Global Well-Posedness for Fractional Navier-Stokes Equations in critical Fourier-Besov-Morrey Spaces

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ABSTRACT. In this paper we study the Cauchy problem of the Fractional Navier-Stokes equations in critical Fourier-Besov-Morrey spaces $\mathcal{FN}_{p,\lambda,q}^s(\mathbb{R}^3)$ with $s = 1 - 2\alpha + \frac{3}{p'} + \frac{\lambda}{p}$. By making use of the Fourier localization method and the Littlewood-Paley theory as in [6] and [21], we get global well-posedness result with small initial data belonging to $\mathcal{FN}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}(\mathbb{R}^3)$. The space $\mathcal{FN}_{p,\lambda,q}^s(\mathbb{R}^3)$ covers the classical spaces $\dot{B}_q^s$ and $\dot{B}_{p,q}^s(\mathbb{R}^3)$ (cf [7],[3], [19], [22]...). The result of this paper extends the works of [6] and [21].

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1. Introduction

We study the mild solutions to the fractional Navier-Stokes equations in $\mathbb{R}^+ \times \mathbb{R}^3$ as follows:

$$\begin{cases} u_t + \mu(-\Delta)^\alpha u + (u \cdot \nabla)u + \nabla \pi = 0 & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \nabla \cdot u = 0, \\ u(0, x) = u_0(x) & x \in \mathbb{R}^3. \end{cases} \quad (1)$$

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Here $u(t, x) = (u_1(t, x), u_2(t, x), u_3(t, x))$ denotes the velocity vector, $\mu > 0$ is the viscosity coefficient, and the scalar function π denotes the pressure. The initial data u_0 is a divergence free vector field and the operator $(-\Delta)^\alpha$ is the Fourier multiplier with symbol $|\xi|^{2\alpha}$. Extensive references may be found in the important new book of P.G. Lemarié-Rieusset [16]. For instance, the study of local existence is studied by several researchers, Leray [17], Kato [10]. The global existence of weak solutions goes back to Leray [17] and Hopf [8]. The global well-posedness of strong solutions for small initial data is due to: Fujita and Kato [13] in the critical Sobolev space $\dot{H}^{1/2}$, Cannone [5] in $\dot{H}^s$, for $s > 1/2$, Kato [11] in the Lebesgue space L^3 , Koch and Tataru [14] in the space BMO^{-1} , and Cannone [4, 5] in $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$. Let us mention that the well-posedness for the Navier-Stokes system in those above spaces can be extended to the generalized Navier-Stokes system [1]. Recently Lei and Lin [15] investigated a new space

$$\chi^{-1} = \{u \in \mathcal{D}'(\mathbb{R}^3); \int_{\mathbb{R}^3} |\xi|^{-1} |\widehat{u}| d\xi < \infty\},$$

contained in BMO^{-1} and equivalent to Fourier-Herz space $\dot{B}_1^{-1}$ [6]. They proved that if the initial data u_0 belongs to χ^{-1} and satisfy $\|u_0\|_{\chi^{-1}} < \mu$ the system (1) admits a global mild solution, and this solution is unique under the condition $\|u\|_{L^\infty(\mathbb{R}^+; \chi^{-1})} < \mu$. Cannone and Wu [6] gave a global well-posedness result for small initial data in a family of critical Fourier-Herz spaces $\dot{B}_q^{-1}(q \in [1, 2])$, and the solution is unique under the condition $\|u\|_{L^\infty(\mathbb{R}^+; \dot{B}_q^{-1}) \cap L^1(\mathbb{R}^+; \dot{B}_q^1)} < \frac{50}{9} \|u_0\|_{\dot{B}_q^{-1}}$.

We recall the definition of $\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^n)$ which cover some classical spaces. For example, $\dot{\mathcal{N}}_{p,0,q}^s = \dot{B}_{p,q}^s$, (cf [7],[3], [12], [19], [22] for details). Next, we present the definition of Fourier-Besov-Morrey spaces $\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^s$ and their basic properties. We observe that Fourier-Besov spaces $\dot{F}B_{p,q}^s$ have received a lot of attention in studying Navier-Stokes equations. In fact Cannone and Wu worked in the Fourier-Herz spaces $\dot{B}_q^s = FB_{1,q}^s$. Recently Xiao, Chen, Fan, and Zhou used $\dot{F}B_{p,q}^s$ to study the Fractional Navier-Stokes Equations [21]. We can notice that $\dot{B}_q^s = \dot{F}B_{1,q}^s = \mathcal{F}\dot{\mathcal{N}}_{1,0,q}^s$. Next, we study the generalized Navier-Stokes equations in the supercritical case $\alpha > \frac{1}{2}$ for a small u_0 belonging to $\mathcal{F}\dot{\mathcal{N}}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}(\mathbb{R}^3)$. The result of this paper extends those of [21] and [15]. Throughout this paper, we use $\dot{B}_{p,q}^s$ to denote the classical homogenous Besov spaces, C will denote constants which can be different at different places, $U \lesssim V$ means that there exists a constant $C > 0$ such that $U \leq CV$, and p' is the conjugate of p satisfying $\frac{1}{p} + \frac{1}{p'} = 1$ for $1 \leq p \leq \infty$.

2. Preliminaries and main result

The proofs of the results presented in this paper are based on a dyadic partition of unity in the Fourier variables, the so-called, homogeneous Littlewood-Paley decomposition. We recall briefly this construction below. We start with a dyadic decomposition of $\mathbb{R}^n$.

Suppose $\chi \in C_0^\infty(\mathbb{R}^n)$, $\varphi \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ satisfying

$$\begin{aligned} \text{supp } \chi &\subset \{\xi \in \mathbb{R}^n : |\xi| \leq \frac{4}{3}\}, \\ \text{supp } \varphi &\subset \{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}, \end{aligned}$$

$$\begin{aligned}\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) &= 1, \quad \xi \in \mathbb{R}^n, \\ \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) &= 1, \quad \xi \in \mathbb{R}^n \setminus \{0\},\end{aligned}$$

and denote $\varphi_j(\xi) = \varphi(2^{-j}\xi)$ and $\mathcal{P}$ the set of all polynomials. The space of tempered distributions is denoted by $\mathcal{S}'$. The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cutoff operators $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\begin{aligned}\dot{\Delta}_j u &= \varphi(2^{-j}D)u = 2^{jn} \int h(2^j y) u(x-y) dy, \\ \dot{S}_j u &= \sum_{k \leq j-1} \dot{\Delta}_k u = \chi(2^{-j}D)u = 2^{jn} \int \tilde{h}(2^j y) u(x-y) dy,\end{aligned}$$

where $h = \mathcal{F}^{-1}\varphi$ and $\tilde{h} = \mathcal{F}^{-1}\chi$.

First, we recall the definition of Morrey spaces which are a complement of L^p spaces.

