu.tr
e-mail: imdat.iscan@giresun.edu.tr;imdati@yahoo.com

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For some results which generalize, improve, and extend the inequalities (1) see [2, 6, 8, 9, 13, 14, 15].

We will now give definitions of the right-hand side and left-hand side Riemann-Liouville fractional integrals which are used throughout this paper.

Definition 1.1. [12]. *Let $f \in L[a,b]$. The right-hand side and left-hand side Riemann-Liouville fractional integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ with $b > a \geq 0$ are defined by*

$$J_{a+}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \;\; x > a$$
$$J_{b-}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \;\; x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-t}t^{\alpha-1}dt$.

Because of comprehensive application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [1, 5, 11, 16].

In [6], İşcan give the definition of harmonically convex function and present the Hermite-Hadamard inequality for harmonically convex functions as follows:

Definition 1.2. *Let $I \subset \mathbb{R} \backslash \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{2}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (2) is reversed, then f is said to be harmonically concave.

Theorem 1.1. [6]. *Let $f : I \subset \mathbb{R} \backslash \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities holds:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \leq \frac{f(a) + f(b)}{2}. \tag{3}$$

In [11], Kunt et al. present Hermite-Hadamard inequality for harmonically convex functions via fractional integrals as follows:

Theorem 1.2. [11]. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$ where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals holds:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^{\alpha} \left[J_{\frac{a+b}{2ab}+}^{\alpha} (f \circ g)(1/a) + J_{\frac{a+b}{2ab}-}^{\alpha} (f \circ g)(1/b)\right] \leq \frac{f(a)+f(b)}{2} \tag{4}$$

with $\alpha > 0$ and $g(x) = \frac{1}{x}$, $x \in \left[\frac{1}{b}, \frac{1}{a}\right]$.

In [17], Zhang and Wan give the definition of p-convex function on $I \subset \mathbb{R}$, and in [9], İşcan give a different definition of p-convex function on $I \subset (0,\infty)$ as follows:

Definition 1.3. *Let $I\subset(0,\infty)$ be a real interval and $p\in\mathbb{R}\backslash\{0\}$. A function $f:I\to\mathbb{R}$ is said to be p-convex, if*

$$f\left(\left[tx^{p}+(1-t)\,y^{p}\right]^{1/p}\right)\leq tf(x)+(1-t)\,f(y) \tag{5}$$

for all $x,y\in I$ and $t\in[0,1]$.

It can be easily seen that for $p=1$ and $p=-1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I\subset(0,\infty)$, respectively.

In [2, Theorem 5], if we take $I\subset(0,\infty)$, $p\in\mathbb{R}\backslash\{0\}$ and $h(t)=t$, then we have the following theorem.

Theorem 1.3. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a p-convex function, $p\in\mathbb{R}\backslash\{0\}$, and $a,b\in I$ with $a<b$. If $f\in L[a,b]$ then the following inequalities holds:*

$$f\left(\left[\frac{a^{p}+b^{p}}{2}\right]^{1/p}\right)\leq\frac{p}{b^{p}-a^{p}}\int_{a}^{b}\frac{f(x)}{x^{1-p}}dx\leq\frac{f(a)+f(b)}{2}. \tag{6}$$

For some results related to p-convex functions and its generalizations, we refer to the reader to see [2, 7, 8, 9, 13, 14, 17].

In this paper, we present Hermite-Hadamard inequality for p-convex functions in fractional integral forms. We obtain an integral identity and some new Hermite-Hadamard type integral inequalities for p-convex functions in fractional integral forms. We give some new Hermite-Hadamard type inequalities for convex, harmonically convex and p-convex functions.

2. Main results

Theorem 2.1. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a p-convex function, $p\in\mathbb{R}\backslash\{0\}$, $\alpha>0$ and $a,b\in I$ with $a<b$. If $f\in L[a,b]$, then the following inequalities for fractional integrals holds:*

(i) If $p>0$,

$$f\left(\left[\frac{a^{p}+b^{p}}{2}\right]^{1/p}\right)\leq\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^{p}-a^{p})^{\alpha}}\left[J_{\frac{a^{p}+b^{p}}{2}+}^{\alpha}(f\circ g)(b^{p})+J_{\frac{a^{p}+b^{p}}{2}-}^{\alpha}(f\circ g)(a^{p})\right]\leq\frac{f(a)+f(b)}{2} \tag{7}$$

with $g(x)=x^{1/p}$, $x\in[a^{p},b^{p}]$,

(ii) If $p<0$,

$$f\left(\left[\frac{a^{p}+b^{p}}{2}\right]^{1/p}\right)\leq\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(a^{p}-b^{p})^{\alpha}}\left[J_{\frac{a^{p}+b^{p}}{2}+}^{\alpha}(f\circ g)(a^{p})+J_{\frac{a^{p}+b^{p}}{2}-}^{\alpha}(f\circ g)(b^{p})\right]\leq\frac{f(a)+f(b)}{2} \tag{8}$$

with $g(x)=x^{1/p}$, $x\in[b^{p},a^{p}]$.

