Hermite-Hadamard type integral inequalities for products of two generalized \((s, m, \xi)\)-preinvex functions

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Abstract. In this paper, the notion of generalized \((s, m, \xi)\)-preinvex function is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving generalized \((s, m, \xi)\)-preinvex functions along with beta function are given. Moreover, we establish some new Hermite-Hadamard type integral inequalities for products of two generalized \((s, m, \xi)\)-preinvex functions via classical and Riemann-Liouville fractional integrals. These results not only extend the results appeared in the literature (see [10],[11]), but also provide new estimates on these types. At the end, some conclusions are given.

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1. Introduction and Preliminaries

The following notations are used throughout this paper. We use \(I\) to denote an interval on the real line \(\mathbb{R} = (-\infty, +\infty)\) and \(I^\circ\) to denote the interior of \(I\). For any subset \(K \subseteq \mathbb{R}^n\), \(K^\circ\) is used to denote the interior of \(K\). \(\mathbb{R}^n\) is used to denote a generic \(n\)-dimensional vector space. The nonnegative real numbers are denoted by \(\mathbb{R}_\circ = [0, +\infty)\). The set of integrable functions on the interval \([a, b]\) is denoted by \(L_1([a, b])\).
The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

**Theorem 1.1.** Let \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) be a convex function on an interval \( I \) of real numbers and \( a, b \in I \) with \( a < b \). Then the following inequality holds:

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}.
\]

In recent years, various generalizations, extensions and variants of such inequalities have been obtained (see [24]-[31]). For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [15]-[20],[23]) and the references cited therein.

In (see [10]), B. G. Pachpatte established two new Hermite-Hadamard type inequalities for products of convex functions as follows.

**Theorem 1.2.** Let \( f \) and \( g \) be real-valued, nonnegative and convex functions on \([a, b]\). Then

\[
\frac{1}{b-a} \int_a^b f(x)g(x) \, dx \leq \frac{1}{3} M(a, b) + \frac{1}{6} N(a, b),
\]

\[
2f \left( \frac{a+b}{2} \right) g \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b f(x)g(x) \, dx + \frac{1}{6} M(a, b) + \frac{1}{3} N(a, b),
\]

where \( M(a, b) = f(a)g(a) + f(b)g(b) \) and \( N(a, b) = f(a)g(a) + f(b)g(b) \).

Some new integral inequalities involving two nonnegative and integrable functions that are related to the Hermite-Hadamard type are also obtained by many authors (see [6]-[8],[12]). Fractional calculus was introduced by Liouville and Riemann (see [23]) and some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [5],[23]).

**Definition 1.1.** Let \( f \in L_1[a, b] \). The Riemann-Liouville integrals \( J^\alpha_{a+}f \) and \( J^\alpha_{b-}f \) of order \( \alpha > 0 \) with \( a \geq 0 \) are defined by

\[
J^\alpha_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt, \quad x > a
\]

and

\[
J^\alpha_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \, dt, \quad b > x,
\]

where \( \Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} \, du \). Here \( J^\alpha_{a+}f(x) = J^\alpha_{b-}f(x) = f(x) \).

In the case of \( \alpha = 1 \), the fractional integral reduces to the classical integral.

**Theorem 1.3.** (see [11]) Let \( f \) and \( g \) be real-valued, nonnegative and convex functions on \([a, b]\). Then

\[
\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[ J^\alpha_{a+}f(b)g(b) + J^\alpha_{b-}f(a)g(a) \right] \\
\leq \left( \frac{\alpha}{\alpha + 2} - \frac{\alpha}{\alpha + 1} + \frac{1}{2} \right) M(a, b) + \frac{\alpha}{(\alpha + 1)(\alpha + 2)} N(a, b),
\]

where \( M(a, b) = f(a)g(a) + f(b)g(b) \) and \( N(a, b) = f(a)g(b) + f(b)g(a) \).
Definition 1.2. (see [2]) A nonnegative function \( f : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) is said to be \( P \)-function or \( P \)-convex, if
\[
f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, \quad t \in [0,1].
\]

Definition 1.3. (see [3]) A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is said to be \( s \)-convex in the second sense, if
\[
f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)
\]
for all \( x, y \in \mathbb{R} \), \( \lambda \in [0,1] \) and \( s \in (0,1] \).

The \( s \)-convex functions in the second sense have been investigated in (see [3]).

Definition 1.4. (see [4]) A set \( K \subseteq \mathbb{R}^n \) is said to be \( \mathcal{m} \)-invex with respect to the mapping \( \eta : K \times K \rightarrow \mathbb{R}^n \), if \( mx + t\eta(y,x) \in K \) for every \( x, y \in K \) and \( t \in [0,1] \).

