Moroccan J. Pure and Appl. Anal.(MJPAA)
 Volume 2(1), 2016, Pages 12–32
 ISSN: 2351-8227

RESEARCH ARTICLE

Generalization of Some Inequalities for Differentiable Co-ordinated Convex Functions With Applications

M. A. LATIF^a, S. S. DRAGOMIR^{a,b} AND E. MOMONIAT^a

ABSTRACT. In this paper, a new weighted identity for functions defined on a rectangle from the plane is established. By using the obtained identity and analysis, some new weighted integral inequalities for the classes of co-ordinated convex, co-ordinated wright-convex and co-ordinated quasi-convex functions on the rectangle from the plane are established which provide weighted generalization of some recent results proved for co-ordinated convex functions. Some applications of our results to random variables and 2D weighted quadrature formula are given as well.

2000 Mathematics Subject Classification. 26D15, 26D20, 26D07.

Key words and phrases. Hermite-Hadamard's inequality, co-ordinated convex function, co-ordinated wright-convex function, co-ordinated quasi-convex function, Hölder's integral inequality, quadrature formula.

1. Introduction

The following definition is well known in mathematical analysis: A function $f : I \rightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$, is said to be convex on I if the inequality

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y),$$

Received February 16, 2016 - Accepted March 30, 2016.

©The Author(s) 2016. This article is published with open access by Sidi Mohamed Ben Abdallah University

^a*School of Computational and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Wits 2050, Johannesburg, South Africa.*

e-mail: m_amer_latif@hotmail.com, ebrahim.momoniat@wits.ac.za.

^b*School of Engineering and Science, Victoria University, PO Box 14428 Melbourne City, MC 8001, Australia.*

e-mail: sever.dragomir@vu.edu.au.

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

A number of results have been established for the class of convex functions but the most famous is the Hermite-Hadamard's inequality (see for instance [7]). This double inequality is stated as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}, \quad (1)$$

where $f : I \rightarrow \mathbb{R}$, $\emptyset \neq I \subseteq \mathbb{R}$ a convex function, $a, b \in I$ with $a < b$. The inequalities in (1) are reversed if f is a concave function.

The inequalities (1) have become an important cornerstone in mathematical analysis and optimization and many uses of these inequalities have been discovered in a variety of settings. Moreover, many inequalities of special means can be obtained for a particular choice of the function f . Due to the rich geometrical significance of Hermite-Hadamard's inequality (1), there is growing literature providing its new proofs, extensions, refinements and generalizations, see for example [2, 4, 5, 6, 9, 21, 22] and the references therein.

Let us consider now a bidimensional interval $[a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$.

A mapping $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is said to be convex on $[a, b] \times [c, d]$ if the inequality

$$f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) \leq \lambda f(x, y) + (1 - \lambda)f(z, w)$$

holds for all $(x, y), (z, w) \in [a, b] \times [c, d]$ and $\lambda \in [0, 1]$.

A modification for convex functions on $[a, b] \times [c, d]$, which are also known as co-ordinated convex functions, was initiated by Dragomir [4, 6] as follows:

A function $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is said to be convex on the co-ordinates on $[a, b] \times [c, d]$ if the partial mappings $f_y : [a, b] \rightarrow \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbb{R}$, $f_x(v) = f(x, v)$ are convex where defined for all $x \in [a, b]$, $y \in [c, d]$.

A formal definition for co-ordinated convex functions may be stated as follows:

Definition 1.1. [13] *A function $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is said to be convex on the co-ordinates on $[a, b] \times [c, d]$ if the inequality*

$$\begin{aligned} & f(tx + (1 - t)y, sz + (1 - s)w) \\ & \leq tsf(x, z) + t(1 - s)f(x, w) + s(1 - t)f(y, z) + (1 - t)(1 - s)f(y, w) \end{aligned}$$

holds for all $(t, s) \in [0, 1] \times [0, 1]$ and $(x, z), (y, w) \in [a, b] \times [c, d]$.

It has been proved in [4] that every convex mapping $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex, (see for example [4, 6]).

The following Hermite-Hadamard type inequality for co-ordinated convex functions on the rectangle from the plane $\mathbb{R}^2$ was also proved in [4]:

Theorem 1.1. [4] *Suppose that $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is co-ordinated convex on $[a, b] \times [c, d]$. Then one has the inequalities:*

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right)$$

$$\begin{aligned}
&\leq \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f \left(x, \frac{c+d}{2} \right) dx + \frac{1}{d-c} \int_c^d f \left(\frac{a+b}{2}, y \right) dy \right] \\
&\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx \\
&\leq \frac{1}{4} \left[\frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \\
&\quad \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \\
&\leq \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4}. \quad (2)
\end{aligned}$$

The above inequalities are sharp.

Sarikaya et al. [23], proved the following Hermite-Hadamard type inequalities.

Theorem 1.2. [23] Let $f : [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $[a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. If $\left| \frac{\partial^2 f}{\partial s \partial t} \right|$ is convex on the co-ordinates on $[a, b] \times [c, d]$, then one has the inequalities:

$$\begin{aligned}
&\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right. \\
&\quad \left. - \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \right. \\
&\quad \left. \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \right| \\
&\leq \frac{(b-a)(d-c)}{16} \left[\left| \frac{\partial^2 f}{\partial s \partial t}(a, c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(a, d) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b, c) \right| + \left| \frac{\partial^2 f}{\partial s \partial t}(b, d) \right| \right]. \quad (3)
\end{aligned}$$

The next two results from [23] involve powers of the absolute value of $\frac{\partial^2 f}{\partial s \partial t}$.

Theorem 1.3. [23] Let $f : [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $[a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. If $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$, $q \geq 1$, is convex on the co-ordinates on $[a, b] \times [c, d]$, then one has the inequalities:

$$\begin{aligned}
&\left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right. \\
&\quad \left. - \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \right. \\
&\quad \left. \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \right|
\end{aligned}$$

$$\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{p}}} \left[\frac{\left| \frac{\partial^2 f}{\partial s \partial t}(a, c) \right|^q + \left| \frac{\partial^2 f}{\partial s \partial t}(a, d) \right|^q + \left| \frac{\partial^2 f}{\partial s \partial t}(b, c) \right|^q + \left| \frac{\partial^2 f}{\partial s \partial t}(b, d) \right|^q}{4} \right]^{\frac{1}{q}}, \quad (4)$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 1.4. [23] *Let $f : [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a partial differentiable mapping on $[a, b] \times [c, d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. If $\left| \frac{\partial^2 f}{\partial s \partial t} \right|^q$, $q > 1$, is convex on the co-ordinates on $[a, b] \times [c, d]$, then one has the inequalities:*

$$\begin{aligned} & \left| \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx + \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right. \\ & \quad \left. - \frac{1}{2} \left[\frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx \right. \right. \\ & \quad \left. \left. + \frac{1}{d-c} \int_c^d f(a, y) dy + \frac{1}{d-c} \int_c^d f(b, y) dy \right] \right| \\ & \leq \frac{(b-a)(d-c)}{16} \left[\frac{\left| \frac{\partial^2 f}{\partial s \partial t}(a, c) \right|^q + \left| \frac{\partial^2 f}{\partial s \partial t}(a, d) \right|^q + \left| \frac{\partial^2 f}{\partial s \partial t}(b, c) \right|^q + \left| \frac{\partial^2 f}{\partial s \partial t}(b, d) \right|^q}{4} \right]^{\frac{1}{q}}. \quad (5) \end{aligned}$$

In a recent paper [22], M. E. Özdemir et al. give the notion of co-ordinated quasi-convex functions which generalize the notion of co-ordinated convex functions.