Proof. (i) Let $p>0$. Since $f:I\subset(0,\infty)\to\mathbb{R}$ is a p-convex function, we have, for all $x,y\in I$ (with $t=\frac{1}{2}$ in the inequality (5))

$$f\left(\left[\frac{x^{p}+y^{p}}{2}\right]^{1/p}\right)\leq\frac{f(x)+f(y)}{2}.$$

Choosing $x=\left[ta^{p}+(1-t)\,b^{p}\right]^{1/p}$ and $y=\left[tb^{p}+(1-t)\,a^{p}\right]^{1/p}$, we get

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \le \frac{f\left(\left[ta^p+(1-t)\,b^p\right]^{1/p}\right)+f\left(\left[tb^p+(1-t)\,a^p\right]^{1/p}\right)}{2}. \tag{9}$$

Multiplying both sides of (9) by $2t^{\alpha-1}$ and integrating with respect to t over $\left[0,\frac{1}{2}\right]$, we get

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\frac{1}{\alpha 2^{\alpha-1}} \le \frac{\Gamma(\alpha)}{(b^p-a^p)^{\alpha}}\left[J^{\alpha}_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^{\alpha}_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right]$$

that provides the left hand side of (7).

For the proof of the second inequality in (7) we first note that if f is a p-convex function, then, for all $t\in[0,1]$, it yields

$$\frac{f\left(\left[ta^p+(1-t)\,b^p\right]^{1/p}\right)+f\left(\left[tb^p+(1-t)\,a^p\right]^{1/p}\right)}{2} \le \frac{f(a)+f(b)}{2}. \tag{10}$$

Multiplying both sides of (10) by $2t^{\alpha-1}$ and integrating with respect to t over $\left[0,\frac{1}{2}\right]$, we get

$$\frac{\Gamma(\alpha)}{(b^p-a^p)^{\alpha}}\left[J^{\alpha}_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^{\alpha}_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right] \le \frac{1}{\alpha 2^{\alpha-1}}\frac{f(a)+f(b)}{2}$$

that provides the right hand side of (7). This completes the proof of i.

(ii) The proof is similar with i. □

Remark 2.1. *In Theorem 2.1, one can see the following.*

(1) If one takes $p=1$ and $\alpha=1$, one has (1),

(2) If one takes $p=-1$, one has (4),

(3) If one takes $p=-1$ and $\alpha=1$, one has (3),

(4) If one takes $\alpha=1$, one has (6).

Corollary 2.1. *In Theorem 2.1, if one takes $p=1$, one has the following Hermite-Hadamard type inequalities for convex functions via fractional integrals:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}+}f(b)+J^{\alpha}_{\frac{a+b}{2}-}f(a)\right] \le \frac{f(a)+f(b)}{2}.$$

Lemma 2.1. *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° and $a,b\in I^{\circ}$ with $a<b$, $p\in\mathbb{R}\backslash\{0\}$ and $\alpha>0$. If $f'\in L[a,b]$ and $w:[a,b]\to\mathbb{R}$ is integrable, then the following equalities for fractional integrals holds:*

(i) If $p>0$,

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\left[J^{\alpha}_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^{\alpha}_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right] \tag{11}$$

$$=\frac{1}{2^{1-\alpha}(b^p-a^p)^{\alpha}}\left[\int_{a^p}^{\frac{a^p+b^p}{2}}(t-a^p)^{\alpha}(f\circ g)'(t)\,dt-\int_{\frac{a^p+b^p}{2}}^{b^p}(b^p-t)^{\alpha}(f\circ g)'(t)\,dt\right]$$

with $g(x)=x^{1/p}$, $x\in[a^p,b^p]$,

(ii) If $p<0$,

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(a^p-b^p)^{\alpha}}\left[J^{\alpha}_{\frac{a^p+b^p}{2}+}(f\circ g)(a^p)+J^{\alpha}_{\frac{a^p+b^p}{2}-}(f\circ g)(b^p)\right] \tag{12}$$

$$= \frac{1}{2^{1-\alpha}(a^p-b^p)^\alpha}\left[\int_{b^p}^{\frac{a^p+b^p}{2}}(t-b^p)^\alpha (f\circ g)'(t)\,dt - \int_{\frac{a^p+b^p}{2}}^{a^p}(a^p-t)^\alpha (f\circ g)'(t)\,dt\right]$$

with $g(x)=x^{1/p}$, $x\in[b^p,a^p]$.