Every convex set is \( \mathcal{m} \)-invex with respect to the mapping \( \eta(y,x) = y - x \), but the converse is not necessarily true. For more details (see [4],[9]) and the references therein.

Definition 1.5. (see [13]) The function \( f \) defined on the \( \mathcal{m} \)-invex set \( K \subseteq \mathbb{R}^n \) is said to be \( \mathcal{m} \)-preinvex with respect \( \eta \), if for every \( x, y \in K \) and \( t \in [0,1] \), we have that
\[
f(x + t\eta(y,x)) \leq (1-t)f(x) + tf(y).
\]
The concept of \( \mathcal{m} \)-preinvexity is more general than convexity since every convex function is \( \mathcal{m} \)-preinvex with respect to the mapping \( \eta(y,x) = y - x \), but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following
\[
\int_a^b (x-a)^p(b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k}f(\gamma_k) + R_m^s|f|,
\]
for certain \( B_{m,k}, \gamma_k \) and rest \( R_m^s|f| \) (see [14]).

Recently, Liu (see [21]) obtained several integral inequalities for the left-hand side of (6) under the Definition 1.2 of \( P \)-function.

Also in (see [22]), Özdemir et al. established several integral inequalities concerning the left-hand side of (6) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of generalized \( (s,m,\xi) \)-preinvex function is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula (6) involving generalized \( (s,m,\xi) \)-preinvex functions along with beta function are given. In Section 3, some generalizations of Hermite-Hadamard type integral inequalities for products of two generalized \( (s,m,\xi) \)-preinvex functions via classical and Riemann-Liouville fractional integrals are given. These results not only extend the results appeared in the literature (see [10],[11]), but also provide new estimates on these types. In Section 4, some conclusions and future research are given.

2. New integral inequalities for generalized \( (s,m,\xi) \)-preinvex functions

Definition 2.1. (see [1]) A set \( K \subseteq \mathbb{R}^n \) is said to be \( \mathcal{m} \)-invex with respect to the mapping \( \eta : K \times K \times (0,1] \rightarrow \mathbb{R}^n \) for some fixed \( m \in (0,1] \), if \( mx + t\eta(y,x,m) \in K \) holds for each \( x, y \in K \) and any \( t \in [0,1] \).
Remark 2.1. In Definition 2.1, under certain conditions, the mapping \( \eta(y, x, m) \) could reduce to \( \eta(y, x) \). For example when \( m = 1 \), then the \( m \)-invex set degenerates an invex set on \( K \).

We are ready to introduce the new definition called generalized \( (s, m, \xi) \)-preinvex function.

**Definition 2.2.** Let \( K \subseteq \mathbb{R}^n \) be an open \( m \)-invex set with respect to \( \eta : K \times K \times (0, 1) \rightarrow \mathbb{R}^n \) and \( \xi : I \rightarrow K \) a continuous function. For \( f : K \rightarrow \mathbb{R} \) and any fixed \( s \in (0, 1] \), if

\[
 f(\Lambda m\xi(x) + \lambda \eta(\xi(y), \xi(x), m)) \leq m(1 - \lambda)^s f(\xi(x)) + \lambda^s f(\xi(y))
\]

is valid for all \( x, y \in I, \lambda \in [0, 1] \), then we say that \( f(x) \) is a generalized \( (s, m, \xi) \)-preinvex function with respect to \( \eta \).

Remark 2.2. It is worthwhile to note that the class of generalized \( (s, m, \xi) \)-preinvex function is a generalization of the class of \( s \)-convex in the second sense.

In this section, in order to prove our main results regarding some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula (6) involving generalized \( (s, m, \xi) \)-preinvex functions along with beta function, we need the following new interesting Lemma:

**Lemma 2.1.** Let \( \xi : I \rightarrow K \) be a continuous function. Assume that \( f : K = [m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)] \rightarrow \mathbb{R} \) is a continuous function on the interval of real numbers \( K^o \) with respect to \( \eta : K \times K \times (0, 1] \rightarrow \mathbb{R} \) and \( m\xi(a) < m\xi(a) + \eta(\xi(b), \xi(a), m) \). Then for any fixed \( p, q > 0 \), we have

\[
\int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} (x - m\xi(a))^p (m\xi(a) + \eta(\xi(b), \xi(a), m) - x)^q f(x)dx
\]

\[
= \eta(\xi(b), \xi(a), m)^{p+q+1} \int_0^1 t^p (1 - t)^q f(m\xi(a) + t\eta(\xi(b), \xi(a), m))dt.
\]

