Hermite-Hadamard type integral inequalities for products of two generalized (s, m, ξ) -preinvex functions

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ABSTRACT. In this paper, the notion of generalized (s, m, ξ) -preinvex function is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula involving generalized (s, m, ξ) -preinvex functions along with beta function are given. Moreover, we establish some new Hermite-Hadamard type integral inequalities for products of two generalized (s, m, ξ) -preinvex functions via classical and Riemann-Liouville fractional integrals. These results not only extend the results appeared in the literature (see [10],[11]), but also provide new estimates on these types. At the end, some conclusions are given.

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1. Introduction and Preliminaries

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I . For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K . $\mathbb{R}^n$ is used to denote a generic n -dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_+ = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

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The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \quad (1)$$

In recent years, various generalizations, extensions and variants of such inequalities have been obtained (see [24]-[31]). For other recent results concerning Hermite-Hadamard type inequalities through various classes of convex functions, (see [15]-[20],[23]) and the references cited therein.

In (see [10]), B. G. Pachpatte established two new Hermite-Hadamard type inequalities for products of convex functions as follows.

Theorem 1.2. *Let f and g be real-valued, nonnegative and convex functions on $[a, b]$. Then*

$$\frac{1}{b-a} \int_a^b f(x)g(x)dx \leq \frac{1}{3}M(a, b) + \frac{1}{6}N(a, b), \quad (2)$$

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)g(x)dx + \frac{1}{6}M(a, b) + \frac{1}{3}N(a, b), \quad (3)$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Some new integral inequalities involving two nonnegative and integrable functions that are related to the Hermite-Hadamard type are also obtained by many authors (see [6]-[8],[12]). Fractional calculus was introduced by Liouville and Riemann (see [23]) and some authors extended to study fractional Hermite-Hadamard type inequalities for functions of different classes (see [5],[23]).

Definition 1.1. *Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by*

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Theorem 1.3. (see [11]) *Let f and g be real-valued, nonnegative and convex functions on $[a, b]$. Then*

$$\begin{aligned} & \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[J_{a+}^\alpha f(b)g(b) + J_{b-}^\alpha f(a)g(a) \right] \\ & \leq \left(\frac{\alpha}{\alpha+2} - \frac{\alpha}{\alpha+1} + \frac{1}{2} \right) M(a, b) + \frac{\alpha}{(\alpha+1)(\alpha+2)} N(a, b), \end{aligned} \quad (4)$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Definition 1.2. (see [2]) A nonnegative function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_o$ is said to be P -function or P -convex, if

$$f(tx + (1-t)y) \leq f(x) + f(y), \quad \forall x, y \in I, t \in [0, 1].$$

Definition 1.3. (see [3]) A function $f : \mathbb{R}_o \longrightarrow \mathbb{R}$ is said to be s -convex in the second sense, if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \quad (5)$$

for all $x, y \in \mathbb{R}_o$, $\lambda \in [0, 1]$ and $s \in (0, 1]$.

The s -convex functions in the second sense have been investigated in (see [3]).

Definition 1.4. (see [4]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in K$ for every $x, y \in K$ and $t \in [0, 1]$.

Every convex set is invex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not necessarily true. For more details (see [4],[9]) and the references therein.

Definition 1.5. (see [13]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η , if for every $x, y \in K$ and $t \in [0, 1]$, we have that

$$f(x + t\eta(y, x)) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y, x) = y - x$, but the converse is not true.

The Gauss-Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p (b-x)^q f(x) dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^* |f|, \quad (6)$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^* |f|$ (see [14]).

Recently, Liu (see [21]) obtained several integral inequalities for the left-hand side of (6) under the Definition 1.2 of P -function.

Also in (see [22]), Özdemir et al. established several integral inequalities concerning the left-hand side of (6) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of generalized (s, m, ξ) -preinvex function is introduced and some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula (6) involving generalized (s, m, ξ) -preinvex functions along with beta function are given. In Section 3, some generalizations of Hermite-Hadamard type integral inequalities for products of two generalized (s, m, ξ) -preinvex functions via classical and Riemann-Liouville fractional integrals are given. These results not only extend the results appeared in the literature (see [10],[11]), but also provide new estimates on these types. In Section 4, some conclusions and future research are given.