Definition 1.2. [20] *A function $f : [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be quasi-convex on $[a, b] \times [c, d]$ if the inequality*

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \max \{f(x, y), f(z, w)\}$$

holds for all $(x, y), (z, w) \in [a, b] \times [c, d]$ and $\lambda \in [0, 1]$.

A function $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is said to be quasi-convex on the co-ordinates on $[a, b] \times [c, d]$ if the partial mappings $f_y : [a, b] \rightarrow \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbb{R}$, $f_x(v) = f(x, v)$ are quasi-convex where defined for all $x \in [a, b]$, $y \in [c, d]$.

Another way of describing the definition of co-ordinated quasi-convex functions is given below.

Definition 1.3. [16] *A function $f : [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be quasi-convex on the co-ordinates on $[a, b] \times [c, d]$ if*

$$f(tx + (1-t)z, sy + (1-s)w) \leq \max \{f(x, y), f(x, w), f(z, y), f(z, w)\}$$

for all $(x, y), (z, w) \in [a, b] \times [c, d]$ and $(s, t) \in [0, 1] \times [0, 1]$.

The class of co-ordinated quasi-convex functions on $[a, b] \times [c, d]$ is denoted by $QC([a, b] \times [c, d])$. It has also been proved in [20] that every quasi-convex functions on

$[a, b] \times [c, d]$ is quasi-convex on the co-ordinates on $[a, b] \times [c, d]$. The following example reveals that there exists quasi-convex function on the co-ordinates which is not quasi-convex.

Example 1.1. [16] *The function $f : [-2, 2]^2 \rightarrow \mathbb{R}$, defined by $f(x, y) = \lfloor x \rfloor \lfloor y \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function. This function is quasi-convex on the co-ordinates on $[-2, 2]^2$ but is not quasi-convex on $[0, 1]^2$.*

For example, take $(x, y) = (-2, 1)$, $(z, w) = (1, -1)$ and $\lambda = \frac{1}{2}$, then

$$f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) = f\left(-\frac{1}{2}, 0\right) = 0,$$

on the other hand

$$\max \{f(x, y), f(z, w)\} = \max \{f(-2, 1), f(1, -1)\} = -1,$$

which shows that $f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) > \max \{f(x, y), f(z, w)\}$.

Another generalization of the notion of the co-ordinated convex functions is the concept of wright-convex functions which is given in the definition below.

Definition 1.4. [20] *A function $f : [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be wright-convex on $[a, b] \times [c, d]$ if the inequality*

$$\begin{aligned} & f(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w) + f((1 - \lambda)x + \lambda z, (1 - \lambda)y + \lambda w) \\ & \leq \max \{f(x, z), f(y, w)\}, \end{aligned}$$

holds for all $(x, z), (y, w) \in [a, b] \times [c, d]$ and $\lambda \in [0, 1]$.

A function $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is said to be wright-convex on the co-ordinates on $[a, b] \times [c, d]$ if the partial mappings $f_y : [a, b] \rightarrow \mathbb{R}$, $f_y(u) = f(u, y)$ and $f_x : [c, d] \rightarrow \mathbb{R}$, $f_x(v) = f(x, v)$ are wright-convex where defined for all $x \in [a, b]$, $y \in [c, d]$.

The above definition of wright-convex functions on the co-ordinates can be reformulated as follows.

Definition 1.5. [20] *A function $f : [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is said to be wright-convex on the co-ordinates on $[a, b] \times [c, d]$ if*

$$\begin{aligned} & f(tx + (1 - t)z, sy + (1 - s)w) + f((1 - t)x + tz, (1 - s)y + sw) \\ & \leq f(x, y) + f(z, y) + f(x, w) + f(z, w) \end{aligned}$$

for all $(x, z), (y, w) \in [a, b] \times [c, d]$ and $(s, t) \in [0, 1] \times [0, 1]$.

The class of co-ordinated wright-convex functions on $[a, b] \times [c, d]$ is represented by $W([a, b] \times [c, d])$. It has also been proved in [20] that every wright-convex functions on $[a, b] \times [c, d]$ is wright-convex on the co-ordinates on $[a, b] \times [c, d]$.

For more recent results on co-ordinated convex, co-ordinated quasi-convex, co-ordinated m -convex, co-ordinated (α, m) -convex and co-ordinated s -convex functions on a rectangle $[a, b] \times [c, d]$ from the plane $\mathbb{R}^2$, we refer the readers to [1, 5, 8], [10]-[20].

In the present paper, we establish a new weighted identity for differentiable mappings defined on a rectangle $[a, b] \times [c, d]$ from the plane $\mathbb{R}^2$ and by using the obtained

identity and analysis, some new weighted integral inequalities for differentiable co-ordinated convex, co-ordinated wright-convex and co-ordinated quasi convex functions are proved. The results proved in the paper provide a weighted generalization of the results given in Theorem 1.2, Theorem 1.3 and Theorem 1.4. Applications of our results to random variables and $2D$ weighted quadrature formula are provided as well.

2. Main Results

We need the following lemma to prove our results. Moreover, the following notions will be used throughout in the section

$$\begin{aligned}
U_1(a, b, t) &= U_1(t) = \frac{1-t}{2}a + \frac{1+t}{2}b, L_1(a, b, t) = L_1(t) = \frac{1+t}{2}a + \frac{1-t}{2}b, \\
U_2(c, d, s) &= U_2(s) = \frac{1-s}{2}c + \frac{1+s}{2}d, L_2(c, d, s) = L_2(s) = \frac{1+s}{2}c + \frac{1-s}{2}d, \\
\Psi(a, b, c, d; |f_{ts}|) &= \frac{|f_{ts}(a, c)| + |f_{ts}(a, d)| + |f_{ts}(b, c)| + |f_{ts}(b, d)|}{4}, \\
\lambda_1\left(b, d, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}|\right) &= \max \left\{ |f_{ts}(b, d)|, \left|f_{ts}\left(b, \frac{c+d}{2}\right)\right|, \left|f_{ts}\left(\frac{a+b}{2}, d\right)\right|, \left|f_{ts}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right| \right\}, \\
\lambda_2\left(a, d, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}|\right) &= \max \left\{ |f_{ts}(a, d)|, \left|f_{ts}\left(a, \frac{c+d}{2}\right)\right|, \left|f_{ts}\left(\frac{a+b}{2}, d\right)\right|, \left|f_{ts}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right| \right\}, \\
\lambda_3\left(b, c, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}|\right) &= \max \left\{ |f_{ts}(b, c)|, \left|f_{ts}\left(b, \frac{c+d}{2}\right)\right|, \left|f_{ts}\left(\frac{a+b}{2}, c\right)\right|, \left|f_{ts}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right| \right\}
\end{aligned}$$

and

$$\begin{aligned}
\lambda_4\left(a, c, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}|\right) &= \max \left\{ |f_{ts}(a, c)|, \left|f_{ts}\left(a, \frac{c+d}{2}\right)\right|, \left|f_{ts}\left(\frac{a+b}{2}, c\right)\right|, \left|f_{ts}\left(\frac{a+b}{2}, \frac{c+d}{2}\right)\right| \right\}.
\end{aligned}$$