**Proof.**

\[
\int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} (x - m\xi(a))^p (m\xi(a) + \eta(\xi(b), \xi(a), m) - x)^q f(x)dx
\]

\[
= \eta(\xi(b), \xi(a), m) \int_0^1 (m\xi(a) + t\eta(\xi(b), \xi(a), m) - m\xi(a))^p
\]

\[
\times (m\xi(a) + \eta(\xi(b), \xi(a), m) - m\xi(a) - t\eta(\xi(b), \xi(a), m))^q f(m\xi(a) + t\eta(\xi(b), \xi(a), m))dt
\]

\[
= \eta(\xi(b), \xi(a), m)^{p+q+1} \int_0^1 t^p (1 - t)^q f(m\xi(a) + t\eta(\xi(b), \xi(a), m))dt.
\]

The following definition will be used in the sequel.

**Definition 2.3.** The Euler Beta function is defined for \( x, y > 0 \) as

\[
\beta(x, y) = \int_0^1 t^{x-1} (1 - t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x + y)}.
\]
Theorem 2.1. Let \( \xi : I \to K \) be a continuous function. Assume that \( f : K = [m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)] \to \mathbb{R} \) is a continuous function on the interval of real numbers \( K^o \) with \( m\xi(a) < m\xi(a) + \eta(\xi(b), \xi(a), m) \). If \( k > 1 \) and \( |f|^{\frac{k}{k-1}} \) is a generalized \((s, m, \xi)\)-preinvex function on an open \( m\)-invekt set \( K \) with respect to \( \eta : K \times K \times (0, 1] \to \mathbb{R} \) for any fixed \( s \in (0, 1] \), then for any fixed \( p, q > 0 \),

\[
\int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} (x - m\xi(a))^p (m\xi(a) + \eta(\xi(b), \xi(a), m) - x)^q f(x) dx \\
\leq \frac{|\eta(\xi(b), \xi(a), m)|^{p+q+1}}{(s + 1)^\frac{1}{k-1}} \left[ \int_0^1 t^k (1 - t)^{kq + 1} dt \right]^\frac{1}{k} \\
\times \left[ \int_0^1 |f(m\xi(a) + t\eta(\xi(b), \xi(a), m))|^{\frac{k}{k-1}} dt \right]^{\frac{1}{k-1}} \\
\leq \frac{|\eta(\xi(b), \xi(a), m)|^{p+q+1}}{(s + 1)^\frac{1}{k-1}} \left[ \beta(kp + 1, kq + 1) \right]^\frac{1}{k} \\
\times \left[ \int_0^1 (m(1 - t)^s |f(\xi(a))|^{\frac{k}{k-1}} + t^s |f(\xi(b))|^{\frac{k}{k-1}}) dt \right]^{\frac{1}{k-1}} \\
= \frac{|\eta(\xi(b), \xi(a), m)|^{p+q+1}}{(s + 1)^\frac{1}{k-1}} \left[ \beta(kp + 1, kq + 1) \right]^\frac{1}{k} \left( m|f(\xi(a))|^{\frac{k}{k-1}} + |f(\xi(b))|^{\frac{k}{k-1}} \right)^{\frac{1}{k-1}}.
\]

The proof of Theorem 2.1 is completed.

Theorem 2.2. Let \( \xi : I \to K \) be a continuous function. Assume that \( f : K = [m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)] \to \mathbb{R} \) is a continuous function on the interval of real numbers \( K^o \) with \( m\xi(a) < m\xi(a) + \eta(\xi(b), \xi(a), m) \). If \( l \geq 1 \) and \( |f|^{\frac{l}{l-1}} \) is a generalized \((s, m, \xi)\)-preinvex function on an open \( m\)-invekt set \( K \) with respect to \( \eta : K \times K \times (0, 1] \to \mathbb{R} \) for any fixed \( s \in (0, 1] \), then for any fixed \( p, q > 0 \),

\[
\int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} (x - m\xi(a))^p (m\xi(a) + \eta(\xi(b), \xi(a), m) - x)^q f(x) dx \\
\leq |\eta(\xi(b), \xi(a), m)|^{p+q+1} \left[ \beta(p + 1, q + 1) \right]^\frac{1}{l-1} \\
\times \left[ m|f(\xi(a))|^{\frac{l}{l-1}} \beta(p + 1, q + s + 1) + |f(\xi(b))|^{\frac{l}{l-1}} \beta(p + s + 1, q + 1) \right]^\frac{1}{l}.
\]
Proof. Since $|f|^l$ is a generalized $(s,m,\xi)$-preinvex function on $K$, combining with Lemma 2.1, Definition 2.3 and the well-known power mean inequality for all $t \in [0,1]$ and for any fixed $s,m \in (0,1]$, we get