Definition 2.1. (see [12] and [20]) For $1 \leq p < \infty$, $0 \leq \lambda < n$, the Morrey spaces $M_p^\lambda = M_p^\lambda(\mathbb{R}^n)$ is defined as the set of functions $f \in L_{loc}^p(\mathbb{R}^n)$ such that

$$\|f\|_{M_p^\lambda} = \sup_{x_0 \in \mathbb{R}^n} \sup_{r > 0} r^{-\frac{\lambda}{p}} \|f\|_{L^p(B(x_0, r))} < \infty, \quad (2)$$

where $B(x_0, r)$ denotes the ball in $\mathbb{R}^n$ with center x_0 and radius r . the space M_p^λ endowed with the norm $\|f\|_{M_p^\lambda}$ is a Banach space. In the case $p = 1$, M_p^λ should be understood as a space of Radon measures and $\|f\|_{L^1(B(x_0, r))}$ denoting the total variation of f on $B(x_0, r)$. For various reasons we find it convenient to include L^∞ among the Morrey spaces, but the indices in the notation M_p^λ will always be restricted to $1 \leq p < \infty$, $0 \leq \lambda < n$, notwithstanding that (2) makes sense for $\lambda = n$ and the resulting space is equivalent to L^∞ (irrespective of the value of p).

It is not difficult to see that the relation $M_{p_1}^\lambda \hookrightarrow M_{p_2}^\mu$ provided $\frac{n-\mu}{p_2} \geq \frac{n-\lambda}{p_1}$ and $p_2 \leq p_1$, and $M_p^0 = L^p$.

If $1 \leq p_1, p_2, p_3 < \infty$ and $0 \leq \lambda_1, \lambda_2, \lambda_3 < n$ with $\frac{1}{p_3} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{\lambda_3}{p_3} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$, then we have the Hölder type inequality

$$\|fg\|_{M_{p_3}^{\lambda_3}} \leq \|f\|_{M_{p_1}^{\lambda_1}} \|g\|_{M_{p_2}^{\lambda_2}}.$$

Also, for $1 \leq p < \infty$ and $0 \leq \lambda < n$,

$$\|\varphi * g\|_{M_p^\lambda} \leq \|\varphi\|_{L^1} \|g\|_{M_p^\lambda}, \quad (3)$$

for all $\varphi \in L^1$ and $g \in M_p^\lambda$.

Definition 2.2. (homogeneous Besov-Morrey spaces) Let $s \in \mathbb{R}$, $1 \leq p < +\infty$, $1 \leq q \leq +\infty$, and $0 \leq \lambda < n$, the space $\dot{\mathcal{N}}_{p, \lambda, q}^s(\mathbb{R}^n)$ is defined by

$$\dot{\mathcal{N}}_{p, \lambda, q}^s(\mathbb{R}^n) = \left\{ u \in \mathcal{Z}'(\mathbb{R}^n); \quad \|u\|_{\dot{\mathcal{N}}_{p, \lambda, q}^s(\mathbb{R}^n)} < \infty \right\}.$$

Here

$$\|u\|_{\dot{\mathcal{N}}_{p,\lambda,q}^s(\mathbb{R}^n)} = \begin{cases} \left\{ \sum_{j \in \mathbb{Z}} 2^{jq} \|\dot{\Delta}_j u\|_{M_p^\lambda}^q \right\}^{1/q} & \text{for } q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{jq} \|\dot{\Delta}_j u\|_{M_p^\lambda} & \text{for } q = \infty. \end{cases}$$

The space $\mathcal{Z}'(\mathbb{R}^n)$ denotes the topological dual of the space $\mathcal{Z}(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n); \partial^\alpha \widehat{f}(0) = 0 \text{ for every multi-index } \alpha\}$, and can be identified to the quotient space $\mathcal{S}'(\mathbb{R}^n)/\mathcal{P}$, where $\mathcal{P}$ denotes the set of all polynomials on $\mathbb{R}^n$. We refer to [22, chap. 8] for more details.

Definition 2.3. (*homogeneous Fourier-Besov-Morrey spaces*)

Let $s \in \mathbb{R}$, $0 \leq \lambda < n$, $1 \leq p < +\infty$ and $1 \leq q \leq +\infty$. The space $\mathcal{FN}_{p,\lambda,q}^s(\mathbb{R}^n)$ denotes the set of all $u \in \mathcal{Z}'(\mathbb{R}^n)$ such that

$$\|u\|_{\mathcal{FN}_{p,\lambda,q}^s(\mathbb{R}^n)} = \left\{ \sum_{j \in \mathbb{Z}} 2^{jq} \|\widehat{\dot{\Delta}_j u}\|_{M_p^\lambda}^q \right\}^{1/q} < +\infty, \quad (4)$$

with suitable modification made when $q = \infty$.

Note that the space $\mathcal{FN}_{p,\lambda,q}^s(\mathbb{R}^n)$ equipped with the norm (4) is a Banach space.

Now, we recall the definition of the mixed space-time spaces used in [6, 21].

Definition 2.4. Let $s \in \mathbb{R}$, $1 \leq p < \infty$, $1 \leq q, \rho \leq \infty$, $0 \leq \lambda < n$, and $I = [0, T)$, $T \in (0, \infty]$. The space-time norm is defined on $u(t, x)$ by

$$\|u(t, x)\|_{\mathcal{L}^\rho(I, \mathcal{FN}_{p,\lambda,q}^s)} = \left\{ \sum_{j \in \mathbb{Z}} 2^{jq} \|\widehat{\dot{\Delta}_j u}\|_{L^\rho(I, M_p^\lambda)}^q \right\}^{1/q},$$

and denote by $\mathcal{L}^\rho(I, \mathcal{FN}_{p,\lambda,q}^s)$ the set of distributions in $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^3)/\mathcal{P}$ with finite $\|\cdot\|_{\mathcal{L}^\rho(I, \mathcal{FN}_{p,\lambda,q}^s)}$ norm.

Our main result is the following theorem.

Theorem 2.1. Let $s \in \mathbb{R}$, $1 \leq p < \infty$, $1 \leq q \leq \infty$, $0 \leq \lambda < n$, and $\frac{1}{2} < \alpha < 1 + \frac{3}{2p'} + \frac{\lambda}{2p}$. Then there exists a constant $C_0(\alpha, p, q)$ such that, for any $u_0 \in$

$\mathcal{FN}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}$ satisfying $\nabla \cdot u_0 = 0$ and $\|u_0\|_{\mathcal{FN}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}} < C_0 \mu$,

the Cauchy problem (1) admits a unique global mild solution u ,

$$u \in \mathcal{C}\left([0, \infty); \mathcal{FN}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}\right) \cap \mathcal{L}^1\left([0, \infty), \mathcal{FN}_{p,\lambda,q}^{1+\frac{3}{p'}+\frac{\lambda}{p}}\right),$$

and it satisfies

$$\|u\|_{\mathcal{L}^\infty([0, \infty); \mathcal{FN}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}})} + \mu \|u\|_{\mathcal{L}^1([0, \infty), \mathcal{FN}_{p,\lambda,q}^{1+\frac{3}{p'}+\frac{\lambda}{p}})} \leq \left(1 + \left(\frac{16}{9}\right)^\alpha\right) \|u_0\|_{\mathcal{FN}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}}.$$

Now, we give some remarks about this result.