2. New integral inequalities for generalized (s, m, ξ) -preinvex functions

Definition 2.1. (see [1]) A set $K \subseteq \mathbb{R}^n$ is said to be m -invex with respect to the mapping $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, x, m) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 2.1. In Definition 2.1, under certain conditions, the mapping $\eta(y, x, m)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m -invex set degenerates an invex set on K .

We are ready to introduce the new definition called generalized (s, m, ξ) -preinvex function.

Definition 2.2. Let $K \subseteq \mathbb{R}^n$ be an open m -invex set with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}^n$ and $\xi : I \rightarrow K$ a continuous function. For $f : K \rightarrow \mathbb{R}$ and any fixed $s \in (0, 1]$, if

$$f(m\xi(x) + \lambda\eta(\xi(y), \xi(x), m)) \leq m(1 - \lambda)^s f(\xi(x)) + \lambda^s f(\xi(y)) \quad (7)$$

is valid for all $x, y \in I, \lambda \in [0, 1]$, then we say that $f(x)$ is a generalized (s, m, ξ) -preinvex function with respect to η .

Remark 2.2. It is worthwhile to note that the class of generalized (s, m, ξ) -preinvex function is a generalization of the class of s -convex in the second sense.

In this section, in order to prove our main results regarding some new integral inequalities for the left-hand side of Gauss-Jacobi type quadrature formula (6) involving generalized (s, m, ξ) -preinvex functions along with beta function, we need the following new interesting Lemma:

Lemma 2.1. Let $\xi : I \rightarrow K$ be a continuous function. Assume that $f : K = [m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)] \rightarrow \mathbb{R}$ is a continuous function on the interval of real numbers K° with respect to $\eta : K \times K \times (0, 1] \rightarrow \mathbb{R}$ and $m\xi(a) < m\xi(a) + \eta(\xi(b), \xi(a), m)$. Then for any fixed $p, q > 0$, we have

$$\begin{aligned} & \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} (x - m\xi(a))^p (m\xi(a) + \eta(\xi(b), \xi(a), m) - x)^q f(x) dx \\ &= \eta(\xi(b), \xi(a), m)^{p+q+1} \int_0^1 t^p (1-t)^q f(m\xi(a) + t\eta(\xi(b), \xi(a), m)) dt. \end{aligned}$$

Proof.

$$\begin{aligned} & \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} (x - m\xi(a))^p (m\xi(a) + \eta(\xi(b), \xi(a), m) - x)^q f(x) dx \\ &= \eta(\xi(b), \xi(a), m) \int_0^1 (m\xi(a) + t\eta(\xi(b), \xi(a), m) - m\xi(a))^p \\ & \quad \times (m\xi(a) + \eta(\xi(b), \xi(a), m) - m\xi(a) - t\eta(\xi(b), \xi(a), m))^q f(m\xi(a) + t\eta(\xi(b), \xi(a), m)) dt \\ &= \eta(\xi(b), \xi(a), m)^{p+q+1} \int_0^1 t^p (1-t)^q f(m\xi(a) + t\eta(\xi(b), \xi(a), m)) dt. \end{aligned}$$

The following definition will be used in the sequel.

Definition 2.3. The Euler Beta function is defined for $x, y > 0$ as

$$\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Theorem 2.1. Let $\xi : I \longrightarrow K$ be a continuous function. Assume that $f : K = [m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)] \longrightarrow \mathbb{R}$ is a continuous function on the interval of real numbers K° with $m\xi(a) < m\xi(a) + \eta(\xi(b), \xi(a), m)$. If $k > 1$ and $|f|^{\frac{k}{k-1}}$ is a generalized (s, m, ξ) -preinvex function on an open m -invex set K with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ for any fixed $s \in (0, 1]$, then for any fixed $p, q > 0$,

$$\begin{aligned} & \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} (x - m\xi(a))^p (m\xi(a) + \eta(\xi(b), \xi(a), m) - x)^q f(x) dx \\ & \leq \frac{|\eta(\xi(b), \xi(a), m)|^{p+q+1}}{(s+1)^{\frac{k-1}{k}}} \left[\beta(kp+1, kq+1) \right]^{\frac{1}{k}} \left(m|f(\xi(a))|^{\frac{k}{k-1}} + |f(\xi(b))|^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}}. \end{aligned}$$