Lemma 2.1. *Let $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice partially differentiable mapping on Δ° and $p : [a, b] \times [c, d] \rightarrow [0, \infty)$ be continuous and symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ for $[a, b] \times [c, d] \subset \Delta^\circ$ with $a < b, c < d$. If $f_{ts} \in L([a, b] \times [c, d])$, then*

$$\Phi(a, b, c, d; p, f) = \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \int_c^d \int_a^b p(x, y) dx dy$$

$$\begin{aligned}
& -\frac{1}{2} \int_c^d \int_a^b [f(x, c) + f(x, d)] p(x, y) dx dy \\
& -\frac{1}{2} \int_c^d \int_a^b [f(a, y) + f(b, y)] p(x, y) dx dy + \int_c^d \int_a^b f(x, y) p(x, y) dx dy \\
& = \frac{(b-a)(d-c)}{16} \int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right] [f_{ts}(U_1(t), U_2(s)) \\
& \quad - f_{ts}(U_1(t), L_2(s)) - f_{ts}(L_1(t), U_2(s)) + f_{ts}(L_1(t), L_2(s))] ds dt. \quad (6)
\end{aligned}$$

Proof. Let

$$\begin{aligned}
I &= \frac{(b-a)(d-c)}{16} \int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right] [f_{ts}(U_1(t), U_2(s)) \\
& \quad - f_{ts}(U_1(t), L_2(s)) - f_{ts}(L_1(t), U_2(s)) + f_{ts}(L_1(t), L_2(s))] ds dt
\end{aligned}$$

and

$$\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy = q(t, s).$$

then

$$\begin{aligned}
I &= \frac{(b-a)(d-c)}{16} \int_0^1 \int_0^1 q(t, s) [f_{ts}(U_1(t), U_2(s)) - f_{ts}(U_1(t), L_2(s)) \\
& \quad - f_{ts}(L_1(t), U_2(s)) + f_{ts}(L_1(t), L_2(s))] ds dt.
\end{aligned}$$

Now by integration by parts and by using the symmetry of $p(x, y)$ about $x = \frac{a+b}{2}$ and $y = \frac{c+d}{2}$, we have

$$\begin{aligned}
& \frac{(b-a)(d-c)}{16} \int_0^1 \int_0^1 q(t, s) f_{ts}(U_1(t), U_2(s)) ds dt \\
& = \frac{(b-a)(d-c)}{16} \int_0^1 \left[\int_0^1 q(t, s) f_{ts}(U_1(t), U_2(s)) ds \right] dt \\
& = \frac{(b-a)(d-c)}{16} \int_0^1 \left[\frac{2}{d-c} q(t, s) f_t(U_1(t), U_2(s)) \Big|_0^1 \right. \\
& \quad \left. - \frac{2}{d-c} \int_0^1 q_s(t, s) f_t(U_1(t), U_2(s)) ds \right] dt \\
& = \frac{(b-a)}{8} \int_0^1 \left[f_t(U_1(t), d) \left(\int_c^d \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right) \right. \\
& \quad \left. - (d-c) \int_0^1 \left(\int_{L_1(t)}^{U_1(t)} p(x, U_2(s)) dx \right) f_t(U_1(t), U_2(s)) ds \right] dt \\
& = \frac{(b-a)}{8} \int_0^1 f_t(U_1(t), d) \left(\int_c^d \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right) dt
\end{aligned}$$

$$\begin{aligned}
& - \frac{(b-a)}{4} \int_{\frac{c+d}{2}}^d \int_0^1 \left(\int_{L_1(t)}^{U_1(t)} p(x, y) dx \right) f_t(U_1(t), y) dt dy \\
& = \frac{1}{4} f(b, d) \int_c^d \int_a^b p(x, y) dx dy - \frac{1}{2} \int_c^d \int_{\frac{a+b}{2}}^b p(x, y) f(x, d) dx dy \\
& \quad - \frac{1}{2} \int_{\frac{c+d}{2}}^d \int_a^b p(x, y) f(b, y) dx dy + \int_{\frac{c+d}{2}}^d \int_{\frac{a+b}{2}}^b p(x, y) f(x, y) dx dy. \quad (7)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& - \frac{(b-a)(d-c)}{16} \int_0^1 \int_0^1 q(t, s) f_{ts}(U_1(t), L_2(s)) ds dt \\
& = \frac{1}{4} f(b, c) \int_c^d \int_a^b p(x, y) dx dy - \frac{1}{2} \int_c^d \int_{\frac{a+b}{2}}^b p(x, y) f(x, c) dx dy \\
& \quad - \frac{1}{2} \int_{\frac{c+d}{2}}^d \int_a^b p(x, y) f(b, y) dx dy + \int_{\frac{c+d}{2}}^d \int_{\frac{a+b}{2}}^b p(x, y) f(x, y) dx dy, \quad (8)
\end{aligned}$$

$$\begin{aligned}
& - \frac{(b-a)(d-c)}{16} \int_0^1 \int_0^1 q(t, s) f_{ts}(L_1(t), U_2(s)) ds dt \\
& = \frac{1}{4} f(a, d) \int_c^d \int_a^b p(x, y) dx dy - \frac{1}{2} \int_c^d \int_a^{\frac{a+b}{2}} p(x, y) f(x, d) dx dy \\
& \quad - \frac{1}{2} \int_{\frac{c+d}{2}}^d \int_a^b p(x, y) f(a, y) dx dy + \int_{\frac{c+d}{2}}^d \int_a^{\frac{a+b}{2}} p(x, y) f(x, y) dx dy \quad (9)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{(b-a)(d-c)}{16} \int_0^1 \int_0^1 q(t, s) f_{ts}(L_1(t), L_2(s)) ds dt \\
& = \frac{1}{4} f(a, c) \int_c^d \int_a^b p(x, y) dx dy - \frac{1}{2} \int_c^d \int_a^{\frac{a+b}{2}} p(x, y) f(x, c) dx dy \\
& \quad - \frac{1}{2} \int_{\frac{c+d}{2}}^d \int_a^b p(x, y) f(a, y) dx dy + \int_{\frac{c+d}{2}}^d \int_a^{\frac{a+b}{2}} p(x, y) f(x, y) dx dy. \quad (10)
\end{aligned}$$

Adding (7)-(10), we get the desired result. $\square$

Remark 2.1. If we take $p(x, y) = \frac{1}{(b-a)(d-c)}$ for all $(x, y) \in [a, b] \times [c, d]$ in Lemma 2.1, we get Lemma 1 from [23, page 139].

Now by using lemma 2.1, we present the main results of this section.

Theorem 2.1. Let $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice partially differentiable mapping on Δ° and $p : [a, b] \times [c, d] \rightarrow [0, \infty)$ be continuous and symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ for

$[a, b] \times [c, d] \subset \Delta^\circ$ with $a < b$, $c < d$. If $f_{ts} \in L([a, b] \times [c, d])$ and $|f_{ts}|$ is convex on the co-ordinates on $[a, b] \times [c, d]$, then

$$|\Phi(a, b, c, d; p, f)| \leq \frac{(b-a)(d-c)}{4} \Psi(a, b, c, d; |f_{ts}|) \int_0^1 \int_0^1 \int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy dt ds. \quad (11)$$

Proof. Taking absolute value on both sides of (6) and using the properties of absolute value, we have

$$\begin{aligned} |\Phi(a, b, c, d; p, f)| &\leq \frac{(b-a)(d-c)}{16} \int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right] [|f_{ts}(U_1(t), U_2(s))| \\ &\quad + |f_{ts}(U_1(t), L_2(s))| + |f_{ts}(L_1(t), U_2(s))| + |f_{ts}(L_1(t), L_2(s))|] ds dt. \end{aligned} \quad (12)$$