Remark 2.1. We notice that the Fourier-Besov-Morrey spaces are independent of the choice of φ_j , and the advantage of working in these spaces lies in they are more adapted than the classical Besov-Morrey-spaces for estimating the bilinear paraproduct using Hölder's inequality directly, instead of Bernstein's inequality.

Theorem 2.1 extends the result of [6] and [21] from Fourier-Herz spaces and Fourier-Besov

spaces to Fourier-Besov-Morrey spaces, in fact, for $\lambda = 0$, $\mathcal{FN}_{p,0,q}^s = \dot{FB}_{p,q}^s$, and $\mathcal{FN}_{1,0,q}^s = \dot{B}_q^s$ where $\dot{FB}_{p,q}^s$ is the homogeneous Fourier-Besov spaces (see definition 2.5).

Remark 2.2. The Fourier-Besov-Morrey space $\mathcal{FN}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}(\mathbb{R}^3)$ is critical for (1). For this, set $u_{0,\gamma}(\xi) = \gamma^{2\alpha-1}u_0(\gamma\xi)$, then $\widehat{u_{0,\gamma}}(\xi) = \gamma^{2\alpha-4}\widehat{u_0}(\gamma^{-1}\xi)$. Next, setting

$$f_j(\xi) = \varphi(2^{-j+[\log_2 \gamma]-\log_2 \gamma}\xi)\widehat{u_{0,\gamma}}(\xi),$$

we can obtain

$$\begin{aligned} 2^{j(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})}\|f_j\|_{M_p^\lambda} &= 2^{j(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})}\sup_{x_0 \in \mathbb{R}^3}\sup_{r>0}r^{-\frac{\lambda}{p}}\|\varphi(2^{-j+[\log_2 \gamma]-\log_2 \gamma}\xi)\widehat{u_{0,\gamma}}(\xi)\|_{L^p(B(x_0,r))} \\ &= 2^{j(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})}\sup_{x_0 \in \mathbb{R}^3}\sup_{r>0}r^{-\frac{\lambda}{p}}\|\varphi(2^{-j+[\log_2 \gamma]-\log_2 \gamma}\xi)\gamma^{2\alpha-4}\widehat{u_0}(\gamma^{-1}\xi)\|_{L^p(B(x_0,r))} \\ &= 2^{j(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})}\gamma^{2\alpha-4}\sup_{x_0 \in \mathbb{R}^3}\sup_{r>0}r^{-\frac{\lambda}{p}}\|\varphi(2^{-j+[\log_2 \gamma]}\xi)\widehat{u_0}(\gamma^{-1}\xi)\|_{L^p(B(x_0,r))} \\ &= 2^{j(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})}\gamma^{2\alpha-4}\gamma^{\frac{3}{p}}\gamma^{-\frac{\lambda}{p}}\sup_{x_0 \in \mathbb{R}^3}\sup_{r>0}(\gamma^{-1}r)^{-\frac{\lambda}{p}}\|\varphi(2^{-j+[\log_2 \gamma]}\eta)\widehat{u_0}(\eta)\|_{L^p(B(\gamma^{-1}x_0,\gamma^{-1}r))} \\ &= 2^{j(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})}2^{(2\alpha-1-\frac{3}{p'}-\frac{\lambda}{p})\log_2 \gamma}\|\varphi(2^{-j+[\log_2 \gamma]}\eta)\widehat{u_0}(\eta)\|_{M_p^\lambda} \\ &= 2^{([\log_2 \gamma]-\log_2 \gamma)(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})}2^{(j-[\log_2 \gamma])(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})}\times \|\varphi(2^{-j+[\log_2 \gamma]}\eta)\widehat{u_0}(\eta)\|_{M_p^\lambda} \\ &\approx 2^{(j-[\log_2 \gamma])(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})}\times \|\varphi(2^{-j+[\log_2 \gamma]}\eta)\widehat{u_0}(\eta)\|_{M_p^\lambda}. \end{aligned}$$

This implies that

$$\left\{ \sum_{j \in \mathbb{Z}} 2^{j(1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p})q} \|f_j\|_{M_p^\lambda}^q \right\}^{1/q} \approx \|u_0\|_{\mathcal{FN}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}},$$

and since

$$\varphi_j(\xi)\widehat{u_{0,\gamma}}(\xi) = \sum_{|k-j| \leq 2} \varphi_k(\xi)f_k(\xi)$$

we easily get

$$\|u_{0,\gamma}\|_{\mathcal{FN}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}} \approx \|u_0\|_{\mathcal{FN}_{p,\lambda,q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}}.$$

Now we introduce the Fourier-Besov spaces which contain some known spaces applied in studying Navier-Stokes equations. The norm of Fourier-Besov spaces $F\dot{B}_{p,q}^s$ [21] is defined as

Definition 2.5. For $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, set

$$\|u\|_{F\dot{B}_{p,q}^s} = \begin{cases} \left(\sum_{j \in \mathbb{Z}} 2^{jq s} \|\widehat{\Delta_j u}\|_{L^p}^q \right)^{1/q} & \text{for } q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{js} \|\widehat{\Delta_j u}\|_{L^p} & \text{for } q = \infty. \end{cases}$$

One defines the homogenous Fourier-Besov spaces $F\dot{B}_{p,q}^s$ as

$$F\dot{B}_{p,q}^s = \{u \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P} : \|u\|_{F\dot{B}_{p,q}^s} < \infty\}.$$

Particularly, for $p = 1$ Cannone and Wu introduced the Fourier-Herz spaces $\dot{\mathcal{B}}_q^s$ [6] with the norm associated

$$\|u\|_{\dot{\mathcal{B}}_q^s} = \begin{cases} \left(\sum_{j \in \mathbb{Z}} 2^{jq_s} \|\widehat{\dot{\Delta}_j u}\|_{L^1}^q \right)^{1/q} & \text{for } q < \infty, \\ \sup_{j \in \mathbb{Z}} 2^{jq_s} \|\widehat{\dot{\Delta}_j u}\|_{L^1} & \text{for } q = \infty. \end{cases}$$

Visibly, we have $\dot{\mathcal{B}}_q^s = F\dot{B}_{1,q}^s$.

The space χ^{-1} introduced by Lei and Lin [15] is

$$\chi^{-1} = \{u \in \mathcal{S}'(\mathbb{R}^3); \int_{\mathbb{R}^3} |\xi|^{-1} |\widehat{u}| d\xi < \infty\}.$$

We have $\chi^{-1} = F\dot{B}_{1,1}^{-1} = \dot{\mathcal{B}}_1^{-1}$.