Proof. Since $|f|^{\frac{k}{k-1}}$ is a generalized (s, m, ξ) -preinvex function on K , combining with Lemma 2.1, Definition 2.3 and Hölder inequality for all $t \in [0, 1]$ and for any fixed $s, m \in (0, 1]$, we get

$$\begin{aligned} & \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} (x - m\xi(a))^p (m\xi(a) + \eta(\xi(b), \xi(a), m) - x)^q f(x) dx \\ & \leq |\eta(\xi(b), \xi(a), m)|^{p+q+1} \left[\int_0^1 t^{kp} (1-t)^{kq} dt \right]^{\frac{1}{k}} \\ & \quad \times \left[\int_0^1 |f(m\xi(a) + t\eta(\xi(b), \xi(a), m))|^{\frac{k}{k-1}} dt \right]^{\frac{k-1}{k}} \\ & \leq |\eta(\xi(b), \xi(a), m)|^{p+q+1} \left[\beta(kp+1, kq+1) \right]^{\frac{1}{k}} \\ & \quad \times \left[\int_0^1 \left(m(1-t)^s |f(\xi(a))|^{\frac{k}{k-1}} + t^s |f(\xi(b))|^{\frac{k}{k-1}} \right) dt \right]^{\frac{k-1}{k}} \\ & = \frac{|\eta(\xi(b), \xi(a), m)|^{p+q+1}}{(s+1)^{\frac{k-1}{k}}} \left[\beta(kp+1, kq+1) \right]^{\frac{1}{k}} \left(m|f(\xi(a))|^{\frac{k}{k-1}} + |f(\xi(b))|^{\frac{k}{k-1}} \right)^{\frac{k-1}{k}}. \end{aligned}$$

The proof of Theorem 2.1 is completed.

Theorem 2.2. Let $\xi : I \longrightarrow K$ be a continuous function. Assume that $f : K = [m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)] \longrightarrow \mathbb{R}$ is a continuous function on the interval of real numbers K° with $m\xi(a) < m\xi(a) + \eta(\xi(b), \xi(a), m)$. If $l \geq 1$ and $|f|^l$ is a generalized (s, m, ξ) -preinvex function on an open m -invex set K with respect to $\eta : K \times K \times (0, 1] \longrightarrow \mathbb{R}$ for any fixed $s \in (0, 1]$, then for any fixed $p, q > 0$,

$$\begin{aligned} & \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} (x - m\xi(a))^p (m\xi(a) + \eta(\xi(b), \xi(a), m) - x)^q f(x) dx \\ & \leq |\eta(\xi(b), \xi(a), m)|^{p+q+1} \left[\beta(p+1, q+1) \right]^{\frac{l-1}{l}} \\ & \quad \times \left[m|f(\xi(a))|^l \beta(p+1, q+s+1) + |f(\xi(b))|^l \beta(p+s+1, q+1) \right]^{\frac{1}{l}}. \end{aligned}$$

Proof. Since $|f|^l$ is a generalized (s, m, ξ) -preinvex function on K , combining with Lemma 2.1, Definition 2.3 and the well-known power mean inequality for all $t \in [0, 1]$ and for any fixed $s, m \in (0, 1]$, we get

$$\begin{aligned}
& \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b),\xi(a),m)} (x - m\xi(a))^p (m\xi(a) + \eta(\xi(b), \xi(a), m) - x)^q f(x) dx \\
&= \eta(\xi(b), \xi(a), m)^{p+q+1} \int_0^1 \left[t^p (1-t)^q \right]^{\frac{l-1}{l}} \left[t^p (1-t)^q \right]^{\frac{1}{l}} f(m\xi(a) + t\eta(\xi(b), \xi(a), m)) dt \\
&\leq |\eta(\xi(b), \xi(a), m)|^{p+q+1} \left[\int_0^1 t^p (1-t)^q dt \right]^{\frac{l-1}{l}} \\
&\quad \times \left[\int_0^1 t^p (1-t)^q |f(m\xi(a) + t\eta(\xi(b), \xi(a), m))|^l dt \right]^{\frac{1}{l}} \\
&\leq |\eta(\xi(b), \xi(a), m)|^{p+q+1} \left[\beta(p+1, q+1) \right]^{\frac{l-1}{l}} \\
&\quad \times \left[\int_0^1 t^p (1-t)^q \left(m(1-t)^s |f(\xi(a))|^l + t^s |f(\xi(b))|^l \right) dt \right]^{\frac{1}{l}} \\
&= |\eta(\xi(b), \xi(a), m)|^{p+q+1} \left[\beta(p+1, q+1) \right]^{\frac{l-1}{l}} \\
&\quad \times \left[m |f(\xi(a))|^l \beta(p+1, q+s+1) + |f(\xi(b))|^l \beta(p+s+1, q+1) \right]^{\frac{1}{l}}.
\end{aligned}$$

The proof of Theorem 2.2 is completed.