Proof. (i) Let $p>0$. It suffices to note that

$$K = \frac{1}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[\int_{a^p}^{\frac{a^p+b^p}{2}}\frac{1}{2^{1-\alpha}(b^p-a^p)^\alpha}dt - \int_{\frac{a^p+b^p}{2}}^{b^p}(b^p-t)^\alpha (f\circ g)'(t)\,dt\right] \tag{13}$$

$$= \frac{1}{2^{1-\alpha}(b^p-a^p)^\alpha}\int_{a^p}^{\frac{a^p+b^p}{2}}(t-a^p)^\alpha (f\circ g)'(t)\,dt - \frac{1}{2^{1-\alpha}(b^p-a^p)^\alpha}\int_{\frac{a^p+b^p}{2}}^{b^p}(b^p-t)^\alpha (f\circ g)'(t)\,dt$$
$$= K_1 - K_2.$$

By integration by parts, we have

$$K_1 = \frac{1}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[(t-a^p)^\alpha (f\circ g)(t)\Big|_{a^p}^{\frac{a^p+b^p}{2}} - \alpha\int_{a^p}^{\frac{a^p+b^p}{2}}(t-a^p)^{\alpha-1}(f\circ g)(t)\,dt\right] \tag{14}$$
$$= \frac{1}{2}f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\frac{1}{\Gamma(\alpha)}\int_{a^p}^{\frac{a^p+b^p}{2}}(t-a^p)^{\alpha-1}(f\circ g)(t)\,dt$$
$$= \frac{1}{2}f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)$$

and similarly

$$K_2 = \frac{1}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[(b^p-t)^\alpha (f\circ g)(t)\Big|_{\frac{a^p+b^p}{2}}^{b^p} + \alpha\int_{\frac{a^p+b^p}{2}}^{b^p}(b^p-t)^{\alpha-1}(f\circ g)(t)\,dt\right] \tag{15}$$
$$= -\frac{1}{2}f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) + \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\frac{1}{\Gamma(\alpha)}\int_{\frac{a^p+b^p}{2}}^{b^p}(b^p-t)^{\alpha-1}(f\circ g)(t)\,dt$$
$$= -\frac{1}{2}f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) + \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p).$$

A combination of (13), (14) and (15) we have (11). This completes the proof of i.
(ii) The proof is similar with i. □

Remark 2.2. *In Lemma 2.1, one can see the following.*
(1) If one takes $p=1$ *and* $\alpha=1$*, one has* [10, Lemma 2.1]*,*
(2) If one takes $\alpha=1$*, one has* [14, Lemma 2.7]*.*

Theorem 2.2. *Let* $f: I\subset(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$*, where* $a,b\in I$ *and* $a<b$*. If* $|f'|$ *is p-convex function on* $[a,b]$ *for* $p\in\mathbb{R}\setminus\{0\}$ *and* $\alpha>0$*, then the following inequality for fractional integrals holds:*
(i) If $p>0$*,*

$$\left|f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p) + J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right]\right|$$
$$\le \frac{b^p-a^p}{2^{1-\alpha}}\left[C_1(\alpha,p)\left|f'(a)\right| + C_2(\alpha,p)\left|f'(b)\right|\right]$$

with $g(x) = x^{1/p}$, $x \in [a^p, b^p]$,

(ii) If $p < 0$,

$$\left| f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(a^p-b^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(a^p) + J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(b^p)\right]\right| \leq \frac{a^p-b^p}{2^{1-\alpha}}\left[-C_1(\alpha,p)\left|f'(a)\right| - C_2(\alpha,p)\left|f'(b)\right|\right]$$

with $g(x) = x^{1/p}$, $x \in [b^p, a^p]$, *where*

$$C_1(\alpha,p) = \int_0^{\frac{1}{2}} \frac{u^{\alpha+1}}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}du + \int_{\frac{1}{2}}^1 \frac{(1-u)^\alpha u}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}du,$$

$$C_2(\alpha,p) = \int_0^{\frac{1}{2}} \frac{u^{\alpha}(1-u)}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}du + \int_{\frac{1}{2}}^1 \frac{(1-u)^{\alpha+1}}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}du.$$