Now, we give some preliminary lemmas used in the sequel.

Lemma 2.1. *Let $s_1, s_2 \in \mathbb{R}$, $1 \leq p_1, p_2 < +\infty$, $1 \leq q_1, q_2 \leq +\infty$, $0 \leq \lambda_1, \lambda_2 < n$, and $0 \leq \theta \leq 1$. If $s = (1 - \theta)s_1 + \theta s_2$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$, $\lambda = (1 - \theta)\lambda_1 + \theta\lambda_2$, then*

$$\|u\|_{\mathcal{FN}_{p,\lambda,q}^s} \leq \|u\|_{\mathcal{FN}_{p_1,\lambda_1,q_1}^{s_1}}^{1-\theta} \|u\|_{\mathcal{FN}_{p_2,\lambda_2,q_2}^{s_2}}^{\theta}.$$

Proof. We have

$$\begin{aligned} \|u\|_{\mathcal{FN}_{p,\lambda,q}^s} &= \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} \|\widehat{\dot{\Delta}_j u}\|_{M_p^\lambda}^q \right\}^{1/q} \\ &= \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} \left(\sup_{x_0 \in \mathbb{R}^n} \sup_{r>0} r^{-\frac{\lambda}{p}} \|\widehat{\dot{\Delta}_j u}\|_{L^p(B(x_0,r))}^{1-\theta} \|\widehat{\dot{\Delta}_j u}\|_{L^p(B(x_0,r))}^{\theta} \right)^q \right\}^{1/q}. \end{aligned}$$

Using Hölder's inequality, with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and for $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ we obtain

$$\begin{aligned} \|u\|_{\mathcal{FN}_{p,\lambda,q}^s} &\leq \left\{ \sum_{j \in \mathbb{Z}} 2^{jsq} \left(\sup_{x_0 \in \mathbb{R}^n} \sup_{r>0} \left(r^{-\frac{\lambda_1}{p_1}} \|\widehat{\dot{\Delta}_j u}\|_{L^{p_1}(B(x_0,r))} \right)^{1-\theta} \left(r^{-\frac{\lambda_2}{p_2}} \|\widehat{\dot{\Delta}_j u}\|_{L^{p_2}(B(x_0,r))} \right)^{\theta} \right)^q \right\}^{1/q} \\ &\leq \left\{ \sum_{j \in \mathbb{Z}} 2^{j(1-\theta)s_1q} \|\widehat{\dot{\Delta}_j u}\|_{M_{p_1}^{\lambda_1}}^{(1-\theta)q} 2^{j\theta s_2q} \|\widehat{\dot{\Delta}_j u}\|_{M_{p_2}^{\lambda_2}}^{\theta q} \right\}^{1/q} \\ &\leq \left(\sum_{j \in \mathbb{Z}} 2^{js_1q_1} \|\widehat{\dot{\Delta}_j u}\|_{M_{p_1}^{\lambda_1}} \right)^{\frac{(1-\theta)}{q_1}} \left(\sum_{j \in \mathbb{Z}} 2^{js_2q_2} \|\widehat{\dot{\Delta}_j u}\|_{M_{p_2}^{\lambda_2}} \right)^{\frac{\theta}{q_2}} \\ &:= \|u\|_{\mathcal{FN}_{p_1,\lambda_1,q_1}^{s_1}}^{1-\theta} \|u\|_{\mathcal{FN}_{p_2,\lambda_2,q_2}^{s_2}}^{\theta}. \end{aligned}$$

Lemma 2.2. *The derivation D_ξ^α is a bounded operator from $\mathcal{FN}_{p,\lambda,q}^{s+|\alpha|}(\mathbb{R}^n)$ to $\mathcal{FN}_{p,\lambda,q}^s(\mathbb{R}^n)$.*

Proof. Using the definition of $\mathcal{FN}_{p,\lambda,q}^s$, and $|\xi| \approx 2^j$ for all $j \in \mathbb{Z}$, we have

$$\|D_\xi^\alpha u\|_{\mathcal{FN}_{p,\lambda,q}^s(\mathbb{R}^n)} = \left\{ \sum_{j \in \mathbb{Z}} 2^{jrs} \|\widehat{\dot{\Delta}_j D_\xi^\alpha u}\|_{M_p^\lambda}^q \right\}^{1/q}$$

$$\begin{aligned}
&= \left\{ \sum_{j \in \mathbb{Z}} 2^{jq_s} \|\varphi_j \widehat{D_\xi^\alpha u}\|_{M_p^\lambda}^q \right\}^{1/q} \\
&= \left\{ \sum_{j \in \mathbb{Z}} 2^{jq_s} \|\varphi_j |\xi|^\alpha \widehat{u}\|_{M_p^\lambda}^q \right\}^{1/q} \\
&\lesssim \left\{ \sum_{j \in \mathbb{Z}} 2^{jq_s} 2^{j|\alpha|q} \|\varphi_j \widehat{u}\|_{M_p^\lambda}^q \right\}^{1/q} \\
&\lesssim \|u\|_{\mathcal{FN}_{p,\lambda,q}^{s+|\alpha|}(\mathbb{R}^n)}.
\end{aligned}$$

We finish this subsection with a Bernstein type lemma in Fourier variables in Morrey spaces.

Lemma 2.3. (see [7]) *Let $1 \leq q \leq p < \infty$, $0 \leq \lambda_1, \lambda_2 < n$, $\frac{n-\lambda_1}{p} \leq \frac{n-\lambda_2}{q}$, and let γ be a multiindex. If $\text{supp}(\widehat{f}) \subset \{|\xi| \leq A2^j\}$ then there is a constant $C > 0$ independent of f and j such that*

$$\|(i\xi)^\gamma \widehat{f}\|_{M_q^{\lambda_2}} \leq C 2^{j|\gamma|+j(\frac{n-\lambda_2}{q}-\frac{n-\lambda_1}{p})} \|\widehat{f}\|_{M_p^{\lambda_1}}. \quad (5)$$

Proof. From the condition on support of $\widehat{f}$ and (3), we obtain

$$\|(i\xi)^\gamma \widehat{f}\|_{M_q^{\lambda_2}} \leq C 2^{j|\gamma|} \|1 \cdot \widehat{f}\|_{M_p^{\lambda_2}} \leq 2^{j|\gamma|} C 2^{j(\frac{n-\lambda_2}{q}-\frac{n-\lambda_1}{p})} \|\widehat{f}\|_{M_p^{\lambda_1}},$$

which gives (5).

3. The Well-Posedness

First, we consider the linear nonhomogeneous dissipative equation

$$\begin{cases} u_t + \mu(-\Delta)^\alpha u = f(t, x) & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3 \\ u(0, x) = u_0(x) & x \in \mathbb{R}^3 \end{cases} \quad (6)$$

for which we can obtain the following result.