Remark 2.3. Clearly, if we replace $f(x)$ with the product of two generalized (s, m, ξ) -preinvex functions $g(x)$ and $h(x)$ in Theorems 2.1 and 2.2, we get some new interesting inequalities for evaluation integrals of the following form

$$\int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b),\xi(a),m)} (x - m\xi(a))^p (m\xi(a) + \eta(\xi(b), \xi(a), m) - x)^q g(x) h(x) dx.$$

The details are left to the interested reader.

3. Hermite-Hadamard type inequalities for products of two generalized (s, m, ξ) -preinvex functions

In this section, some generalizations of Hermite-Hadamard type inequalities for products of two generalized (s, m, ξ) -preinvex functions via classical and Riemann-Liouville fractional integrals are given.

Theorem 3.1. Let $\xi : I \longrightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}_o$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \longrightarrow \mathbb{R}_o$ for any fixed $s \in (0, 1]$ and let $m\xi(a) < m\xi(a) + \eta(\xi(b), \xi(a), m)$. Assume that $f, g : A \longrightarrow \mathbb{R}_o$ are generalized (s, m, ξ) -preinvex functions on $[m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)]$. Then

$$\frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b),\xi(a),m)} f(x) g(x) dx$$

$$\leq \frac{P(m, a, b)}{2s+1} + m\beta(s+1, s+1)Q(a, b), \quad (8)$$

where

$$P(m, a, b) = m^2 f(\xi(a))g(\xi(a)) + f(\xi(b))g(\xi(b)),$$

and

$$Q(a, b) = f(\xi(a))g(\xi(b)) + f(\xi(b))g(\xi(a)).$$

Proof. Since f and g are generalized (s, m, ξ) -preinvex functions, then for all $t \in [0, 1]$ we get

$$f(m\xi(a) + t\eta(\xi(b), \xi(a), m)) \leq m(1-t)^s f(\xi(a)) + t^s f(\xi(b)), \quad (9)$$

and

$$g(m\xi(a) + t\eta(\xi(b), \xi(a), m)) \leq m(1-t)^s g(\xi(a)) + t^s g(\xi(b)). \quad (10)$$

Changing the variable $x = m\xi(a) + t\eta(\xi(b), \xi(a), m)$ and using inequalities (9) and (10), we obtain

$$\begin{aligned} & \frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} f(x)g(x)dx \\ &= \int_0^1 f(m\xi(a) + t\eta(\xi(b), \xi(a), m)) g(m\xi(a) + t\eta(\xi(b), \xi(a), m)) dt \\ &\leq \int_0^1 [m(1-t)^s f(\xi(a)) + t^s f(\xi(b))] [m(1-t)^s g(\xi(a)) + t^s g(\xi(b))] dt \\ &= \frac{P(m, a, b)}{2s+1} + m\beta(s+1, s+1)Q(a, b). \end{aligned}$$

The proof of Theorem 3.1 is completed.

Remark 3.1. For $m = s = 1$, $\xi(x) = x$, $\forall x \in I$ and $\eta(b, a, 1) = b - a$ in Theorem 3.1, then we get inequality (2). Moreover, the details for finding the analogous of inequality (3) for products of two generalized (s, m, ξ) -preinvex functions is left to the interested reader.

Corollary 3.1. Under the same conditions as in Theorem 3.1 for $f(x) = g(x)$, we have

$$\begin{aligned} & \frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} f^2(x)dx \\ &\leq \frac{P_1(m, a, b)}{2s+1} + m\beta(s+1, s+1)Q_1(a, b), \end{aligned} \quad (11)$$

where

$$P_1(m, a, b) = m^2 f^2(\xi(a)) + f^2(\xi(b)),$$

and

$$Q_1(a, b) = 2f(\xi(a))f(\xi(b)).$$

Theorem 3.2. Let $\xi : I \rightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}$ for any fixed $s \in (0, 1]$ and let $m\xi(a) < m\xi(a) + \eta(\xi(b), \xi(a), m)$. Assume that $f, g : A \rightarrow \mathbb{R}$. If $|f|^p, |g|^q$ are integrable and generalized (s, m, ξ) -preinvex functions on $[m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)]$, then for $q > 1$ and $p^{-1} + q^{-1} = 1$, we have