$$
\int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b),\xi(a),m)} (x-m\xi(a))^p (m\xi(a)+\eta(\xi(b),\xi(a),m) - x)^q f(x)dx
$$

$$
= \eta(\xi(b),\xi(a),m)^{p+q+1} \int_0^1 \left[ t^p (1-t)^q \right]^{\frac{1}{p+1}} \left[ t^p (1-t)^q \right]^{\frac{1}{q+1}} f(m\xi(a) + \eta(\xi(b),\xi(a),m))dt
$$

$$
\leq |\eta(\xi(b),\xi(a),m)|^{p+q+1} \left[ \int_0^1 t^p (1-t)^q dt \right]^{\frac{1}{p+1}}
$$

$$
\times \left[ \int_0^1 t^p (1-t)^q f(m\xi(a) + \eta(\xi(b),\xi(a),m))dt \right]^{\frac{1}{q+1}}
$$

$$
\leq |\eta(\xi(b),\xi(a),m)|^{p+q+1} \left[ \beta(p+1,q+1) \right]^{\frac{1}{p+1}}
$$

$$
\times \left[ \int_0^1 t^p (1-t)^q \left( m(1-t)^s |f(\xi(a))|^l + t^s |f(\xi(b))|^l \right) dt \right]^{\frac{1}{q+1}}
$$

$$
= |\eta(\xi(b),\xi(a),m)|^{p+q+1} \left[ \beta(p+1,q+1) \right]^{\frac{1}{p+1}}
$$

$$
\times \left[ m |f(\xi(a))|^l \beta(p+1,q+s+1) + |f(\xi(b))|^l \beta(p+s+1,q+1) \right]^{\frac{1}{q+1}}.
$$

The proof of Theorem 2.2 is completed.

Remark 2.3. Clearly, if we replace $f(x)$ with the product of two generalized $(s,m,\xi)$-preinvex functions $g(x)$ and $h(x)$ in Theorems 2.1 and 2.2, we get some new interesting inequalities for evaluation integrals of the following form

$$
\int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b),\xi(a),m)} (x-m\xi(a))^p (m\xi(a)+\eta(\xi(b),\xi(a),m) - x)^q g(x)h(x)dx.
$$

The details are left to the interested reader.

3. Hermite-Hadamard type inequalities for products of two generalized $(s,m,\xi)$-preinvex functions

In this section, some generalizations of Hermite-Hadamard type inequalities for products of two generalized $(s,m,\xi)$-preinvex functions via classical and Riemann-Liouville fractional integrals are given.

Theorem 3.1. Let $\xi : I \to A$ be a continuous function. Suppose $A \subseteq \mathbb{R}_+$ be an open $m$-invex subset with respect to $\eta : A \times A \times (0,1) \to \mathbb{R}_+$ for any fixed $s \in (0,1]$ and let $m\xi(a) < m\xi(a) + \eta(\xi(b),\xi(a),m)$. Assume that $f,g : A \to \mathbb{R}_+$ are generalized $(s,m,\xi)$-preinvex functions on $[m\xi(a), m\xi(a)+\eta(\xi(b),\xi(a),m)]$. Then

$$
\frac{1}{\eta(\xi(b),\xi(a),m)} \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b),\xi(a),m)} f(x)g(x)dx
$$
Theorem 3.2. Since changing the variable and for products of two generalized \(\xi\)-preinvex functions, then we get inequality (2). Moreover, the details for finding the analogous of inequality (3) are as follows:

\[
P(m, a, b) = m^2 f(\xi(a))g(\xi(a)) + f(\xi(b))g(\xi(b)),
\]
and

\[
Q(a, b) = f(\xi(a))g(\xi(b)) + f(\xi(b))g(\xi(a)).
\]

Proof. Since \(f\) and \(g\) are generalized \((s, m, \xi)\)-preinvex functions, then for all \(t \in [0, 1]\) we get

\[
f(m\xi(a) + t\eta(\xi(b), \xi(a), m)) \leq m(1 - t)^s f(\xi(a)) + t^s f(\xi(b)),
\]
and

\[
g(m\xi(a) + t\eta(\xi(b), \xi(a), m)) \leq m(1 - t)^s g(\xi(a)) + t^s g(\xi(b)).
\]

Changing the variable \(x = m\xi(a) + t\eta(\xi(b), \xi(a), m)\) and using inequalities (9) and (10), we obtain

\[
\frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} f(x)g(x)dx
\]

\[
= \int_0^1 f(m\xi(a) + t\eta(\xi(b), \xi(a), m)) g(m\xi(a) + t\eta(\xi(b), \xi(a), m)) dt
\]

\[
\leq \int_0^1 [m(1 - t)^s f(\xi(a)) + t^s f(\xi(b))] [m(1 - t)^s g(\xi(a)) + t^s g(\xi(b))] dt
\]

\[
= \frac{P(m, a, b)}{2s + 1} + m\beta(s + 1, s + 1)Q(a, b).
\]

The proof of Theorem 3.1 is completed.