By the convexity of $|f_{ts}|$ on the co-ordinates on $[a, b] \times [c, d]$, we have

$$\begin{aligned} |f_{ts}(U_1(t), U_2(s))| &\leq \left(\frac{1-t}{2}\right) \left(\frac{1-s}{2}\right) |f_{ts}(a, c)| + \left(\frac{1-t}{2}\right) \left(\frac{1+s}{2}\right) |f_{ts}(a, d)| \\ &\quad + \left(\frac{1+t}{2}\right) \left(\frac{1-s}{2}\right) |f_{ts}(b, c)| + \left(\frac{1+t}{2}\right) \left(\frac{1+s}{2}\right) |f_{ts}(b, d)|, \end{aligned} \quad (13)$$

$$\begin{aligned} |f_{ts}(U_1(t), L_2(s))| &\leq \left(\frac{1-t}{2}\right) \left(\frac{1+s}{2}\right) |f_{ts}(a, c)| + \left(\frac{1-t}{2}\right) \left(\frac{1-s}{2}\right) |f_{ts}(a, d)| \\ &\quad + \left(\frac{1+t}{2}\right) \left(\frac{1+s}{2}\right) |f_{ts}(b, c)| + \left(\frac{1+t}{2}\right) \left(\frac{1-s}{2}\right) |f_{ts}(b, d)|, \end{aligned} \quad (14)$$

$$\begin{aligned} |f_{ts}(L_1(t), U_2(s))| &\leq \left(\frac{1+t}{2}\right) \left(\frac{1-s}{2}\right) |f_{ts}(a, c)| + \left(\frac{1+t}{2}\right) \left(\frac{1+s}{2}\right) |f_{ts}(a, d)| \\ &\quad + \left(\frac{1-t}{2}\right) \left(\frac{1-s}{2}\right) |f_{ts}(b, c)| + \left(\frac{1-t}{2}\right) \left(\frac{1+s}{2}\right) |f_{ts}(b, d)|, \end{aligned} \quad (15)$$

and

$$\begin{aligned} |f_{ts}(L_1(t), L_2(s))| &\leq \left(\frac{1+t}{2}\right) \left(\frac{1+s}{2}\right) |f_{ts}(a, c)| + \left(\frac{1+t}{2}\right) \left(\frac{1-s}{2}\right) |f_{ts}(a, d)| \\ &\quad + \left(\frac{1-t}{2}\right) \left(\frac{1+s}{2}\right) |f_{ts}(b, c)| + \left(\frac{1-t}{2}\right) \left(\frac{1-s}{2}\right) |f_{ts}(b, d)|. \end{aligned} \quad (16)$$

Using (13)-(16) in (12), we get (11). $\square$

Remark 2.2. If we take $p(x, y) = \frac{1}{(b-a)(d-c)}$ for all $(x, y) \in [a, b] \times [c, d]$ in Theorem 2.1, we get Theorem 1.2 from [23].

A more general result is given in the following theorem.

Theorem 2.2. Let $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice partially differentiable mapping on Δ° and $p : [a, b] \times [c, d] \rightarrow [0, \infty)$ be continuous and symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ for $[a, b] \times [c, d] \subset \Delta^\circ$ with $a < b$, $c < d$. If $f_{ts} \in L([a, b] \times [c, d])$ and $|f_{ts}|^q$ is convex on the co-ordinates on $[a, b] \times [c, d]$ for $q \geq 1$, then

$$|\Phi(a, b, c, d; p, f)| \leq \frac{(b-a)(d-c)}{4} [\Psi(a, b, c, d; |f_{ts}|^q)]^{\frac{1}{q}} \int_0^1 \int_0^1 \int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy ds dt. \quad (17)$$

Proof. Taking absolute value on both sides of (6), by using the properties of absolute value and the Hölder inequality, we have

$$\begin{aligned} |\Phi(a, b, c, d; p, f)| &\leq \frac{(b-a)(d-c)}{16} \left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right] ds dt \right)^{1-\frac{1}{q}} \\ &\times \left[\left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right] |f_{ts}(U_1(t), U_2(s))|^q ds dt \right)^{\frac{1}{q}} \right. \\ &+ \left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right] |f_{ts}(U_1(t), L_2(s))|^q ds dt \right)^{\frac{1}{q}} \\ &+ \left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right] |f_{ts}(L_1(t), U_2(s))|^q ds dt \right)^{\frac{1}{q}} \\ &\left. + \left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right] |f_{ts}(L_1(t), L_2(s))|^q ds dt \right)^{\frac{1}{q}} \right]. \quad (18) \end{aligned}$$

By the power-mean inequality ($a_1^r + a_2^r + a_3^r + a_4^r \leq 4^{1-r}(a_1 + a_2 + a_3 + a_4)^r$ for $a_1, a_2, a_3, a_4 > 0$ and $r < 1$) and using the convexity of $|f_{ts}|^q$ on the co-ordinates on $[a, b] \times [c, d]$ for $q \geq 1$, we have

$$\begin{aligned} &\left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right] |f_{ts}(U_1(t), U_2(s))|^q ds dt \right)^{\frac{1}{q}} \\ &+ \left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right] |f_{ts}(U_1(t), L_2(s))|^q ds dt \right)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right] |f_{ts}(L_1(t), U_2(s))|^q ds dt \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right] |f_{ts}(L_1(t), L_2(s))|^q ds dt \right)^{\frac{1}{q}} \Bigg] \\
& \leq 4^{1-\frac{1}{q}} [|f_{ts}(a, c)| + |f_{ts}(a, d)| + |f_{ts}(b, c)| + |f_{ts}(b, d)|]^{\frac{1}{q}} \\
& \quad \times \left(\int_0^1 \int_0^1 \int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy ds dt \right)^{\frac{1}{q}}. \quad (19)
\end{aligned}$$

A usage of (19) in (18) yields the desired result. $\square$

Remark 2.3. If we take $p(x, y) = \frac{1}{(b-a)(d-c)}$ for all $(x, y) \in [a, b] \times [c, d]$ in Theorem 2.2, we get Theorem 1.4.

A different approach leads to the following result.

Theorem 2.3. Let $f : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ be a twice differentiable mapping on Δ° and $p : [a, b] \times [c, d] \rightarrow [0, \infty)$ be continuous and symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ for $[a, b] \times [c, d] \subset \Delta^\circ$ with $a < b$, $c < d$. If $f_{ts} \in L([a, b] \times [c, d])$ and $|f_{ts}|^q$ is convex on the co-ordinates on $[a, b] \times [c, d]$ for $q > 1$, then

$$\begin{aligned}
& |\Phi(a, b, c, d; p, f)| \\
& \leq \frac{(b-a)(d-c)}{4} [\Psi(a, b, c, d; |f_{ts}|^q)]^{\frac{1}{q}} \left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right]^p ds dt \right)^{\frac{1}{p}}, \quad (20)
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. From Lemma 2.1 and the Hölder inequality, we have

$$\begin{aligned}
|\Phi(a, b, c, d; p, f)| & \leq \frac{(b-a)(d-c)}{16} \left(\int_0^1 \int_0^1 \left[\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \right]^p ds dt \right)^{\frac{1}{p}} \\
& \times \left[\left(\int_0^1 \int_0^1 |f_{ts}(U_1(t), U_2(s))|^q ds dt \right)^{\frac{1}{q}} + \left(\int_0^1 \int_0^1 |f_{ts}(U_1(t), L_2(s))|^q ds dt \right)^{\frac{1}{q}} \right. \\
& \left. + \left(\int_0^1 \int_0^1 |f_{ts}(L_1(t), U_2(s))|^q ds dt \right)^{\frac{1}{q}} + \left(\int_0^1 \int_0^1 |f_{ts}(L_1(t), L_2(s))|^q ds dt \right)^{\frac{1}{q}} \right]. \quad (21)
\end{aligned}$$