$$\frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} |f(x)g(x)|dx$$

$$\leq \frac{\left[m|f(\xi(a))|^p + |f(\xi(b))|^p \right]^{\frac{1}{p}} \left[m|g(\xi(a))|^q + |g(\xi(b))|^q \right]^{\frac{1}{q}}}{s+1}. \quad (12)$$

Proof. Suppose that $q > 1$. Since $|f|^p, |g|^q$ are integrable and generalized (s, m, ξ) -preinvex functions, then for all $t \in [0, 1]$ we get

$$|f(m\xi(a) + t\eta(\xi(b), \xi(a), m))|^p \leq m(1-t)^s |f(\xi(a))|^p + t^s |f(\xi(b))|^p, \quad (13)$$

and

$$|g(m\xi(a) + t\eta(\xi(b), \xi(a), m))|^q \leq m(1-t)^s |g(\xi(a))|^q + t^s |g(\xi(b))|^q. \quad (14)$$

Changing the variable $x = m\xi(a) + t\eta(\xi(b), \xi(a), m)$, using Hölder integral inequality and inequalities (13), (14), we obtain

$$\begin{aligned} & \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} |f(x)g(x)| dx \\ & \leq \left(\int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} |f(x)|^p dx \right)^{\frac{1}{p}} \left(\int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} |g(x)|^q dx \right)^{\frac{1}{q}} \\ & = \eta^{\frac{1}{p}}(\xi(b), \xi(a), m) \left(\int_0^1 |f(m\xi(a) + t\eta(\xi(b), \xi(a), m))|^p dt \right)^{\frac{1}{p}} \\ & \quad \times \eta^{\frac{1}{q}}(\xi(b), \xi(a), m) \left(\int_0^1 |g(m\xi(a) + t\eta(\xi(b), \xi(a), m))|^q dt \right)^{\frac{1}{q}} \\ & \leq \eta^{\frac{1}{p}}(\xi(b), \xi(a), m) \left(\int_0^1 [m(1-t)^s |f(\xi(a))|^p + t^s |f(\xi(b))|^p] dt \right)^{\frac{1}{p}} \\ & \quad \times \eta^{\frac{1}{q}}(\xi(b), \xi(a), m) \left(\int_0^1 [m(1-t)^s |g(\xi(a))|^q + t^s |g(\xi(b))|^q] dt \right)^{\frac{1}{q}} \\ & = \left(\frac{\eta(\xi(b), \xi(a), m)}{s+1} \right)^{\frac{1}{p}} \left[m|f(\xi(a))|^p + |f(\xi(b))|^p \right]^{\frac{1}{p}} \\ & \quad \times \left(\frac{\eta(\xi(b), \xi(a), m)}{s+1} \right)^{\frac{1}{q}} \left[m|g(\xi(a))|^q + |g(\xi(b))|^q \right]^{\frac{1}{q}}. \end{aligned}$$

The proof of Theorem 3.2 is completed.

Corollary 3.2. *Under the same conditions as in Theorem 3.2 for $f(x) = g(x)$, we have*

$$\begin{aligned} & \frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b), \xi(a), m)} f^2(x) dx \\ & \leq \frac{\left[m|f(\xi(a))|^p + |f(\xi(b))|^p \right]^{\frac{1}{p}} \left[m|f(\xi(a))|^q + |f(\xi(b))|^q \right]^{\frac{1}{q}}}{s+1}. \end{aligned} \quad (15)$$

Corollary 3.3. *Under the same conditions as in Corollary 3.2 for $p = q = 2$, we have*

$$\begin{aligned} & \frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} f^2(x) dx \\ & \leq \frac{m|f(\xi(a))|^2 + |f(\xi(b))|^2}{s+1}. \end{aligned} \quad (16)$$