Proof. (i) Let $p > 0$. Using Lemma 2.1-i, it follows that

$$\begin{aligned}
&\left| f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p) + J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right]\right| \\
&\leq \frac{1}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[\int_{a^p}^{\frac{a^p+b^p}{2}} (t-a^p)^\alpha \left|(f\circ g)'(t)\right| dt + \int_{\frac{a^p+b^p}{2}}^{b^p} (b^p-t)^\alpha \left|(f\circ g)'(t)\right| dt\right] \\
&\leq \frac{b^p-a^p}{2^{1-\alpha}}\left[\int_{a^p}^{\frac{a^p+b^p}{2}} (t-a^p)^\alpha \frac{1}{pt^{1-(1/p)}}\left|f'\left(t^{1/p}\right)\right| dt + \int_{\frac{a^p+b^p}{2}}^{b^p} (b^p-t)^\alpha \frac{1}{pt^{1-(1/p)}}\left|f'\left(t^{1/p}\right)\right| dt\right].
\end{aligned}$$

Setting $t = ua^p + (1-u)b^p$ and $dt = (a^p - b^p)\,du$ gives

$$\begin{aligned}
&\left| f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p) + J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right]\right| \qquad (16) \\
&\leq \frac{b^p-a^p}{2^{1-\alpha}}\left[\begin{array}{c} \int_0^{\frac{1}{2}} \frac{u^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right| du \\ +\int_{\frac{1}{2}}^1 \frac{(1-u)^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right| du \end{array}\right].
\end{aligned}$$

Since $|f'|$ is p-convex function on $[a,b]$, we have

$$\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right| \leq u\left|f'(a)\right| + (1-u)\left|f'(b)\right|. \qquad (17)$$

A combination of (16) and (17), we have

$$\begin{aligned}
&\left| f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p) + J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right]\right| \\
&\leq \frac{b^p-a^p}{2^{1-\alpha}}\left[\begin{array}{c} \int_0^{\frac{1}{2}} \frac{u^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\left(u\left|f'(a)\right| + (1-u)\left|f'(b)\right|\right) du \\ +\int_{\frac{1}{2}}^1 \frac{(1-u)^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\left(u\left|f'(a)\right| + (1-u)\left|f'(b)\right|\right) du \end{array}\right] \\
&= \frac{b^p-a^p}{2^{1-\alpha}}\left[\begin{array}{c} \left(\int_0^{\frac{1}{2}} \frac{u^{\alpha+1}}{p[ua^p+(1-u)b^p]^{1-(1/p)}}du + \int_{\frac{1}{2}}^1 \frac{(1-u)^\alpha u}{p[ua^p+(1-u)b^p]^{1-(1/p)}}du\right)\left|f'(a)\right| \\ +\left(\int_0^{\frac{1}{2}} \frac{u^{\alpha}(1-u)}{p[ua^p+(1-u)b^p]^{1-(1/p)}}du + \int_{\frac{1}{2}}^1 \frac{(1-u)^{\alpha+1}}{p[ua^p+(1-u)b^p]^{1-(1/p)}}du\right)\left|f'(b)\right| \end{array}\right]
\end{aligned}$$

$$= \frac{b^p - a^p}{2^{1-\alpha}} \left[C_1(\alpha, p) \left| f'(a) \right| + C_2(\alpha, p) \left| f'(b) \right| \right].$$

This completes the proof of i.

(ii) The proof is similar with i. □

Remark 2.3. *In Theorem 2.2, one can see the following.*

(1) If one takes $p = 1$ and $\alpha = 1$, one has [10, Theorem 2.2],

(2) If one takes $\alpha = 1$, one has [14, Teheorem 3.3].

Corollary 2.2. *In Theorem 2.2, one can see the following.*

(1) If one takes $p = 1$, one has the following Hermite-Hadamard inequality for convex functions via fractional integrals:

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}} \left[J^{\alpha}_{\frac{a+b}{2}+} f(b) + J^{\alpha}_{\frac{a+b}{2}-} f(a) \right] \right| \leq \frac{b-a}{4(\alpha+1)} \left[\left| f'(a) \right| + \left| f'(b) \right| \right],$$

(2) If one takes $p = -1$, one has the following Hermite-Hadamard inequality for harmonically convex functions via fractional integrals:

$$\left| f\left(\frac{2ab}{a+b}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}} \left(\frac{ab}{b-a}\right)^{\alpha} \left[J^{\alpha}_{\frac{a+b}{2ab}+} (f \circ g)(1/a) + J^{\alpha}_{\frac{a+b}{2ab}-} (f \circ g)(1/b) \right] \right|$$

$$\leq \frac{1}{4(\alpha+1)} \left(\frac{b-a}{ab}\right) \left[\left| f'(a) \right| + \left| f'(b) \right| \right],$$