By the power-mean inequality ($a_1^r + a_2^r + a_3^r + a_4^r \leq 4^{1-r}(a_1 + a_2 + a_3 + a_4)^r$ for $a_1, a_2, a_3, a_4 > 0$ and $r < 1$) and using the convexity of $|f_{ts}|^q$ on the co-ordinates on $[a, b] \times [c, d]$ for $q > 1$, we have

$$\begin{aligned}
& \left(\int_0^1 \int_0^1 |f_{ts}(U_1(t), U_2(s))|^q ds dt \right)^{\frac{1}{q}} + \left(\int_0^1 \int_0^1 |f_{ts}(U_1(t), L_2(s))|^q ds dt \right)^{\frac{1}{q}} \\
& + \left(\int_0^1 \int_0^1 |f_{ts}(L_1(t), U_2(s))|^q ds dt \right)^{\frac{1}{q}} + \left(\int_0^1 \int_0^1 |f_{ts}(L_1(t), L_2(s))|^q ds dt \right)^{\frac{1}{q}} \\
& \leq 4^{1-\frac{1}{q}} \left[\int_0^1 \int_0^1 |f_{ts}(U_1(t), U_2(s))|^q ds dt + \int_0^1 \int_0^1 |f_{ts}(U_1(t), L_2(s))|^q ds dt \right. \\
& \quad \left. + \int_0^1 \int_0^1 |f_{ts}(L_1(t), U_2(s))|^q ds dt + \int_0^1 \int_0^1 |f_{ts}(L_1(t), L_2(s))|^q ds dt \right]^{\frac{1}{q}} \\
& \leq 4 \left[\frac{|f_{ts}(a, c)|^q + |f_{ts}(a, d)|^q + |f_{ts}(b, c)|^q + |f_{ts}(b, d)|^q}{4} \right]^{\frac{1}{q}}. \quad (22)
\end{aligned}$$

From (21) and (22), we get (20). $\square$

Remark 2.4. If we take $p(x, y) = \frac{1}{(b-a)(d-c)}$ for all $(x, y) \in [a, b] \times [c, d]$ in Theorem 2.3, we get Theorem 1.3.

Remark 2.5. Theorem 2.1-Theorem 2.3 continue to hold true if in their statements we replace the condition “convex on the co-ordinates” with the condition “wright-convex on the co-ordinates”. However, the details are left to the interested reader.

In what follows we give our results for the quasi-convex mappings on the co-ordinates on $[a, b] \times [c, d]$.

Theorem 2.4. Suppose the assumptions of Theorem 2.1 are satisfied. If the mapping $|f_{ts}|$ is quasi-convex on the co-ordinates on $[a, b] \times [c, d]$, then the following inequality holds

$$\begin{aligned}
|\Phi(a, b, c, d; p, f)| & \leq \frac{(b-a)(d-c)}{16} \left[\lambda_1 \left(b, d, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}| \right) \right. \\
& \quad + \lambda_2 \left(a, d, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}| \right) + \lambda_3 \left(b, c, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}| \right) \\
& \quad \left. + \lambda_4 \left(a, c, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}| \right) \right] \int_0^1 \int_0^1 \int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy dt ds. \quad (23)
\end{aligned}$$

Proof. We continue inequality (12) in the proof of Theorem 2.1. Now, by the quasi-convexity on the co-ordinates of $|f_{ts}|$ on $[a, b] \times [c, d]$, we obtain

$$\begin{aligned}
& |f_{ts}(U_1(t), U_2(s))| \\
& \leq \max \left\{ |f_{ts}(b, d)|, \left| f_{ts} \left(b, \frac{c+d}{2} \right) \right|, \left| f_{ts} \left(\frac{a+b}{2}, d \right) \right|, \left| f_{ts} \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right| \right\}, \quad (24)
\end{aligned}$$

$$\begin{aligned}
& |f_{ts}(L_1(t), U_2(s))| \\
& \leq \max \left\{ |f_{ts}(a, d)|, \left| f_{ts} \left(a, \frac{c+d}{2} \right) \right|, \left| f_{ts} \left(\frac{a+b}{2}, d \right) \right|, \left| f_{ts} \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right| \right\}, \quad (25)
\end{aligned}$$

$$|f_{ts}(U_1(t), L_2(s))| \leq \max \left\{ |f_{ts}(b, c)|, \left| f_{ts}\left(b, \frac{c+d}{2}\right) \right|, \left| f_{ts}\left(\frac{a+b}{2}, c\right) \right|, \left| f_{ts}\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right| \right\}, \quad (26)$$

and

$$|f_{ts}(L_1(t), L_2(s))| \leq \max \left\{ |f_{ts}(a, c)|, \left| f_{ts}\left(a, \frac{c+d}{2}\right) \right|, \left| f_{ts}\left(\frac{a+b}{2}, c\right) \right|, \left| f_{ts}\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right| \right\}, \quad (27)$$

for all $(t, s) \in [0, 1] \times [0, 1]$. A combination of (24)-(27) and (12) gives the required inequality (23). $\square$

Corollary 2.1. *Suppose the assumptions of Theorem 2.4 are fulfilled and if $p(x, y) = \frac{1}{(b-a)(d-c)}$ for all $(x, y) \in [a, b] \times [c, d]$, then the following inequality holds valid*

$$\begin{aligned} & \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right. \\ & \quad - \frac{1}{2(b-a)} \int_a^b [f(x, c) + f(x, d)] dx - \frac{1}{2(d-c)} \int_c^d [f(a, y) + f(b, y)] dy \\ & \quad \left. + \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy \right| \leq \frac{(b-a)(d-c)}{16} \\ & \quad \times \left[\lambda_1 \left(b, d, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}| \right) + \lambda_2 \left(a, d, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}| \right) \right. \\ & \quad \left. + \lambda_3 \left(b, c, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}| \right) + \lambda_4 \left(a, c, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}| \right) \right]. \quad (28) \end{aligned}$$

Corollary 2.2. *Suppose the assumptions of Theorem 2.4 are satisfied and additionally*

- (1) *If $|f_{ts}|$ is non-decreasing on the co-ordinates on $[a, b] \times [c, d]$, then the following inequality holds true*

$$\begin{aligned} & |\Phi(a, b, c, d; p, f)| \\ & \leq \frac{(b-a)(d-c)}{16} \left[|f_{ts}(b, d)| + \left| f_{ts}\left(\frac{a+b}{2}, d\right) \right| + \left| f_{ts}\left(b, \frac{c+d}{2}\right) \right| \right. \\ & \quad \left. + \left| f_{ts}\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right| \right] \int_0^1 \int_0^1 \int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy dt ds. \quad (29) \end{aligned}$$

- (2) *If $|f_{ts}|$ is non-increasing on the co-ordinates on $[a, b] \times [c, d]$, then the following inequality holds true*

$$\begin{aligned} & |\Phi(a, b, c, d; p, f)| \\ & \leq \frac{(b-a)(d-c)}{16} \left[|f_{ts}(a, c)| + \left| f_{ts}\left(a, \frac{c+d}{2}\right) \right| + \left| f_{ts}\left(\frac{a+b}{2}, c\right) \right| \right. \end{aligned}$$

$$+ \left| f_{ts} \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right| \int_0^1 \int_0^1 \int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy dt ds. \quad (30)$$

Corollary 2.3. *If we take $p(x, y) = \frac{1}{(b-a)(d-c)}$ for all $(x, y) \in [a, b] \times [c, d]$ in Corollary 2.2 and additionally*