Theorem 3.3. *Let $\xi : I \rightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}_o$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}_o$ for any fixed $s \in (0, 1]$ and let $m\xi(a) < m\xi(a) + \eta(\xi(b), \xi(a), m)$. Assume that $f, g : A \rightarrow \mathbb{R}_o$. If f^p, g^p, fg are integrable and f^p, g^p are generalized (s, m, ξ) -preinvex functions on $[m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)]$, where*

$$0 < m_1 \leq \frac{f(x)}{g(x)} \leq M_1, \quad \forall x \in [m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)], \quad (17)$$

then for $p > 1$ and $p^{-1} + q^{-1} = 1$, we have the following inequality

$$\begin{aligned} & \frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} f(x)g(x) dx \\ & \leq \frac{c_1}{2(s+1)} \left[m(f^p(\xi(a)) + g^p(\xi(a))) + f^p(\xi(b)) + g^p(\xi(b)) \right] \\ & + \frac{c_2}{2(s+1)} \left[m(f^q(\xi(a)) + g^q(\xi(a))) + f^q(\xi(b)) + g^q(\xi(b)) \right], \end{aligned} \quad (18)$$

where

$$c_1 = \frac{2^p}{p} \left(\frac{M_1}{M_1 + 1} \right)^p, \quad c_2 = \frac{2^q}{q} \left(\frac{1}{m_1 + 1} \right)^q. \quad (19)$$

Proof. To prove our theorem we need the following Young's inequality:

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}, \quad \forall x, y \geq 0, \quad p > 1, \quad p^{-1} + q^{-1} = 1. \quad (20)$$

From (17), we have

$$f(x) \leq \frac{M_1}{M_1 + 1}(f(x) + g(x)), \quad g(x) \leq \frac{1}{m_1 + 1}(f(x) + g(x)). \quad (21)$$

From (20) and (21), we have

$$\begin{aligned} & \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} f(x)g(x) dx \\ & \leq \frac{1}{p} \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} f^p(x) dx + \frac{1}{q} \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} g^q(x) dx \\ & \leq \frac{1}{p} \left(\frac{M_1}{M_1 + 1} \right)^p \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} (f(x) + g(x))^p dx \\ & + \frac{1}{q} \left(\frac{1}{m_1 + 1} \right)^q \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} (f(x) + g(x))^q dx. \end{aligned} \quad (22)$$

Using the elementary inequality $(c + d)^p \leq 2^{p-1}(c^p + d^p)$, ($p > 1$ and $c, d \in \mathbb{R}_+$) in (22) and the fact that f^p, g^p are generalized (s, m, ξ) -preinvex functions on $[m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)]$, we get

$$\begin{aligned}
& \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b),\xi(a),m)} f(x)g(x)dx \\
& \leq \frac{1}{p} \left(\frac{M_1}{M_1+1} \right)^p 2^{p-1} \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b),\xi(a),m)} [f^p(x) + g^p(x)] dx \\
& \quad + \frac{1}{q} \left(\frac{1}{m_1+1} \right)^q 2^{q-1} \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b),\xi(a),m)} [f^q(x) + g^q(x)] dx \\
& = \frac{c_1}{2} \eta(\xi(b), \xi(a), m) \\
& \quad \times \left[\int_0^1 f^p(m\xi(a) + t\eta(\xi(b), \xi(a), m)) dt + \int_0^1 g^p(m\xi(a) + t\eta(\xi(b), \xi(a), m)) dt \right] \\
& \quad + \frac{c_2}{2} \eta(\xi(b), \xi(a), m) \\
& \quad \times \left[\int_0^1 f^q(m\xi(a) + t\eta(\xi(b), \xi(a), m)) dt + \int_0^1 g^q(m\xi(a) + t\eta(\xi(b), \xi(a), m)) dt \right] \\
& \leq \frac{c_1}{2} \eta(\xi(b), \xi(a), m) \\
& \quad \times \left[\int_0^1 (m(1-t)^s f^p(\xi(a)) + t^s f^p(\xi(b))) dt + \int_0^1 (m(1-t)^s g^p(\xi(a)) + t^s g^p(\xi(b))) dt \right] \\
& \quad + \frac{c_2}{2} \eta(\xi(b), \xi(a), m) \\
& \quad \times \left[\int_0^1 (m(1-t)^s f^q(\xi(a)) + t^s f^q(\xi(b))) dt + \int_0^1 (m(1-t)^s g^q(\xi(a)) + t^s g^q(\xi(b))) dt \right] \\
& = \frac{c_1 \eta(\xi(b), \xi(a), m)}{2(s+1)} \left[m(f^p(\xi(a)) + g^p(\xi(a))) + f^p(\xi(b)) + g^p(\xi(b)) \right] \\
& \quad + \frac{c_2 \eta(\xi(b), \xi(a), m)}{2(s+1)} \left[m(f^q(\xi(a)) + g^q(\xi(a))) + f^q(\xi(b)) + g^q(\xi(b)) \right].
\end{aligned}$$

The proof of Theorem 3.3 is completed.