(3) If one takes $p = -1$ and $\alpha = 1$, one has the following Hermite-Hadamard inequality for harmonically convex functions:

$$\left| f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \leq \frac{1}{8} \left(\frac{b-a}{ab}\right) \left[\left| f'(a) \right| + \left| f'(b) \right| \right].$$

Theorem 2.3. *Let $f : I \subset (0, \infty) \to \mathbb{R}$ be a differentiable function on I° such that $f' \in L[a, b]$, where $a, b \in I$ and $a < b$. If $|f'|^q$, $q \geq 1$, is p-convex function on $[a, b]$ for $p \in \mathbb{R} \setminus \{0\}$, $\alpha > 0$, then the following inequality for fractional integrals holds:*

(i) If $p > 0$,

$$\left| f\left(\left[\frac{a^p + b^p}{2}\right]^{1/p}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p - a^p)^{\alpha}} \left[J^{\alpha}_{\frac{a^p+b^p}{2}+} (f \circ g)(b^p) + J^{\alpha}_{\frac{a^p+b^p}{2}-} (f \circ g)(a^p) \right] \right|$$

$$\leq \frac{b^p - a^p}{2^{1-\alpha}} \left[\begin{array}{c} (C_5(\alpha, p))^{1-\frac{1}{q}} \left[C_6(\alpha, p) |f'(a)|^q + C_7(\alpha, p) |f'(b)|^q \right]^{\frac{1}{q}} \\ + (C_8(\alpha, p))^{1-\frac{1}{q}} \left[C_9(\alpha, p) |f'(a)|^q + C_{10}(\alpha, p) |f'(b)|^q \right]^{\frac{1}{q}} \end{array} \right],$$

with $g(x) = x^{1/p}$, $x \in [a^p, b^p]$,

(ii) If $p < 0$,

$$\left| f\left(\left[\frac{a^p + b^p}{2}\right]^{1/p}\right) - \frac{\Gamma(\alpha+1)}{2^{1-\alpha}(a^p - b^p)^{\alpha}} \left[J^{\alpha}_{\frac{a^p+b^p}{2}+} (f \circ g)(a^p) + J^{\alpha}_{\frac{a^p+b^p}{2}-} (f \circ g)(b^p) \right] \right|$$

$$\leq \frac{a^p - b^p}{2^{1-\alpha}} \left[\begin{array}{c} (-C_5(\alpha, p))^{1-\frac{1}{q}} \left[-C_6(\alpha, p) |f'(a)|^q - C_7(\alpha, p) |f'(b)|^q \right]^{\frac{1}{q}} \\ + (-C_8(\alpha, p))^{1-\frac{1}{q}} \left[-C_9(\alpha, p) |f'(a)|^q - C_{10}(\alpha, p) |f'(b)|^q \right]^{\frac{1}{q}} \end{array} \right],$$

with $g(x)=x^{1/p}$, $x\in[b^p,a^p]$, *where*

$$C_5(\alpha,p)=\int_0^{\frac{1}{2}}\frac{u^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}du,\quad C_6(\alpha,p)=\int_0^{\frac{1}{2}}\frac{u^{\alpha+1}}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}du,$$

$$C_7(\alpha,p)=\int_0^{\frac{1}{2}}\frac{u^\alpha(1-u)}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}du,\quad C_8(\alpha,p)=\int_{\frac{1}{2}}^1\frac{(1-u)^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}du,$$

$$C_9(\alpha,p)=\int_{\frac{1}{2}}^1\frac{(1-u)^\alpha u}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}du,\quad C_{10}(\alpha,p)=\int_{\frac{1}{2}}^1\frac{(1-u)^{\alpha+1}}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}du.$$

Proof. (i) Let $p>0$. Using (16), power mean inequality and the p-convexity of $|f'|^q$ it follows that

$$\left|f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right]\right|$$

$$\le\frac{b^p-a^p}{2^{1-\alpha}}\left[\begin{array}{c}\int_0^{\frac{1}{2}}\frac{u^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\left|f'\left([ua^p+(1-u)b^p]^{1/p}\right)\right|du\\+\int_{\frac{1}{2}}^1\frac{(1-u)^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\left|f'\left([ua^p+(1-u)b^p]^{1/p}\right)\right|du\end{array}\right]$$

$$\le\frac{b^p-a^p}{2^{1-\alpha}}\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}\frac{u^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}du\right)^{1-\frac{1}{q}}\\\times\left(\int_0^{\frac{1}{2}}\frac{u^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\left|f'\left([ua^p+(1-u)b^p]^{1/p}\right)\right|^q du\right)^{\frac{1}{q}}\\+\quad\left(\int_{\frac{1}{2}}^1\frac{(1-u)^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\right)^{1-\frac{1}{q}}\\\times\left(\int_{\frac{1}{2}}^1\frac{(1-u)^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\left|f'\left([ua^p+(1-u)b^p]^{1/p}\right)\right|^q du\right)^{\frac{1}{q}}\end{array}\right]$$