Lemma 3.1. (Linear Estimate) *Let $I = [0, T)$, $0 < T \leq \infty$, $s \in \mathbb{R}$, $0 \leq \lambda < n$, $1 \leq p < \infty$, and $1 \leq q \leq \infty$. Assume that $u_0 \in \mathcal{FN}_{p,\lambda,q}^s$ and $f \in \mathcal{L}^1(I, \mathcal{FN}_{p,\lambda,q}^s)$. Then the solution $u(t, x)$ to the Cauchy problem (6) satisfies*

$$\|u\|_{\mathcal{L}^\infty(I, \mathcal{FN}_{p,\lambda,q}^s)} + \mu \|u\|_{\mathcal{L}^1(I, \mathcal{FN}_{p,\lambda,q}^{s+2\alpha})} \leq (1 + (\frac{16}{9})^\alpha) (\|u_0\|_{\mathcal{FN}_{p,\lambda,q}^s} + \|f\|_{\mathcal{L}^1(I, \mathcal{FN}_{p,\lambda,q}^s)}).$$

Proof. Taking the Fourier transform with respect to ξ on both sides of (6), we have

$$\partial_t \widehat{u} + \mu |\xi|^{2\alpha} \widehat{u} = \widehat{f}.$$

Due to Duhamel's formula, we can write

$$\widehat{u}(t, \xi) = e^{-\mu |\xi|^{2\alpha} t} \widehat{u}_0(\xi) + \int_0^t e^{-\mu(t-t')|\xi|^{2\alpha}} \widehat{f}(t', \xi) dt'.$$

Then

$$|\widehat{u}(t, \xi)| \leq e^{-\mu |\xi|^{2\alpha} t} |\widehat{u}_0(\xi)| + \int_0^t e^{-\mu(t-t')|\xi|^{2\alpha}} |\widehat{f}(t', \xi)| dt'.$$

Multiplying by φ_j on both sides, and taking the L^p -norm, we get

$$\|\widehat{\Delta_j u}\|_{M_p^\lambda} \leq \|h\|_{M_p^\lambda} + \|\|g\|_{L^1([0,t])}\|_{M_p^\lambda} \leq \|h\|_{M_p^\lambda} + \|\|g\|_{M_p^\lambda}\|_{L^1([0,t])},$$

where $h(t, \xi) = e^{-\mu|\xi|^{2\alpha}t} |\widehat{\dot{\Delta}_j u_0}|(\xi)$, $g(t', \xi) = e^{-\mu(t-t')|\xi|^{2\alpha}} |\widehat{\dot{\Delta}_j f}|(t', \xi)$.

Since $\text{supp} \varphi_j \subset \{\xi \in \mathbb{R}^n : \frac{3}{4}2^j \leq |\xi| \leq \frac{8}{3}2^j\}$, we have $e^{-\mu(t-t')|\xi|^{2\alpha}} \leq e^{-\mu(t-t')(\frac{9}{16})^\alpha 2^{2\alpha j}}$.

Therefore,

$$\|\widehat{\dot{\Delta}_j u}\|_{M_p^\lambda} \leq e^{-\mu t (\frac{9}{16})^\alpha 2^{2\alpha j}} \|\widehat{\dot{\Delta}_j u_0}\|_{M_p^\lambda} + \int_0^t e^{-\mu(t-t')(\frac{9}{16})^\alpha 2^{2\alpha j}} \|\widehat{\dot{\Delta}_j f}\|_{M_p^\lambda} dt'.$$

Integrating with respect to time on $[0, t_0]$, we obtain

$$\begin{aligned} \|\widehat{\dot{\Delta}_j u}\|_{L^1([0, t_0], M_p^\lambda)} &\leq \int_0^{t_0} e^{-\mu t (\frac{9}{16})^\alpha 2^{2\alpha j}} dt \|\widehat{\dot{\Delta}_j u_0}\|_{M_p^\lambda} \\ &\quad + \int_0^{t_0} \int_0^t e^{-\mu(t-t')(\frac{9}{16})^\alpha 2^{2\alpha j}} \|\widehat{\dot{\Delta}_j f}\|_{M_p^\lambda} dt' dt \\ &\leq \frac{1}{\mu(\frac{9}{16})^\alpha 2^{2\alpha j}} \|\widehat{\dot{\Delta}_j u_0}\|_{M_p^\lambda} + \int_0^{t_0} \left(\int_{t'}^{t_0} e^{-\mu(t-t')(\frac{9}{16})^\alpha 2^{2\alpha j}} dt \right) \|\widehat{\dot{\Delta}_j f}\|_{M_p^\lambda} dt' \\ &\leq \frac{1}{\mu(\frac{9}{16})^\alpha 2^{2\alpha j}} \|\widehat{\dot{\Delta}_j u_0}\|_{M_p^\lambda} + \frac{1}{\mu(\frac{9}{16})^\alpha 2^{2\alpha j}} \|\widehat{\dot{\Delta}_j f}\|_{L^1([0, t_0], M_p^\lambda)}. \end{aligned}$$

Consequently

$$\begin{aligned} \mu \left\{ \sum_{j \in \mathbb{Z}} 2^{jq(s+2\alpha)} \|\widehat{\dot{\Delta}_j u}\|_{L^1(I, M_p^\lambda)}^q \right\}^{1/q} &\leq \left(\frac{16}{9} \right)^\alpha \left(\left\{ \sum_{j \in \mathbb{Z}} 2^{jq s} \|\widehat{\dot{\Delta}_j u_0}\|_{M_p^q}^q \right\}^{1/q} + \right. \\ &\quad \left. \left\{ \sum_{j \in \mathbb{Z}} 2^{jq s} \|\widehat{\dot{\Delta}_j f}\|_{L^1(I, M_p^\lambda)}^q \right\}^{1/q} \right). \end{aligned}$$

Finally, we obtain the following estimate

$$\mu \|u\|_{\mathcal{L}^1(I, \mathcal{FN}_{p, \lambda, q}^{s+2\alpha})} \leq \left(\frac{16}{9} \right)^\alpha \left(\|u_0\|_{\mathcal{FN}_{p, \lambda, q}^{s+2\alpha}} + \|f\|_{\mathcal{L}^1(I, \mathcal{FN}_{p, \lambda, q}^{s+2\alpha})} \right). \quad (7)$$

On the other hand

$$|\widehat{\dot{\Delta}_j u}| \leq e^{-\mu|\xi|^{2\alpha}t} |\widehat{\dot{\Delta}_j u_0}| + \int_0^t e^{-\mu(t-t')|\xi|^{2\alpha}} |\widehat{\dot{\Delta}_j f}|(t', \xi) dt'.$$

Similarly, taking the L^q -norm on both sides with respect to ξ , we get

$$\|\widehat{\dot{\Delta}_j u}\|_{M_p^\lambda} \leq \|h\|_{M_p^\lambda} + \|g\|_{L^1([0, t])} \| \cdot \|_{M_p^\lambda},$$

where $h(t, \xi) = e^{-\mu|\xi|^{2\alpha}t} |\widehat{\dot{\Delta}_j u_0}|$ and $g(t', \xi) = e^{-\mu(t-t')|\xi|^{2\alpha}} |\widehat{\dot{\Delta}_j f}|(t', \xi)$.