- (1) *If $|f_{ts}|$ is non-decreasing on the co-ordinates on $[a, b] \times [c, d]$, then the following inequality holds true*

$$\begin{aligned} & \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right. \\ & \quad - \frac{1}{2(b-a)} \int_a^b [f(x, c) + f(x, d)] dx - \frac{1}{2(d-c)} \int_c^d [f(a, y) + f(b, y)] dy \\ & \quad \left. + \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy \right| \leq \frac{(b-a)(d-c)}{16} \\ & \quad \times \left[|f_{ts}(b, d)| + \left| f_{ts} \left(\frac{a+b}{2}, d \right) \right| + \left| f_{ts} \left(b, \frac{c+d}{2} \right) \right| + \left| f_{ts} \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right| \right]. \quad (31) \end{aligned}$$

- (2) *If $|f_{ts}|$ is non-increasing on the co-ordinates on $[a, b] \times [c, d]$, then the following inequality holds true*

$$\begin{aligned} & \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right. \\ & \quad - \frac{1}{2(b-a)} \int_a^b [f(x, c) + f(x, d)] dx - \frac{1}{2(d-c)} \int_c^d [f(a, y) + f(b, y)] dy \\ & \quad \left. + \frac{1}{(b-a)(d-c)} \int_c^d \int_a^b f(x, y) dx dy \right| \leq \frac{(b-a)(d-c)}{16} \\ & \quad \times \left[|f_{ts}(a, c)| + \left| f_{ts} \left(a, \frac{c+d}{2} \right) \right| + \left| f_{ts} \left(\frac{a+b}{2}, c \right) \right| + \left| f_{ts} \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right| \right]. \quad (32) \end{aligned}$$

Theorem 2.5. *Suppose the assumptions of Theorem 2.1 are satisfied. If the mapping $|f_{ts}|^q$ is quasi-convex on the co-ordinates on $[a, b] \times [c, d]$ for $q \geq 1$, then the following inequality holds*

$$\begin{aligned} |\Phi(a, b, c, d; p, f)| & \leq \frac{(b-a)(d-c)}{16} \left\{ \left[\lambda_1 \left(b, d, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}|^q \right) \right]^{\frac{1}{q}} \right. \\ & \quad + \left[\lambda_2 \left(a, d, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}|^q \right) \right]^{\frac{1}{q}} + \left[\lambda_3 \left(b, c, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}|^q \right) \right]^{\frac{1}{q}} \\ & \quad \left. + \left[\lambda_4 \left(a, c, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}|^q \right) \right]^{\frac{1}{q}} \right\} \int_0^1 \int_0^1 \int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy dt ds. \quad (33) \end{aligned}$$

Proof. We continue inequality (18) in the proof of Theorem 2.2. Now, by the quasi-convexity on the co-ordinates of $|f_{ts}|^q$ on $[a, b] \times [c, d]$ for $q \geq 1$ and the power-mean

inequality, we obtain

$$\begin{aligned} & |f_{ts}(U_1(t), U_2(s))|^q \\ & \leq \max \left\{ |f_{ts}(b, d)|^q, \left| f_{ts} \left(b, \frac{c+d}{2} \right) \right|^q, \left| f_{ts} \left(\frac{a+b}{2}, d \right) \right|^q, \left| f_{ts} \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|^q \right\}, \end{aligned} \quad (34)$$

$$\begin{aligned} & |f_{ts}(L_1(t), U_2(s))|^q \\ & \leq \max \left\{ |f_{ts}(a, d)|^q, \left| f_{ts} \left(a, \frac{c+d}{2} \right) \right|^q, \left| f_{ts} \left(\frac{a+b}{2}, d \right) \right|^q, \left| f_{ts} \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|^q \right\}, \end{aligned} \quad (35)$$

$$\begin{aligned} & |f_{ts}(U_1(t), L_2(s))|^q \\ & \leq \max \left\{ |f_{ts}(b, c)|^q, \left| f_{ts} \left(b, \frac{c+d}{2} \right) \right|^q, \left| f_{ts} \left(\frac{a+b}{2}, c \right) \right|^q, \left| f_{ts} \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|^q \right\}, \end{aligned} \quad (36)$$

and

$$\begin{aligned} & |f_{ts}(L_1(t), L_2(s))|^q \\ & \leq \max \left\{ |f_{ts}(a, c)|^q, \left| f_{ts} \left(a, \frac{c+d}{2} \right) \right|^q, \left| f_{ts} \left(\frac{a+b}{2}, c \right) \right|^q, \left| f_{ts} \left(\frac{a+b}{2}, \frac{c+d}{2} \right) \right|^q \right\}, \end{aligned} \quad (37)$$

for all $(t, s) \in [0, 1] \times [0, 1]$. Using (34)-(37) in (18) we get the desired result. $\square$

Corollary 2.4. *Suppose the assumptions of Theorem 2.5 are fulfilled and if $p(x, y) = \frac{1}{(b-a)(d-c)}$ for all $(x, y) \in [a, b] \times [c, d]$, then the following inequality holds valid*

$$\begin{aligned} & \left| \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \right. \\ & \quad \left. - \frac{1}{2} \int_a^b [f(x, c) + f(x, d)] dx - \frac{1}{2} \int_c^d [f(a, y) + f(b, y)] dy \right. \\ & \quad \left. + \int_c^d \int_a^b f(x, y) dx dy \right| \leq \frac{(b-a)(d-c)}{16} \left\{ \left[\lambda_1 \left(b, d, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}|^q \right) \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\lambda_2 \left(a, d, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}|^q \right) \right]^{\frac{1}{q}} + \left[\lambda_3 \left(b, c, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}|^q \right) \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\lambda_4 \left(a, c, \frac{a+b}{2}, \frac{c+d}{2}; |f_{ts}|^q \right) \right]^{\frac{1}{q}} \right\} \int_0^1 \int_0^1 \int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy dt ds. \end{aligned} \quad (38)$$

Remark 2.6. *Suppose the assumptions of Theorem 2.5 are satisfied and additionally*

- (1) If $|f_{ts}|^q$ is non-decreasing on the co-ordinates on $[a, b] \times [c, d]$, then (29) holds valid.
- (2) If $|f_{ts}|^q$ is non-increasing on the co-ordinates on $[a, b] \times [c, d]$, then (30) holds true.

Remark 2.7. In Corollary 2.4

- (1) If $|f_{ts}|^q$ is non-decreasing on the co-ordinates on $[a, b] \times [c, d]$, then (31) holds valid.
- (2) If $|f_{ts}|^q$ is non-increasing on the co-ordinates on $[a, b] \times [c, d]$, then (32) holds true.

3. Applications to Random Variables

Let $0 < a < b$, $0 < c < d$, $\alpha, \beta \in \mathbb{R}$ and let X and Y be two independent continuous random variables having the bi-variate continuous probability density function $p : [a, b] \times [c, d] \rightarrow [0, \infty)$ which is symmetric to $\frac{a+b}{2}$ and $\frac{c+d}{2}$ the α -moment of X and the β -moment of Y about the origin are respectively defined as follows

$$E_\alpha(X) = \int_c^b t^\alpha p_1(t) dt, E_\beta(Y) = \int_c^d s^\beta p_2(s) ds$$

which are assumed to be finite, here $p_1 : [a, b] \rightarrow [0, \infty)$ and $p_2 : [c, d] \rightarrow [0, \infty)$ are the marginal probability density functions of X and Y . Since X and Y are independent random variables, we have

$$p(t, s) = p_1(t) p_2(s)$$

for all $(t, s) \in [a, b] \times [c, d]$.

Now we give some applications of our results to random variables.