$$\le\frac{b^p-a^p}{2^{1-\alpha}}\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}\frac{u^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}du\right)^{1-\frac{1}{q}}\\\times\left(\left(\int_0^{\frac{1}{2}}\frac{u^{\alpha+1}}{p[ua^p+(1-u)b^p]^{1-(1/p)}}du\right)|f'(a)|^q+\left(\int_0^{\frac{1}{2}}\frac{u^\alpha(1-u)}{p[ua^p+(1-u)b^p]^{1-(1/p)}}du\right)|f'(b)|^q\right)^{\frac{1}{q}}\\+\quad\left(\int_{\frac{1}{2}}^1\frac{(1-u)^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}du\right)^{1-\frac{1}{q}}\\\times\left(\left(\int_{\frac{1}{2}}^1\frac{(1-u)^\alpha u}{p[ua^p+(1-u)b^p]^{1-(1/p)}}du\right)|f'(a)|^q+\left(\int_{\frac{1}{2}}^1\frac{(1-u)^{\alpha+1}}{p[ua^p+(1-u)b^p]^{1-(1/p)}}du\right)|f'(b)|^q\right)^{\frac{1}{q}}\end{array}\right]$$

$$=\frac{b^p-a^p}{2^{1-\alpha}}\left[\begin{array}{c}(C_5(\alpha,p))^{1-\frac{1}{q}}\left[C_6(\alpha,p)|f'(a)|^q+C_7(\alpha,p)|f'(b)|^q\right]^{\frac{1}{q}}\\+(C_8(\alpha,p))^{1-\frac{1}{q}}\left[C_9(\alpha,p)|f'(a)|^q+C_{10}(\alpha,p)|f'(b)|^q\right]^{\frac{1}{q}}\end{array}\right].$$

This completes the proof of i.

(ii) The proof is similar with i. □

Corollary 2.3. *In Theorem 2.3, one can see the following.*

(1) If one takes $p=1$*, one has the following Hermite-Hadamard inequality for convex functions via fractional integrals:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^\alpha}\left[J^\alpha_{\frac{a+b}{2}+}f(b)+J^\alpha_{\frac{a+b}{2}-}f(a)\right]\right|$$

$$\leq \frac{b-a}{2^{1-\alpha}}\left[\begin{array}{c}\left(C_5(\alpha,1)\right)^{1-\frac{1}{q}}\left[C_6(\alpha,1)\left|f'(a)\right|^q+C_7(\alpha,1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\\+\left(C_8(\alpha,1)\right)^{1-\frac{1}{q}}\left[C_9(\alpha,1)\left|f'(a)\right|^q+C_{10}(\alpha,1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\end{array}\right],$$

(2) If one takes $p=1$ and $\alpha=1$, one has the following Hermite-Hadamard inequality for convex functions:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{2^{-3}}{3^{\frac{1}{q}}}\left[\begin{array}{c}\left[\left|f'(a)\right|^q+2\left|f'(b)\right|^q\right]^{\frac{1}{q}}\\+\left[2\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{\frac{1}{q}}\end{array}\right],$$

(3) If one takes $p=-1$, one has the following Hermite-Hadamard inequality for harmonically convex functions via fractional integrals:

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ g)(1/b)\right]\right|$$

$$\leq\frac{1}{2^{1-\alpha}}\left(\frac{b-a}{ab}\right)\left[\begin{array}{c}\left(-C_5(\alpha,-1)\right)^{1-\frac{1}{q}}\left[-C_6(\alpha,-1)\left|f'(a)\right|^q-C_7(\alpha,-1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\\+\left(-C_8(\alpha,-1)\right)^{1-\frac{1}{q}}\left[-C_9(\alpha,-1)\left|f'(a)\right|^q-C_{10}(\alpha,-1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\end{array}\right],$$

(4) If one takes $p=-1$ and $\alpha=1$, one has the following Hermite-Hadamard inequality for harmonically convex functions:

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$

$$\leq\left(\frac{b-a}{ab}\right)\left[\begin{array}{c}\left(-C_5(1,-1)\right)^{1-\frac{1}{q}}\left[-C_6(1,-1)\left|f'(a)\right|^q-C_7(1,-1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\\+\left(-C_8(1,-1)\right)^{1-\frac{1}{q}}\left[-C_9(1,-1)\left|f'(a)\right|^q-C_{10}(1,-1)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\end{array}\right],$$