Now,

$$\|\widehat{\dot{\Delta}_j u}\|_{M_p^\lambda} \leq \|h\|_{M_p^\lambda} + \|g\|_{M_p^\lambda} \| \cdot \|_{L^1([0, t])}.$$

Thus

$$\|\widehat{\dot{\Delta}_j u}\|_{M_p^\lambda} \leq e^{-\mu t (\frac{9}{16})^\alpha 2^{2\alpha j}} \|\widehat{\dot{\Delta}_j u_0}\|_{M_p^\lambda} + \int_0^t e^{-\mu(t-t')(\frac{9}{16})^\alpha 2^{2\alpha j}} \|\widehat{\dot{\Delta}_j f}\|_{M_p^\lambda} dt'.$$

For any $0 \leq t_0 \leq T$, we get

$$\sup_{0 \leq t \leq t_0} \|\widehat{\dot{\Delta}_j u}\|_{M_p^\lambda} \leq \|\widehat{\dot{\Delta}_j u_0}\|_{M_p^\lambda} + \|\widehat{\dot{\Delta}_j f}\|_{L^1([0, t_0], M_p^\lambda)}.$$

Therefore

$$\begin{aligned} \left(\sum_{j \in \mathbb{Z}} 2^{jq_s} \|\widehat{\dot{\Delta}_j u}\|_{L^\infty(I, M_p^\lambda)}^q \right)^{1/q} &\leq \left(\sum_{j \in \mathbb{Z}} 2^{jq_s} \|\widehat{\dot{\Delta}_j u_0}\|_{M_p^\lambda}^q \right)^{1/q} \\ &\quad + \left(\sum_{j \in \mathbb{Z}} 2^{jq_s} \|\widehat{\dot{\Delta}_j f}\|_{L^1([0, t_0], M_p^\lambda)}^q \right)^{1/q}. \end{aligned}$$

We conclude that

$$\|u\|_{\mathcal{L}^\infty(I, \dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^s)} \leq \|u_0\|_{\dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^s} + \|u\|_{\mathcal{L}^1(I, \dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^s)}.$$

Collecting the above estimate and the inequality (7), we get the desired result.

Proposition 3.1. *Let $1 \leq p < \infty$, $1 \leq \rho \leq \infty$, $1 \leq q \leq 2$, $\frac{1}{2} < \alpha < \frac{2 + \frac{3}{p'} + \frac{\lambda}{p}}{4 - \frac{2}{\rho}}$, $0 \leq \lambda < n$, and set*

$$X = \mathcal{L}^\infty(I, \dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^{1-2\alpha + \frac{3}{p'} + \frac{\lambda}{p}}) \cap \mathcal{L}^\rho(I, \dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^{1-2\alpha + \frac{3}{p'} + \frac{2\alpha}{\rho} + \frac{\lambda}{p}}),$$

with the norm

$$\|u\|_X = \|u\|_{\mathcal{L}^\infty(I, \dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^{1-2\alpha + \frac{3}{p'} + \frac{\lambda}{p}})} + \mu \|u\|_{\mathcal{L}^\rho(I, \dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^{1-2\alpha + \frac{3}{p'} + \frac{2\alpha}{\rho} + \frac{\lambda}{p}})}.$$

There exists a constant $C = C(\alpha, p, q) > 0$ depending on α, p, q such that

$$\|\nabla \cdot (u \otimes v)\|_{\mathcal{L}^\infty(I, \dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^{1-4\alpha + \frac{3}{p'} + \frac{2\alpha}{\rho} + \frac{\lambda}{p}})} \leq C\mu^{-1} \|u\|_X \|v\|_X. \quad (8)$$

Proof. Let us introduce some notations about the standard localization operators. We set

$$u_j = \dot{\Delta}_j u, \quad \dot{S}_j u = \sum_{k \leq j-1} \dot{\Delta}_k u, \quad \widetilde{\dot{\Delta}}_j u = \sum_{|k-j| \leq 1} \dot{\Delta}_k u, \quad \forall j \in \mathbb{Z}.$$

Using Bony's paraproduct decomposition and the quasi-orthogonality property for the Littlewood-Paley decomposition, for fixed j , we have

$$\begin{aligned} \dot{\Delta}_j(uv) &= \sum_{|k-j| \leq 4} \dot{\Delta}_j(\dot{S}_{k-1} u \dot{\Delta}_k v) + \sum_{|k-j| \leq 4} \dot{\Delta}_j(\dot{S}_{k-1} v \dot{\Delta}_k u) + \sum_{k \geq j-3} \dot{\Delta}_j(\dot{\Delta}_k u \widetilde{\dot{\Delta}}_k v) \\ &= I_j + II_j + III_j. \end{aligned}$$

For the proof of this proposition, we can write

$$\begin{aligned} &\|\partial(uv)\|_{\mathcal{L}^\rho(I, \dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^{1-4\alpha + \frac{3}{p'} + \frac{2\alpha}{\rho} + \frac{\lambda}{p}})} \\ &\lesssim \left\{ \sum_{j \in \mathbb{Z}} 2^{j(-2(2\alpha-1) + \frac{3}{p'} + \frac{2\alpha}{\rho} + \frac{\lambda}{p})q} \|\widehat{\dot{\Delta}_j(uv)}\|_{L^\rho(I, M_p^\lambda)}^q \right\}^{1/q} \\ &\lesssim \left\{ \sum_{j \in \mathbb{Z}} 2^{j(-2(2\alpha-1) + \frac{3}{p'} + \frac{2\alpha}{\rho} + \frac{\lambda}{p})q} \|\widehat{I_j}\|_{L^\rho(I, M_p^\lambda)}^q \right\}^{1/q} \\ &\quad + \left\{ \sum_{j \in \mathbb{Z}} 2^{j(-2(2\alpha-1) + \frac{3}{p'} + \frac{2\alpha}{\rho} + \frac{\lambda}{p})q} \|\widehat{II_j}\|_{L^\rho(I, M_p^\lambda)}^q \right\}^{1/q} \end{aligned}$$

$$+ \left\{ \sum_{j \in \mathbb{Z}} 2^{j(-2(2\alpha-1) + \frac{3}{p'} + \frac{2\alpha}{\rho} + \frac{\lambda}{p})q} \|\widehat{III_j}\|_{L^\rho(I, M_p^\lambda)}^q \right\}^{1/q}. \quad (9)$$