Corollary 3.4. *Under the same conditions as in Theorem 3.3 for $f(x) = g(x)$, we have*

$$\begin{aligned}
& \frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(a)+\eta(\xi(b),\xi(a),m)} f^2(x)dx \\
& \leq \frac{c_1 [m f^p(\xi(a)) + f^p(\xi(b))] + c_2 [m f^q(\xi(a)) + f^q(\xi(b))]}{s+1}.
\end{aligned} \tag{23}$$

Corollary 3.5. *Under the same conditions as in Corollary 3.4 for $p = q = 2$, we have*

$$\begin{aligned} & \frac{1}{\eta(\xi(b), \xi(a), m)} \int_{m\xi(a)}^{m\xi(a) + \eta(\xi(b), \xi(a), m)} f^2(x) dx \\ & \leq \frac{2}{s+1} \left[\left(\frac{M_1}{M_1 + 1} \right)^2 + \left(\frac{1}{m_1 + 1} \right)^2 \right] [mf^2(\xi(a)) + f^2(\xi(b))]. \end{aligned} \quad (24)$$

In the following theorem we give Hermite-Hadamard type integral inequality for products of two generalized (s, m, ξ) -preinvex functions via Riemann-Liouville fractional integrals.

Theorem 3.4. *Let $\xi : I \rightarrow A$ be a continuous function. Suppose $A \subseteq \mathbb{R}_\circ$ be an open m -invex subset with respect to $\eta : A \times A \times (0, 1] \rightarrow \mathbb{R}_\circ$ for any fixed $s \in (0, 1]$ and let $m\xi(a) < m\xi(a) + \eta(\xi(b), \xi(a), m)$ and $m\xi(b) < m\xi(b) + \eta(\xi(a), \xi(b), m)$. Assume that $f, g : A \rightarrow \mathbb{R}_\circ$ are generalized (s, m, ξ) -preinvex functions on $[m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)]$ and $[m\xi(b), m\xi(b) + \eta(\xi(a), \xi(b), m)]$. Then for $\alpha > 0$, we have*

$$\begin{aligned} & \frac{\Gamma(\alpha)}{\eta^\alpha(\xi(b), \xi(a), m)} J_{(m\xi(a) + \eta(\xi(b), \xi(a), m))}^\alpha f(m\xi(a))g(m\xi(a)) \\ & + \frac{\Gamma(\alpha)}{\eta^\alpha(\xi(a), \xi(b), m)} J_{(m\xi(b) + \eta(\xi(a), \xi(b), m))}^\alpha f(m\xi(b))g(m\xi(b)) \\ & \leq \left[m^2\beta(\alpha, 2s+1) + \frac{1}{\alpha+2s} \right] M(a, b) + m\beta(\alpha+s, s+1)N(a, b), \end{aligned} \quad (25)$$

where

$$M(a, b) = f(\xi(a))g(\xi(a)) + f(\xi(b))g(\xi(b)),$$

and

$$N(a, b) = f(\xi(a))g(\xi(b)) + f(\xi(b))g(\xi(a)).$$

Proof. Since f and g are generalized (s, m, ξ) -preinvex functions on $[m\xi(a), m\xi(a) + \eta(\xi(b), \xi(a), m)]$, then for all $t \in [0, 1]$ we get

$$f(m\xi(a) + t\eta(\xi(b), \xi(a), m)) \leq m(1-t)^s f(\xi(a)) + t^s f(\xi(b)), \quad (26)$$

and

$$g(m\xi(a) + t\eta(\xi(b), \xi(a), m)) \leq m(1-t)^s g(\xi(a)) + t^s g(\xi(b)). \quad (27)$$