(5) If one takes $\alpha=1$, one has the following Hermite-Hadamard inequalities for p-convex functions:

(i) If $p>0$,

$$\left|f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|$$

$$\leq\left(b^p-a^p\right)\left[\begin{array}{c}\left(C_5(1,p)\right)^{1-\frac{1}{q}}\left[C_6(1,p)\left|f'(a)\right|^q+C_7(1,p)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\\+\left(C_8(1,p)\right)^{1-\frac{1}{q}}\left[C_9(1,p)\left|f'(a)\right|^q+C_{10}(1,p)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\end{array}\right],$$

(ii) If $p<0$,

$$\left|f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|$$

$$\leq\left(a^p-b^p\right)\left[\begin{array}{c}\left(-C_5(1,p)\right)^{1-\frac{1}{q}}\left[-C_6(1,p)\left|f'(a)\right|^q-C_7(1,p)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\\+\left(-C_8(1,p)\right)^{1-\frac{1}{q}}\left[-C_9(1,p)\left|f'(a)\right|^q-C_{10}(1,p)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\end{array}\right].$$

Theorem 2.4. *Let $f:I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ and $a<b$. If $|f'|^q$, $q>1$, is p-convex function on $[a,b]$ for $p\in\mathbb{R}\backslash\{0\}$, $\alpha>0$, $\frac{1}{q}+\frac{1}{r}=1$, then the following inequality for fractional integrals holds:*

(i) If $p>0$,

$$\left|f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}\left(b^p-a^p\right)^{\alpha}}\left[J^{\alpha}_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^{\alpha}_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right]\right|$$

$$\leq \frac{b^p-a^p}{2^{1-\alpha}}\left[\left(C_{11}(\alpha,p,r)\right)^{\frac{1}{r}}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}+\left(C_{12}(\alpha,p,r)\right)^{\frac{1}{r}}\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right]$$

where

$$C_{11}(\alpha,p,r)=\int_0^{\frac{1}{2}}\left(\frac{u^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}\right)^r du,\quad C_{12}(\alpha,p,r)=\int_{\frac{1}{2}}^1\left(\frac{(1-u)^\alpha}{p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}\right)^r du,$$

with $g(x)=x^{1/p}$, $x\in[a^p,b^p]$,

(ii) If $p<0$,

$$\left|f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(a^p-b^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(a^p)+J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(b^p)\right]\right|$$

$$\leq \frac{a^p-b^p}{2^{1-\alpha}}\left[\left(C_{13}(\alpha,p,r)\right)^{\frac{1}{r}}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}+\left(C_{14}(\alpha,p,r)\right)^{\frac{1}{r}}\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right]$$

where

$$C_{13}(\alpha,p,r)=\int_0^{\frac{1}{2}}\left(\frac{u^\alpha}{-p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}\right)^r du,\quad C_{14}(\alpha,p,r)=\int_{\frac{1}{2}}^1\left(\frac{(1-u)^\alpha}{-p\left[ua^p+(1-u)b^p\right]^{1-(1/p)}}\right)^r du,$$

with $g(x)=x^{1/p}$, $x\in[b^p,a^p]$.

Proof. (i) Let $p>0$. Using (16), Hölder's inequality and the p-convexity of $|f'|^q$it follows that

$$\left|f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b^p-a^p)^\alpha}\left[J^\alpha_{\frac{a^p+b^p}{2}+}(f\circ g)(b^p)+J^\alpha_{\frac{a^p+b^p}{2}-}(f\circ g)(a^p)\right]\right|$$

$$\leq \frac{b^p-a^p}{2^{1-\alpha}}\left[\begin{array}{c}\int_0^{\frac{1}{2}}\frac{u^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\left|f'\left([ua^p+(1-u)b^p]^{1/p}\right)\right|du\\+\int_{\frac{1}{2}}^1\frac{(1-u)^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\left|f'\left([ua^p+(1-u)b^p]^{1/p}\right)\right|du\end{array}\right]$$

$$\leq \frac{b^p-a^p}{2^{1-\alpha}}\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left(\frac{u^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\right)^r du\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{2}}\left|f'\left([ua^p+(1-u)b^p]^{1/p}\right)\right|^q du\right)^{\frac{1}{q}}\\+\left(\int_{\frac{1}{2}}^1\left(\frac{(1-u)^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\right)^r\right)^{\frac{1}{r}}\left(\int_{\frac{1}{2}}^1\left|f'\left([ua^p+(1-u)b^p]^{1/p}\right)\right|^q du\right)^{\frac{1}{q}}\end{array}\right]$$