We estimate the above three terms separately. First, using Young's inequality (3) in Morrey spaces, and Lemma 2.3 with $|\gamma| = 0$, we get

$$\begin{aligned} \|\widehat{I_j}\|_{L^\rho(I, M_p^\lambda)} &\leq \sum_{|k-j| \leq 4} \|\widehat{\dot{S}_{k-1} u \dot{\Delta}_k v}\|_{L^\rho(I, M_p^\lambda)} \\ &\leq \sum_{|k-j| \leq 4} \|\widehat{v_k}\|_{L^\rho(I, M_p^\lambda)} \sum_{l \leq k-2} \|\widehat{u_l}\|_{L^\infty(I, L^1)} \\ &\leq \sum_{|k-j| \leq 4} \|\widehat{v_k}\|_{L^\rho(I, M_p^\lambda)} \sum_{l \leq k-2} 2^{l(\frac{3}{p'} + \frac{\lambda}{p})} \|\widehat{u_l}\|_{L^\infty(I, M_p^\lambda)} \\ &\leq \sum_{|k-j| \leq 4} \|\widehat{v_k}\|_{L^\rho(I, M_p^\lambda)} \sum_{l \leq k-2} 2^{l(\frac{3}{p'} + \frac{\lambda}{p})} 2^{-l(2\alpha-1)} 2^{l(2\alpha-1)} \|\widehat{u_l}\|_{L^\infty(I, M_p^\lambda)} \\ &\leq \sum_{|k-j| \leq 4} \|\widehat{v_k}\|_{L^\rho(I, M_p^\lambda)} \left(\sum_{l \leq k-2} 2^{l(2\alpha-1)q'} \right)^{\frac{1}{q'}} \|u\|_{\mathcal{L}^\infty(I, \mathcal{FN}_{p, \lambda, q}^{1-2\alpha + \frac{3}{p'} + \frac{\lambda}{p}})} \\ &\lesssim \sum_{|k-j| \leq 4} 2^{k(2\alpha-1)} \|\widehat{v_k}\|_{L^\rho(I, M_p^\lambda)} \|u\|_{\mathcal{L}^\infty(I, \mathcal{FN}_{p, \lambda, q}^{1-2\alpha + \frac{3}{p'} + \frac{\lambda}{p}})}. \end{aligned}$$

Multiplying by $2^{j(-2(2\alpha-1) + \frac{3}{p'} + \frac{2\alpha}{\rho} + \frac{\lambda}{p})}$, and taking l^q -norm of both sides in the above estimate, we obtain

$$\begin{aligned} &\left\{ \sum_{j \in \mathbb{Z}} 2^{j(-2(2\alpha-1) + \frac{3}{p'} + \frac{2\alpha}{\rho} + \frac{\lambda}{p})q} \|\widehat{I_j}\|_{L^\rho(I, M_p^\lambda)}^q \right\}^{1/q} \\ &\lesssim \|v\|_{\mathcal{L}^\rho(I, \mathcal{FN}_{p, \lambda, q}^{1-2\alpha + \frac{3}{p'} + \frac{2\alpha}{\rho} + \frac{\lambda}{p}})} \|u\|_{\mathcal{L}^\infty(I, \mathcal{FN}_{p, \lambda, q}^{1-2\alpha + \frac{3}{p'} + \frac{\lambda}{p}})}. \end{aligned} \quad (10)$$

In a similar way we prove that

$$\begin{aligned} &\left\{ \sum_{j \in \mathbb{Z}} 2^{j(-2(2\alpha-1) + \frac{3}{p'} + \frac{2\alpha}{\rho} + \frac{\lambda}{p})q} \|\widehat{II_j}\|_{L^\rho(I, M_p^\lambda)}^q \right\}^{1/q} \\ &\lesssim \|u\|_{\mathcal{L}^\rho(I, \mathcal{FN}_{p, \lambda, q}^{1-2\alpha + \frac{3}{p'} + \frac{2\alpha}{\rho} + \frac{\lambda}{p}})} \|v\|_{\mathcal{L}^\infty(I, \mathcal{FN}_{p, \lambda, q}^{1-2\alpha + \frac{3}{p'} + \frac{\lambda}{p}})}. \end{aligned} \quad (11)$$

To estimate III_j , let

$$III_{jk} := \dot{\Delta}_j \left(\sum_{|i-k| \leq 1} \dot{\Delta}_i v \dot{\Delta}_k u \right) = \sum_{i=-1}^1 \dot{\Delta}_j (\dot{\Delta}_k u \dot{\Delta}_{i+k} v).$$

The estimate of the so-called "remainder term" requires a different approach. First we use the Young inequality (3) in Morrey spaces, and Lemma 2.3 with $|\gamma| = 0$, to obtain

$$2^{j(-2(2\alpha-1) + \frac{3}{p'} + \frac{2\alpha}{\rho} + \frac{\lambda}{p})} \|\widehat{III_{jk}}\|_{L^\rho(I, M_p^\lambda)}$$

$$\begin{aligned}
&\leq \sum_{i=-1}^1 2^{j(-2(2\alpha-1)+\frac{3}{p'}+\frac{2\alpha}{\rho}+\frac{\lambda}{p})} \left\| \widehat{\Delta_k u \Delta_{i+k} v} \right\|_{L^\rho(I, M_p^\lambda)} \\
&\leq \sum_{i=-1}^1 2^{j(-2(2\alpha-1)+\frac{3}{p'}+\frac{2\alpha}{\rho}+\frac{\lambda}{p})} \left\| \widehat{u}_k \right\|_{L^\rho(I, M_p^\lambda)} \left\| \widehat{v}_{i+k} \right\|_{L^\infty(I, L^1)} \\
&\leq \sum_{i=-1}^1 2^{j(-2(2\alpha-1)+\frac{3}{p'}+\frac{2\alpha}{\rho}+\frac{\lambda}{p})} 2^{(i+k)(\frac{3}{p'}+\frac{\lambda}{p})} \left\| \widehat{u}_k \right\|_{L^\rho(I, M_p^\lambda)} \left\| \widehat{v}_{i+k} \right\|_{L^\infty(I, M_p^\lambda)} \\
&= \sum_{i=-1}^1 2^{(-2(2\alpha-1)+\frac{3}{p'}+\frac{2\alpha}{\rho}+\frac{\lambda}{p})(j-k)} 2^{(2\alpha-1)i} \left(2^{(-2(2\alpha-1)+\frac{3}{p'}+\frac{2\alpha}{\rho}+\frac{\lambda}{p})k} \left\| \widehat{u}_k \right\|_{L^\rho(I, M_p^\lambda)} \right) \\
&\quad \times \left(2^{(-2(2\alpha-1)+\frac{3}{p'}+\frac{\lambda}{p})(i+k)} \left\| \widehat{v}_{i+k} \right\|_{L^\infty(I, M_p^\lambda)} \right).
\end{aligned}$$