Remark 3.1. For \(m = s = 1\), \(\xi(x) = x\), \(\forall x \in I\) and \(\eta(b, a, 1) = b - a\) in Theorem 3.1, then we get inequality (2). Moreover, the details for finding the analogous of inequality (3) for products of two generalized \((s, m, \xi)\)-preinvex functions is left to the interested reader.

Corollary 3.1. Under the same conditions as in Theorem 3.1 for \(f(x) = g(x)\), we have

\[
\frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} f^2(x)dx
\]

\[
\leq \frac{P_1(m, a, b)}{2s + 1} + m\beta(s + 1, s + 1)Q_1(a, b),
\]

where

\[
P_1(m, a, b) = m^2 f^2(\xi(a)) + f^2(\xi(b)),
\]
and

\[
Q_1(a, b) = 2f(\xi(a))f(\xi(b)).
\]

Theorem 3.2. Let \(\xi : I \rightarrow A\) be a continuous function. Suppose \(A \subseteq \mathbb{R}\) be an open \(m\)-invex subset with respect to \(\eta : A \times A \times [0, 1] \rightarrow \mathbb{R}\) for any fixed \(s \in (0, 1]\) and let \(m\xi(a) < m\xi(a) + \eta(\xi(b), \xi(a), m)\). Assume that \(f, g : A \rightarrow \mathbb{R}\). If \(|f|^p, |g|^q\) are integrable and generalized \((s, m, \xi)\)-preinvex functions on \([m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)\), then for \(q > 1\) and \(p^{-1} + q^{-1} = 1\), we have

\[
\frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} |f(x)g(x)|dx
\]
Changing the variable $x = mξ(a) + tnξ(b), ξ(a), m)$, using Hölder integral inequality and inequalities (13), (14), we obtain

$$\int_{mξ(a)}^{mξ(a)+ηξ(b),ξ(a),m} |f(x)g(x)|dx \leq \left( \int_{mξ(a)}^{mξ(a)+ηξ(b),ξ(a),m} |f(x)|^pdx \right)^{\frac{1}{p}} \left( \int_{mξ(a)}^{mξ(a)+ηξ(b),ξ(a),m} |g(x)|^qdx \right)^{\frac{1}{q}}$$

$$\leq \eta^{\frac{1}{p}}(ξ(b), ξ(a), m) \left( \int_0^1 |f(mξ(a) + tηξ(b), ξ(a), m)|^pdt \right)^{\frac{1}{p}} \times η^\frac{1}{q}(ξ(b), ξ(a), m) \left( \int_0^1 |g(mξ(a) + tηξ(b), ξ(a), m)|^qdt \right)^{\frac{1}{q}}$$

$$\leq η^{\frac{1}{p}}(ξ(b), ξ(a), m) \left( \int_0^1 [m(1 - t)^s|f(ξ(a))|^p + t^s|f(ξ(b))|^p]dt \right)^{\frac{1}{p}} \times η^\frac{1}{q}(ξ(b), ξ(a), m) \left( \int_0^1 [m(1 - t)^s|g(ξ(a))|^q + t^s|g(ξ(b))|^q]dt \right)^{\frac{1}{q}}$$

$$= \left( η(ξ(b), ξ(a), m) \right)^{\frac{1}{p}} \left[ m|f(ξ(a))|^p + |f(ξ(b))|^p \right]^\frac{1}{p} \times \left( η(ξ(b), ξ(a), m) \right)^{\frac{1}{q}} \left[ m|g(ξ(a))|^q + |g(ξ(b))|^q \right]^\frac{1}{q}.$$
Corollary 3.3. Under the same conditions as in Corollary 3.2 for $p = q = 2$, we have
\[
\frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} f^2(x) dx \\
\leq \frac{m|f(\xi(a))|^2 + |f(\xi(b))|^2}{s + 1}.
\] (16)