$$\leq \frac{b^p-a^p}{2^{1-\alpha}}\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left(\frac{u^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\right)^r du\right)^{\frac{1}{r}}\left(\int_0^{\frac{1}{2}}u|f'(a)|^q+(1-u)|f'(b)|^q du\right)^{\frac{1}{q}}\\+\left(\int_{\frac{1}{2}}^1\left(\frac{(1-u)^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\right)^r du\right)^{\frac{1}{r}}\left(\int_{\frac{1}{2}}^1 u|f'(a)|^q+(1-u)|f'(b)|^q du\right)^{\frac{1}{q}}\end{array}\right]$$

$$=\frac{b^p-a^p}{2^{1-\alpha}}\left[\begin{array}{c}\left(\int_0^{\frac{1}{2}}\left(\frac{u^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\right)^r du\right)^{\frac{1}{r}}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}\\+\left(\int_{\frac{1}{2}}^1\left(\frac{(1-u)^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\right)^r du\right)^{\frac{1}{r}}\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}\end{array}\right]$$

$$=\frac{b^p-a^p}{2^{1-\alpha}}\left[\left(C_{11}(\alpha,p,r)\right)^{\frac{1}{r}}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}+\left(C_{12}(\alpha,p,r)\right)^{\frac{1}{r}}\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}\right].$$

This completes the proof of i.

(ii) The proof is similar with i. □

Corollary 2.4. *In Theorem 2.4, one can see the following.*

(1) If one takes $p=1$, one has the following Hermite-Hadamard inequality for convex functions via fractional integrals:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}(b-a)^{\alpha}}\left[J^{\alpha}_{\frac{a+b}{2}+}f(b)+J^{\alpha}_{\frac{a+b}{2}-}f(a)\right]\right|\leq\frac{b-a}{4(\alpha r+1)^{\frac{1}{r}}}\left[\begin{array}{c}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\\+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\end{array}\right],$$

(2) If one takes $p=1$ and $\alpha=1$, one has the following Hermite-Hadamard inequality for convex functions:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,dx\right|\leq\frac{b-a}{4(r+1)^{\frac{1}{r}}}\left[\begin{array}{c}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\\+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\end{array}\right],$$

(3) If one takes $p=-1$, one has the following Hermite-Hadamard inequality for harmonically convex functions via fractional integrals:

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{\Gamma(\alpha+1)}{2^{1-\alpha}}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{\frac{a+b}{2ab}+}(f\circ g)(1/a)+J^{\alpha}_{\frac{a+b}{2ab}-}(f\circ g)(1/b)\right]\right|$$
$$\leq\frac{1}{4(\alpha r+1)^{\frac{1}{r}}}\left(\frac{b-a}{ab}\right)\left[\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right],$$

(4) If one takes $p=-1$ and $\alpha=1$, one has the following Hermite-Hadamard inequality for harmonically convex functions:

$$\left|f\left(\frac{2ab}{a+b}\right)-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|\leq\frac{1}{4(r+1)^{\frac{1}{r}}}\left(\frac{b-a}{ab}\right)\left[\begin{array}{c}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\\+\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{\frac{1}{q}}\end{array}\right],$$

(5) If one takes $\alpha=1$, one has the following Hermite-Hadamard inequalities for p-convex functions:

(i) If $p>0$,

$$\left|f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|\leq(b^p-a^p)\left[\begin{array}{c}(C_{11}(1,p,r))^{\frac{1}{r}}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}\\+(C_{12}(1,p,r))^{\frac{1}{r}}\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}\end{array}\right],$$

(ii) If $p<0$,

$$\left|f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\right|\leq(a^p-b^p)\left[\begin{array}{c}(C_{13}(1,p,r))^{\frac{1}{r}}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{\frac{1}{q}}\\+(C_{14}(1,p,r))^{\frac{1}{r}}\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{\frac{1}{q}}\end{array}\right].$$

Conclusion. In Theorem 2.1, new Hermite-Hadamard type inequalities for p-convex functions in fractional integral forms are built. In Lemma 2.1, an integral identity and in Theorem 2.2, Theorem 2.3 and Theorem 2.4, some new Hermite-Hadamard type integral inequalities for p-convex functions in fractional integral forms are obtained. In Corollary 2.2, Corollary 2.3 and Corollary 2.4, some new Hermite-Hadamard type inequalities for convex, harmonically convex and p-convex functions are given. Some results presented Remark 2.1, Remark 2.2 and Remark 2.3, provide extensions of others given in earlier works for convex, harmonically convex and p-convex functions.

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