Theorem 3.1. *The inequality*

$$\begin{aligned} & \left| \left(E_\alpha(X) - \frac{a^\alpha + b^\alpha}{2} \right) \left(E_\beta(Y) - \frac{c^\beta + d^\beta}{2} \right) \right| \\ & \leq \frac{(b-a)(d-c)}{4} \alpha \beta \left(\frac{a^{\alpha-1} + b^{\alpha-1}}{2} \right) \left(\frac{c^{\beta-1} + d^{\beta-1}}{2} \right). \end{aligned} \quad (39)$$

holds holds for $0 < a < b$, $0 < c < d$ and $\alpha, \beta \geq 2$.

Proof. Let $f(t, s) = t^\alpha s^\beta$ on $[a, b] \times [c, d]$ for $\alpha, \beta \geq 2$, we observe that $|f_{ts}(t, s)| = \alpha\beta t^{\alpha-1} s^{\beta-1}$ is convex on the co-ordinates on $[a, b] \times [c, d]$. Since

$$\begin{aligned} & |f_{ts}(a, c)| + |f_{ts}(a, d)| + |f_{ts}(b, c)| + |f_{ts}(b, d)| \\ & = \alpha\beta (a^{\alpha-1} + b^{\alpha-1}) (c^{\beta-1} + d^{\beta-1}), \end{aligned}$$

$$\int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy \leq \int_c^d \int_a^b p(x, y) dx dy = 1$$

and hence

$$\int_0^1 \int_0^1 \int_{L_2(s)}^{U_2(s)} \int_{L_1(t)}^{U_1(t)} p(x, y) dx dy dt ds \leq 1.$$

Also

$$\begin{aligned} & \frac{f(a, c) + f(a, d) + f(b, c) + f(b, d)}{4} \int_c^d \int_a^b p(x, y) dx dy \\ &= \frac{a^\alpha c^\beta + a^\alpha d^\beta + b^\alpha c^\beta + b^\alpha d^\beta}{4} = \frac{(a^\alpha + b^\alpha)(c^\beta + d^\beta)}{4}, \end{aligned}$$

$$\begin{aligned} & \frac{1}{2} \int_c^d \int_a^b [f(x, c) + f(x, d)] p(x, y) dx dy + \frac{1}{2} \int_c^d \int_a^b [f(a, y) + f(b, y)] p(x, y) dx dy \\ &= \left(\frac{c^\beta + d^\beta}{2} \right) E_\alpha(X) + \left(\frac{a^\alpha + b^\alpha}{2} \right) E_\beta(Y) \end{aligned}$$

and

$$\int_c^d \int_a^b f(x, y) p(x, y) dx dy = E_\alpha(X) E_\beta(Y).$$

The result follows immediately from the inequality (11). $\square$

Theorem 3.2. *The inequality*

$$\begin{aligned} & \left| \left(E_\alpha(X) - \frac{a^\alpha + b^\alpha}{2} \right) \left(E_\beta(Y) - \frac{c^\beta + d^\beta}{2} \right) \right| \\ & \leq \frac{(b-a)(d-c)}{16} \alpha \beta \left(b^{\alpha-1} + \left(\frac{a+b}{2} \right)^{\alpha-1} \right) \left(d^{\beta-1} + \left(\frac{c+d}{2} \right)^{\beta-1} \right). \end{aligned} \quad (40)$$

holds holds for $0 < a < b$, $0 < c < d$ and $\alpha, \beta \geq 1$.

Proof. Let $f(t, s) = t^\alpha s^\beta$ on $[a, b] \times [c, d]$ for $\alpha, \beta \geq 1$, we observe that $|f_{ts}(t, s)| = \alpha\beta t^{\alpha-1} s^{\beta-1}$ is non-decreasing and quasi-convex on the co-ordinates on $[a, b] \times [c, d]$. The proof is similar to that of Theorem 3.1 by using the inequality (29) we obtain the required result. $\square$

Remark 3.1. For $\alpha = \beta = 1$, we have from Theorem 3.2 that

$$\left| \left(E(X) - \frac{a+b}{2} \right) \left(E(Y) - \frac{c+d}{2} \right) \right| \leq \frac{(b-a)(d-c)}{4}, \quad (41)$$

where $E_1(X) = E(X)$ and $E_1(Y) = E(Y)$ are the expectation of the random variables X and Y respectively.

4. Applications to 2D weighted trapezoidal formula

Let $[a, b] \times [c, d]$ be a rectangle from the plane $\mathbb{R}^2$. Suppose d_1 and d_2 are the divisions $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and $c = y_0 < y_1 < \cdots < y_{m-1} < y_m = b$ of the intervals $[a, b]$ and $[c, d]$ respectively and let $\Omega = \{[x_i, x_{i+1}] \times [y_j, y_{j+1}] : 0 \leq i \leq n-1, 0 \leq j \leq m-1\}$ be a corresponding division of the rectangle $[a, b] \times [c, d]$ from the plane $\mathbb{R}^2$.

Consider the following 2D weighted quadrature formula

$$\int_c^d \int_a^b f(x, y) p(x, y) dx dy = T(f, p, \Omega) + E(f, p, \Omega), \quad (42)$$

where

$$\begin{aligned} T(f, p, \Omega) = & - \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \left[\frac{f(x_i, y_j) + f(x_i, y_{j+1}) + f(x_{i+1}, y_j) + f(x_{i+1}, y_{j+1})}{4} \right. \\ & \times \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} p(x, y) dx dy + \frac{1}{2} \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} [f(x, y_j) + f(x, y_{j+1})] p(x, y) dx dy \\ & \left. + \frac{1}{2} \int_{y_j}^{y_{j+1}} \int_{x_i}^{x_{i+1}} [f(x_i, y) + f(x_{i+1}, y)] p(x, y) dx dy \right] \quad (43) \end{aligned}$$

for the trapezoidal version and $E(f, p, \Omega)$ denotes the associated approximation error.

The following results provide some estimates of the remainder term $E(f, p, \Omega)$.

Theorem 4.1. *Suppose the assumptions of Theorem 2.2 are satisfied. If $|f_{ts}|^q$ is convex on the co-ordinates on $[a, b] \times [c, d]$ for $q \geq 1$, then in (42), for every division Ω of the rectangle $[a, b] \times [c, d]$ from the plane $\mathbb{R}^2$, the following holds*

$$\begin{aligned} |E(f, p, \Omega)| \leq & \frac{1}{4} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (x_{i+1} - x_i) (y_{j+1} - y_j) \\ & \times [\Psi(x_i, x_{i+1}, y_j, y_{j+1}; |f_{ts}|^q)]^{\frac{1}{q}} \int_0^1 \int_0^1 \int_{L_2(y_j, y_{j+1}, s)}^{U_2(y_j, y_{j+1}, s)} \int_{L_1(x_i, x_{i+1}, t)}^{U_1(x_i, x_{i+1}, t)} p(x, y) dx dy ds dt. \quad (44) \end{aligned}$$

Proof. Applying Theorem 2.2 on the rectangles $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ ($0 \leq i \leq n-1, 0 \leq j \leq m-1$) of the division Ω of the rectangle $[a, b] \times [c, d]$ from the plane $\mathbb{R}^2$, we get

$$\begin{aligned} |\Phi(x_i, x_{i+1}, y_j, y_{j+1}; p, f)| \leq & \frac{(x_{i+1} - x_i) (y_{j+1} - y_j)}{4} \\ & \times \left[\frac{|f_{ts}(x_i, y_j)|^q + |f_{ts}(x_i, y_{j+1})|^q + |f_{ts}(x_{i+1}, y_j)|^q + |f_{ts}(x_{i+1}, y_{j+1})|^q}{4} \right]^{\frac{1}{q}} \\ & \times \int_0^1 \int_0^1 \int_{L_2(y_j, y_{j+1}, s)}^{U_2(y_j, y_{j+1}, s)} \int_{L_1(x_i, x_{i+1}, t)}^{U_1(x_i, x_{i+1}, t)} p(x, y) dx dy ds dt. \quad (45) \end{aligned}$$