Theorem 3.3. Let $\xi : I \rightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}_0$ be an open $m$-invex subset with respect to $\eta : A \times A \times (0, 1) \rightarrow \mathbb{R}_0$ for any fixed $s \in (0, 1]$ and let $m\xi(a) < m\xi(a) + \eta(\xi(b), \xi(a), m)$. Assume that $f, g : A \rightarrow \mathbb{R}_0$. If $f^p, g^p, fg$ are integrable and $f^p, g^p$ are generalized $(s, m, \xi)$-preinvex functions on $[m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)]$, where
\[
0 < m_1 \leq \frac{f(x)}{g(x)} \leq M_1, \forall x \in [m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)],
\] (17)
then for $p > 1$ and $p^{-1} + q^{-1} = 1$, we have the following inequality
\[
\frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} f(x)g(x) dx \\
\leq \frac{c_1}{2(s + 1)} \left[ m (f^p(\xi(a)) + g^p(\xi(a))) + f^p(\xi(b)) + g^p(\xi(b)) \right] \\
+ \frac{c_2}{2(s + 1)} \left[ m (f^q(\xi(a)) + g^q(\xi(a))) + f^q(\xi(b)) + g^q(\xi(b)) \right],
\] (18)
where
\[
c_1 = \frac{2^p}{p} \left( \frac{M_1}{M_1 + 1} \right)^p, \quad c_2 = \frac{2^q}{q} \left( \frac{1}{m_1 + 1} \right)^q.
\] (19)

Proof. To prove our theorem we need the following Young’s inequality:
\[
xy \leq \frac{x^p}{p} + \frac{y^q}{q}, \forall x, y \geq 0, p > 1, p^{-1} + q^{-1} = 1.
\] (20)

From (17), we have
\[
f(x) \leq \frac{M_1}{M_1 + 1} (f(x) + g(x)), \quad g(x) \leq \frac{1}{m_1 + 1} (f(x) + g(x)).
\] (21)

From (20) and (21), we have
\[
\int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} f(x)g(x) dx \\
\leq \frac{1}{p} \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} f^p(x) dx + \frac{1}{q} \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} g^q(x) dx \\
\leq \frac{1}{p} \left( \frac{M_1}{M_1 + 1} \right)^p \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} (f(x) + g(x))^p dx \\
+ \frac{1}{q} \left( \frac{1}{m_1 + 1} \right)^q \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} (f(x) + g(x))^q dx.
\] (22)
Using the elementary inequality \((c + d)^p \leq 2^{p-1}(c^p + d^p)\), \((p > 1\) and \(c,d \in \mathbb{R}_+\) in (22) and the fact that \(f^p, g^p\) are generalized \((s, m, \xi, \eta)\)-preinvex functions on \([m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)]\), we get

\[
\int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} f(x)g(x)dx \\
\leq \frac{1}{p} \left(\frac{M_1}{M_1 + 1}\right)^p 2^{p-1} \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} \left[f^p(x) + g^p(x)\right]dx \\
+ \frac{1}{q} \left(\frac{1}{m_1 + 1}\right)^q 2^{q-1} \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} \left[f^q(x) + g^q(x)\right]dx \\
= \frac{c_1}{2} \eta(\xi(b), \xi(a), m) \\
\times \left[\int_0^1 f^p (m\xi(a) + t\eta(\xi(b), \xi(a), m)) \, dt + \int_0^1 g^p (m\xi(a) + t\eta(\xi(b), \xi(a), m)) \, dt\right] \\
+ \frac{c_2}{2} \eta(\xi(b), \xi(a), m) \\
\times \left[\int_0^1 f^q (m\xi(a) + t\eta(\xi(b), \xi(a), m)) \, dt + \int_0^1 g^q (m\xi(a) + t\eta(\xi(b), \xi(a), m)) \, dt\right] \\
\leq \frac{c_1}{2} \eta(\xi(b), \xi(a), m) \\
\times \left[\int_0^1 (m(1 - t)^s f^p(\xi(a)) + t^s f^p(\xi(b))) \, dt + \int_0^1 (m(1 - t)^s g^p(\xi(a)) + t^s g^p(\xi(b))) \, dt\right] \\
+ \frac{c_2}{2} \eta(\xi(b), \xi(a), m) \\
\times \left[\int_0^1 (m(1 - t)^s f^q(\xi(a)) + t^s f^q(\xi(b))) \, dt + \int_0^1 (m(1 - t)^s g^q(\xi(a)) + t^s g^q(\xi(b))) \, dt\right] \\
= \frac{c_1 \eta(\xi(b), \xi(a), m)}{2(s + 1)} \left[m (f^p(\xi(a)) + g^p(\xi(a))) + f^p(\xi(b)) + g^p(\xi(b))\right] \\
+ \frac{c_2 \eta(\xi(b), \xi(a), m)}{2(s + 1)} \left[m (f^q(\xi(a)) + g^q(\xi(a))) + f^q(\xi(b)) + g^q(\xi(b))\right].
\]

The proof of Theorem 3.3 is completed.