Taking the l^q -norm on both sides in the above estimate, Hölder's inequalities for series, and noticing that $\alpha < \frac{2+\frac{3}{p'}+\frac{\lambda}{p}}{4-\frac{2}{\rho}}$ implies that $-2(2\alpha-1) + \frac{2\alpha}{\rho} + \frac{3}{p'} + \frac{\lambda}{q} > 0$, we get

$$\begin{aligned}
&\left(\sum_{j \in \mathbb{Z}} 2^{j(-2(2\alpha-1)+\frac{3}{p'}+\frac{2\alpha}{\rho}+\frac{\lambda}{p})q} \left\| \widehat{II I_j} \right\|_{L^\rho(I, M_p^\lambda)}^q \right)^{\frac{1}{q}} \\
&\leq \left(\sum_{j \in \mathbb{Z}} \left(\sum_{l \leq 3} \sum_{i=-1}^1 2^{(-2(2\alpha-1)+\frac{3}{p'}+\frac{2\alpha}{\rho}+\frac{\lambda}{p})l} 2^{(2\alpha-1)i} 2^{(-2(2\alpha-1)+\frac{3}{p'}+\frac{2\alpha}{\rho}+\frac{\lambda}{p})(j-l)} \right. \right. \\
&\quad \times \left. \left\| \widehat{u}_{j-l} \right\|_{L^\rho(I, M_p^\lambda)} 2^{(-2(2\alpha-1)+\frac{3}{p'}+\frac{\lambda}{p})(j-l+i)} \left\| \widehat{v}_{j-l+i} \right\|_{L^\infty(I, M_p^\lambda)} \right)^q \right)^{\frac{1}{q}} \\
&\leq \sum_{i=-1}^1 \sum_{l \leq 3} 2^{(-2(2\alpha-1)+\frac{3}{p'}+\frac{2\alpha}{\rho}+\frac{\lambda}{p})l} 2^{2(\alpha-1)i} \|u\|_{\mathcal{L}^\rho(I, \dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^{1-2\alpha+\frac{3}{p'}+\frac{2\alpha}{\rho}+\frac{\lambda}{p}})} \\
&\quad \times \|v\|_{\mathcal{L}^\infty(I, \dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q'}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}})}.
\end{aligned}$$

Since $1 \leq q \leq 2$ we have $q \leq q'$ and $\dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}} \hookrightarrow \dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q'}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}$, we obtain

$$\begin{aligned}
&\left(\sum_{j \in \mathbb{Z}} 2^{j(-2(2\alpha-1)+\frac{3}{p'}+\frac{2\alpha}{\rho}+\frac{\lambda}{q})q} \left\| \widehat{II I_j} \right\|_{L^\rho(I, M_p^\lambda)}^q \right)^{\frac{1}{q}} \\
&\lesssim \|u\|_{\mathcal{L}^\rho(I, \dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^{1-2\alpha+\frac{3}{p'}+\frac{2\alpha}{\rho}+\frac{\lambda}{p}})} \|v\|_{\mathcal{L}^\infty(I, \dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}})}. \tag{12}
\end{aligned}$$

Combing estimates (9), (10), (11), (12) yield (8). $\square$

The following lemma can be found in [21].

Lemma 3.2. *Let X be a Banach space with norm $\|\cdot\|_X$ and $B : X \times X \mapsto X$ be a bounded bilinear operator satisfying*

$$\|B(u, v)\|_X \leq \eta \|u\|_X \|v\|_X$$

for all $u, v \in X$ and a constant $\eta > 0$. Then, if $0 < \varepsilon < \frac{1}{4\eta}$ and if $y \in X$ such that $\|y\|_X \leq \varepsilon$, the equation $x := y + B(x, x)$ has a solution $\bar{x}$ in X such that $\|\bar{x}\|_X \leq 2\varepsilon$. This solution is

the only one in the ball $\overline{B}(0, 2\varepsilon)$. Moreover, the solution depends continuously on y in the sense: if $\|y'\|_X < \varepsilon$, $x' = y' + B(x', x')$, and $\|x'\|_X \leq 2\varepsilon$, then

$$\|\bar{x} - x'\|_X \leq \frac{1}{1 - 4\varepsilon\eta} \|y - y'\|_X.$$

Proof of Theorem 2.1. We will use Lemma 3.2 to ensure the existence of global mild solutions with small initial data. Note that the functions here are vector fields, whose norm is the sum of the norms of the three components. Here, as usual, we begin with the mild integral equation

$$\begin{aligned} u(t, x) &= e^{-\mu t(-\Delta)^\alpha} u_0 - \int_0^t e^{-\mu(t-\tau)(-\Delta)^\alpha} \mathbb{P}\nabla \cdot (u \otimes u)(\tau, x) d\tau \\ &= e^{-\mu t(-\Delta)^\alpha} u_0 + B(u, u) \end{aligned} \quad (13)$$

where $\mathbb{P} = Id - \nabla \Delta^{-1} \nabla$ is the Leray-Hopf projector, which is a pseudo differential operator of order 0.

We notice that $B(u, v)$ can be thought as the solution to the heat equation (6) with $u_0 = 0$ and force $f = \mathbb{P}\nabla \cdot (u \otimes v)$. According to Lemma 3.1 with $s = 1 - 2\alpha + \frac{3}{p'} + \frac{\lambda}{p}$ and Proposition 3.1 with $\rho = 1$, we obtain

$$\begin{aligned} \|B(u, v)\|_X &\leq \left(1 + \left(\frac{16}{9}\right)^\alpha\right) \|\mathbb{P}\nabla \cdot (u \otimes v)\|_{\mathcal{L}^1(I, \dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}})} \\ &\leq \left(1 + \left(\frac{16}{9}\right)^\alpha\right) C\mu^{-1} \|u\|_X \|v\|_X. \end{aligned}$$

By lemma 3.2, we know that if $\|e^{-\mu t(-\Delta)^\alpha} u_0\|_X < R$ with $R = \frac{\mu}{4(1+(\frac{16}{9})^\alpha)C}$

then the equation (13) has a unique solution in

$B(0, 2R) := \{x \in X : \|x\|_X \leq 2R\}$. To show $\|e^{-\mu t(-\Delta)^\alpha} u_0\|_X < R$, notice that $e^{-\mu t(-\Delta)^\alpha} u_0$ is the solution to the dissipative equation with $u_0 = u_0$ and $f = 0$. So, Lemma 3.1 yields

$$\|e^{-\mu t(-\Delta)^\alpha} u_0\|_X \leq \left(1 + \left(\frac{16}{9}\right)^\alpha\right) \|u_0\|_{\dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}}.$$

If $\|u_0\|_{\dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}} \leq C_0\mu$ with $C_0 = \frac{1}{4(1+(\frac{16}{9})^\alpha)2C}$, then (13) has a unique global solution $u \in X$ satisfying

$$\|u\|_X \leq 2 \left(1 + \left(\frac{16}{9}\right)^\alpha\right) \|u_0\|_{\dot{\mathcal{F}}\dot{\mathcal{N}}_{p, \lambda, q}^{1-2\alpha+\frac{3}{p'}+\frac{\lambda}{p}}}.$$

The continuity with respect to time is standard. $\square$

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