Similarly, since f and g are generalized (s, m, ξ) -preinvex functions on $[m\xi(b), m\xi(b) + \eta(\xi(a), \xi(b), m)]$, then for all $t \in [0, 1]$ we get

$$f(m\xi(b) + t\eta(\xi(a), \xi(b), m)) \leq m(1-t)^s f(\xi(b)) + t^s f(\xi(a)), \quad (28)$$

and

$$g(m\xi(b) + t\eta(\xi(a), \xi(b), m)) \leq m(1-t)^s g(\xi(b)) + t^s g(\xi(a)). \quad (29)$$

Now, we using the following steps for proof our theorem:

Changing the variable $u = m\xi(a) + t\eta(\xi(b), \xi(a), m)$ and $w = m\xi(b) + t\eta(\xi(a), \xi(b), m)$. Multiplying inequalities (26) with (27) and inequalities (28) with (29). After, multiplying both sides of inequalities obtained by $t^{\alpha-1}$ and then integrating with respect to t over $[0, 1]$, we get

$$\frac{\Gamma(\alpha)}{\eta^\alpha(\xi(b), \xi(a), m)} J_{(m\xi(a) + \eta(\xi(b), \xi(a), m))}^\alpha f(m\xi(a))g(m\xi(a))$$

$$\begin{aligned}
& + \frac{\Gamma(\alpha)}{\eta^\alpha(\xi(a), \xi(b), m)} J_{(m\xi(b)+\eta(\xi(a), \xi(b), m))}^\alpha f(m\xi(b))g(m\xi(b)) \\
& = \int_0^1 t^{\alpha-1} f(m\xi(a) + t\eta(\xi(b), \xi(a), m)) g(m\xi(a) + t\eta(\xi(b), \xi(a), m)) dt \\
& \quad + \int_0^1 t^{\alpha-1} f(m\xi(b) + t\eta(\xi(a), \xi(b), m)) g(m\xi(b) + t\eta(\xi(a), \xi(b), m)) dt \\
& \leq \int_0^1 t^{\alpha-1} [m(1-t)^s f(\xi(a)) + t^s f(\xi(b))] [m(1-t)^s g(\xi(a)) + t^s g(\xi(b))] dt \\
& \quad + \int_0^1 t^{\alpha-1} [m(1-t)^s f(\xi(b)) + t^s f(\xi(a))] [m(1-t)^s g(\xi(b)) + t^s g(\xi(a))] dt \\
& = \left[m^2 \beta(\alpha, 2s+1) + \frac{1}{\alpha+2s} \right] M(a, b) + m\beta(\alpha+s, s+1)N(a, b).
\end{aligned}$$

The proof of Theorem 3.4 is completed.

Remark 3.2. For $m = s = 1$, $\xi(x) = x$, $\forall x \in I$ and $\eta(y, x, 1) = y - x$ in Theorem 3.4, then we get Theorem 1.3.

Corollary 3.6. Under the same conditions as in Theorem 3.4 for $f(x) = g(x)$, we have

$$\begin{aligned}
& \frac{\Gamma(\alpha)}{\eta^\alpha(\xi(b), \xi(a), m)} J_{(m\xi(a)+\eta(\xi(b), \xi(a), m))}^\alpha f^2(m\xi(a)) \\
& \quad + \frac{\Gamma(\alpha)}{\eta^\alpha(\xi(a), \xi(b), m)} J_{(m\xi(b)+\eta(\xi(a), \xi(b), m))}^\alpha f^2(m\xi(b)) \\
& \leq \left[m^2 \beta(\alpha, 2s+1) + \frac{1}{\alpha+2s} \right] M_1(a, b) + m\beta(\alpha+s, s+1)N_1(a, b), \tag{30}
\end{aligned}$$

where

$$M_1(a, b) = f^2(\xi(a)) + f^2(\xi(b)),$$

and

$$N_1(a, b) = 2f(\xi(a))f(\xi(b)).$$

4. Conclusions

In the present paper, we established some new Hermite-Hadamard type integral inequalities for products of two generalized (s, m, ξ) -preinvex functions via classical and Riemann-Liouville fractional integrals.

Motivated by this new interesting class of generalized (s, m, ξ) -preinvex functions we can indeed see to be vital for fellow researchers and scientists working in the same domain.

We conclude that our methods may be a stimulant for further investigations concerning Hermite-Hadamard type integral inequalities for products of various kinds of preinvex functions involving classical integrals, Riemann-Liouville fractional integrals, k -fractional integrals, local fractional integrals, fractional integral operators, q -calculus, (p, q) -calculus, time scale calculus and conformable fractional integrals.

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