**Corollary 3.4.** Under the same conditions as in Theorem 3.3 for \(f(x) = g(x)\), we have

\[
\frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} f^2(x)dx \\
\leq \frac{c_1}{s + 1} \left[m f^p(\xi(a)) + f^p(\xi(b))\right] \\
+ \frac{c_2}{s + 1} \left[m f^q(\xi(a)) + f^q(\xi(b))\right].
\] (23)
Corollary 3.5. Under the same conditions as in Corollary 3.4 for \( p = q = 2 \), we have

\[
\frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(\alpha)+\eta(\xi(b), \xi(a), m)} f^2(x) \, dx
\leq \frac{2}{s+1} \left[ \left( \frac{M_1}{M_1 + 1} \right)^2 + \left( \frac{1}{m_1 + 1} \right)^2 \right] \left[ mf^2(\xi(a)) + f^2(\xi(b)) \right].
\]

(24)

In the following theorem we give Hermite-Hadamard type integral inequality for products of two generalized \((s, m, \xi)\)-preinvex functions via Riemann-Liouville fractional integrals.

Theorem 3.4. Let \( \xi : I \rightarrow A \) be a continuous function. Suppose \( A \subseteq \mathbb{R}_0 \) be an open \( m \)-invex subset with respect to \( \eta : A \times A \times (0, 1] \rightarrow \mathbb{R}_0 \) for any fixed \( s \in (0, 1] \) and let \( m(1) < m(\xi(a)) + \eta(\xi(b), \xi(a), m) \) and \( m(1) < m(\xi(b)) + \eta(\xi(a), \xi(b), m) \). Assume that \( f, g : A \rightarrow \mathbb{R}_0 \) are generalized \((s, m, \xi)\)-preinvex functions on \( [m(\xi(a)), m(\xi(a)) + \eta(\xi(b), \xi(a), m)] \) and \( [m(\xi(b)), m(\xi(b)) + \eta(\xi(a), \xi(b), m)] \). Then for \( \alpha > 0 \), we have

\[
\frac{\Gamma(\alpha)}{\eta^\alpha(\xi(b), \xi(a), m)} \int_{\eta^\alpha(\xi(b), \xi(a), m)}^{\eta^\alpha(\xi(b), \xi(a), m) + \eta(\xi(b), \xi(a), m) - f(m(\xi(a)))g(m(\xi(a)))} \, dx
\]

\[
+ \frac{\Gamma(\alpha)}{\eta^\alpha(\xi(a), \xi(b), m)} \int_{\eta^\alpha(\xi(a), \xi(b), m)}^{\eta^\alpha(\xi(a), \xi(b), m) + \eta(\xi(a), \xi(b), m) - f(m(\xi(b)))g(m(\xi(b)))} \, dx
\]

\[
\leq \left[ m^2 \beta(\alpha, 2s + 1) + \frac{1}{\alpha + 2s} \right] M(a, b) + m\beta(\alpha + s, s + 1) N(a, b),
\]

(25)

where

\[
M(a, b) = f(\xi(a))g(\xi(a)) + f(\xi(b))g(\xi(b)),
\]

and

\[
N(a, b) = f(\xi(a))g(\xi(b)) + f(\xi(b))g(\xi(a)).
\]

Proof. Since \( f \) and \( g \) are generalized \((s, m, \xi)\)-preinvex functions on \( [m(\xi(a)), m(\xi(a)) + \eta(\xi(b), \xi(a), m)] \), then for all \( t \in [0, 1] \) we get

\[
f(m(\xi(a)) + t\eta(\xi(b), \xi(a), m)) \leq m(1 - t)^s f(\xi(a)) + t^s f(\xi(b)),
\]

(26)

and

\[
g(m(\xi(a)) + t\eta(\xi(b), \xi(a), m)) \leq m(1 - t)^s g(\xi(a)) + t^s g(\xi(b)).
\]

(27)

Similarly, since \( f \) and \( g \) are generalized \((s, m, \xi)\)-preinvex functions on \( [m(\xi(b)), m(\xi(b)) + \eta(\xi(a), \xi(b), m)] \), then for all \( t \in [0, 1] \) we get

\[
f(m(\xi(b)) + t\eta(\xi(a), \xi(b), m)) \leq m(1 - t)^s f(\xi(b)) + t^s f(\xi(a)),
\]

(28)

and

\[
g(m(\xi(b)) + t\eta(\xi(a), \xi(b), m)) \leq m(1 - t)^s g(\xi(b)) + t^s g(\xi(a)).
\]

(29)

Now, we using the following steps for proof our theorem:

Changing the variable \( u = m(\xi(a)) + t\eta(\xi(b), \xi(a), m) \) and \( w = m(\xi(b)) + t\eta(\xi(a), \xi(b), m) \). Multiplying inequalities (26) with (27) and inequalities (28) with (29). After, multiplying both sides of inequalities obtained by \( t^{\alpha-1} \) and then integrating with respect to \( t \) over \([0, 1] \), we get

\[
\frac{\Gamma(\alpha)}{\eta^\alpha(\xi(b), \xi(a), m)} \int_{\eta^\alpha(\xi(b), \xi(a), m)}^{\eta^\alpha(\xi(b), \xi(a), m) + \eta(\xi(b), \xi(a), m) - f(m(\xi(a)))g(m(\xi(a)))} \, dx
\]

\[
+ \frac{\Gamma(\alpha)}{\eta^\alpha(\xi(a), \xi(b), m)} \int_{\eta^\alpha(\xi(a), \xi(b), m)}^{\eta^\alpha(\xi(a), \xi(b), m) + \eta(\xi(a), \xi(b), m) - f(m(\xi(b)))g(m(\xi(b)))} \, dx
\]

\[
\leq \left[ m^2 \beta(\alpha, 2s + 1) + \frac{1}{\alpha + 2s} \right] M(a, b) + m\beta(\alpha + s, s + 1) N(a, b),
\]
Under the same conditions as in Theorem 3.4 for Corollary 3.6.

Remark 3.2. For The proof of Theorem 3.4 is completed.

\[
\Gamma(\alpha) + \frac{\Gamma(\alpha)}{\eta^s(\xi(b), \xi(a), m)} J^{\alpha}_{m \xi(b) + \eta(\xi(a), \xi(b), m)} f(m \xi(b)) g(m \xi(b))
\]
\[
= \int_0^1 t^{\alpha-1} f(m \xi(a) + t \eta(\xi(b), \xi(a), m)) g(m \xi(a) + t \eta(\xi(b), \xi(a), m)) dt
\]
\[
+ \int_0^1 t^{\alpha-1} f(m \xi(b) + t \eta(\xi(a), \xi(b), m)) g(m \xi(b) + t \eta(\xi(a), \xi(b), m)) dt
\]
\[
\leq \int_0^1 t^{\alpha-1} [m(1-t)^s f(\xi(a)) + t^s f(\xi(b))] [m(1-t)^s g(\xi(a)) + t^s g(\xi(b))] dt
\]
\[
+ \int_0^1 t^{\alpha-1} [m(1-t)^s f(\xi(b)) + t^s f(\xi(a))] [m(1-t)^s g(\xi(b)) + t^s g(\xi(a))] dt
\]
\[
= \left[ m^2 \beta(\alpha, 2s + 1) + \frac{1}{\alpha + 2s} \right] M(a, b) + m \beta(\alpha + s, s + 1) N(a, b).
\]

The proof of Theorem 3.4 is completed.

Remark 3.2. For \( m = s = 1, \xi(x) = x, \forall x \in I \) and \( \eta(y, x, 1) = y - x \) in Theorem 3.4, then we get Theorem 1.3.

Corollary 3.6. Under the same conditions as in Theorem 3.4 for \( f(x) = g(x) \), we have

\[
\Gamma(\alpha) + \frac{\Gamma(\alpha)}{\eta^s(\xi(b), \xi(a), m)} J^{\alpha}_{m \xi(b) + \eta(\xi(a), \xi(b), m)} f^2(m \xi(a))
\]
\[
+ \frac{\Gamma(\alpha)}{\eta^s(\xi(a), \xi(b), m)} J^{\alpha}_{m \xi(a) + \eta(\xi(a), \xi(b), m)} f^2(m \xi(b))
\]
\[
\leq \left[ m^2 \beta(\alpha, 2s + 1) + \frac{1}{\alpha + 2s} \right] M_1(a, b) + m \beta(\alpha + s, s + 1) N_1(a, b),
\]

where

\[
M_1(a, b) = f^2(\xi(a)) + f^2(\xi(b)),
\]

and

\[
N_1(a, b) = 2f(\xi(a)) f(\xi(b)).
\]

4. Conclusions

In the present paper, we established some new Hermite-Hadamard type integral inequalities for products of two generalized \((s, m, \xi)\)-preinvex functions via classical and Riemann-Liouville fractional integrals.

Motivated by this new interesting class of generalized \((s, m, \xi)\)-preinvex functions we can indeed see to be vital for fellow researchers and scientists working in the same domain.

We conclude that our methods may be a stimulant for further investigations concerning Hermite-Hadamard type integral inequalities for products of various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, \(k\)-fractional integrals, local fractional integrals, fractional integral operators, \(q\)-calculus, \((p, q)\)-calculus, time scale calculus and conformable fractional integrals.
References