Summing over i from 0 to $n - 1$ and j over 0 to $m - 1$, we deduce, by the triangle inequality, that (44) holds. $\square$

Remark 4.1. *The inequality holds if the condition of convexity of $|f_{ts}|^q$ on the co-ordinates on $[a, b] \times [c, d]$ is replaced with the condition of wright-convexity of $|f_{ts}|^q$ on the co-ordinates on $[a, b] \times [c, d]$ for $q \geq 1$.*

Theorem 4.2. *Suppose the assumptions of Theorem 2.2 are satisfied. If $|f_{ts}|^q$ is convex on the co-ordinates on $[a, b] \times [c, d]$ for $q \geq 1$, then in (42), for every division Ω of the rectangle $[a, b] \times [c, d]$ from the plane $\mathbb{R}^2$, the following holds*

$$\begin{aligned}
|E(f, p, \Omega)| &\leq \frac{1}{16} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (x_{i+1} - x_i) (y_{j+1} - y_j) \\
&\quad \times \left\{ \left[\lambda_1 \left(x_{i+1}, y_{j+1}, \frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}; |f_{ts}|^q \right) \right]^{\frac{1}{q}} \right. \\
&\quad + \left[\lambda_2 \left(x_i, y_{j+1}, \frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}; |f_{ts}|^q \right) \right]^{\frac{1}{q}} \\
&\quad + \left[\lambda_3 \left(x_{i+1}, y_j, \frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}; |f_{ts}|^q \right) \right]^{\frac{1}{q}} \\
&\quad \left. + \left[\lambda_4 \left(x_i, y_j, \frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2}; |f_{ts}|^q \right) \right]^{\frac{1}{q}} \right\} \\
&\quad \times \int_0^1 \int_0^1 \int_{L_2(y_j, y_{j+1}, s)}^{U_2(y_j, y_{j+1}, s)} \int_{L_1(x_i, x_{i+1}, t)}^{U_1(x_i, x_{i+1}, t)} p(x, y) dx dy ds dt. \quad (46)
\end{aligned}$$

Proof. The proof follows from (33) by using the similar arguments as that of the proof of Theorem 4.1. $\square$

Remark 4.2. *If $|f_{ts}|$ is non-decreasing in Theorem 4.2, then the following inequality holds*

$$\begin{aligned}
|E(f, p, \Omega)| &\leq \frac{1}{16} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (x_{i+1} - x_i) (y_{j+1} - y_j) \\
&\quad \times \left[\left| f_{ts}(x_{i+1}, y_{j+1}) \right| + \left| f_{ts} \left(\frac{x_i + x_{i+1}}{2}, y_{j+1} \right) \right| + \left| f_{ts} \left(x_{i+1}, \frac{y_j + y_{j+1}}{2} \right) \right| \right. \\
&\quad \left. + \left| f_{ts} \left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2} \right) \right| \right] \int_0^1 \int_0^1 \int_{L_2(y_j, y_{j+1}, s)}^{U_2(y_j, y_{j+1}, s)} \int_{L_1(x_i, x_{i+1}, t)}^{U_1(x_i, x_{i+1}, t)} p(x, y) dx dy ds dt. \quad (47)
\end{aligned}$$

and if $|f_{ts}|$ is non-increasing in Theorem 4.2, then the following inequality holds

$$\begin{aligned}
|E(f, p, \Omega)| &\leq \frac{1}{16} \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} (x_{i+1} - x_i) (y_{j+1} - y_j) \\
&\quad \times \left[|f_{ts}(x_i, y_j)| + \left| f_{ts} \left(x_i, \frac{y_j + y_{j+1}}{2} \right) \right| + \left| f_{ts} \left(\frac{x_i + x_{i+1}}{2}, y_j \right) \right| \right. \\
&\quad \left. + \left| f_{ts} \left(\frac{x_i + x_{i+1}}{2}, \frac{y_j + y_{j+1}}{2} \right) \right| \right] \int_0^1 \int_0^1 \int_{L_2(y_j, y_{j+1}, s)}^{U_2(y_j, y_{j+1}, s)} \int_{L_1(x_i, x_{i+1}, t)}^{U_1(x_i, x_{i+1}, t)} p(x, y) dx dy ds dt.
\end{aligned} \tag{48}$$

References

- [1] M. Alomari and M. Darus, Fejer inequality for double integrals, *Facta Universitatis (NIS): Ser. Math. Inform.* 24(2009), 15-28.
- [2] M. Alomari, M. Darus, U.S. Kirmaci, Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, *Computers & Mathematics with Applications*, Volume 59, Issue 1, January 2010, Pages 225-232
- [3] S. S. Dragomir and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and to Trapezoidal formula, *Appl. Math. Lett.* 11(5) (1998) 91-95.
- [4] S.S. Dragomir, On Hadamard's inequality for convex functions on the co-ordinates in a rectangle from the plane, *Taiwanese Journal of Mathematics*, 4 (2001), 775-788.
- [5] S. S. Dragomir, Two mappings in connection to Hadamard's inequalities, *Journal of Mathematical Analysis and Applications*, 167, 49-56. [http://dx.doi.org/10.1016/0022-247X\(92\)90233-4](http://dx.doi.org/10.1016/0022-247X(92)90233-4)
- [6] S.S. Dragomir and C.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. Online: [http://www.staff.vu.edu.au/RGMIA/monographs/hermite_hadamard.html].
- [7] J. Hadamard, Étude sur les Propriétés des Fonctions Entières en Particulier d'une Fonction Considérée par Riemann. *Journal de Mathématiques Pures et Appliquées*, 58, 171-215.
- [8] D. Y. Hwang, K. L. Tseng, and G. S. Yang, Some Hadamard's inequalities for co-ordinated convex functions in a rectangle from the plane, *Taiwanese Journal of Mathematics*, 11(2007), 63-73.
- [9] D. Y. Hwang, Some inequalities for differentiable convex mapping with application to weighted trapezoidal formula and higher moments of random variables, *Applied Mathematics and Computation* 217 (2011) 9598-9605.
- [10] D. -Y. Hwang, K.-C. Hsu and K.-L. Tseng, Hadamard-Type inequalities for Lipschitzian functions in one and two variables with applications, *Journal of Mathematical Analysis and Applications*, 405, 546-554. <http://dx.doi.org/10.1016/j.jmaa.2013.04.032>.
- [11] K.-C. Hsu, Some Hermite-Hadamard type inequalities for differentiable co-ordinated convex functions and applications, *Advances in Pure Mathematics*, 2014, 4, 326-340.
- [12] K.-C. Hsu, Refinements of Hermite-Hadamard type inequalities for differentiable co-ordinated convex functions and applications, *Taiwanese Journal of Mathematics*, (In press). <http://dx.doi.org/10.1142/9261>.
- [13] M. A. Latif and M. Alomari, Hadamard-type inequalities for product of two convex functions on the co-ordinates, *Int. Math. Forum*, 4(47), 2009, 2327-2338.
- [14] M. A. Latif and M. Alomari, On the Hadamard-type inequalities for h -convex functions on the co-ordinates, *Int. J. of Math. Analysis*, 3(33), 2009, 1645-1656.
- [15] M. A. Latif, S. S. Dragomir, On some new inequalities for differentiable co-ordinated convex functions, *Journal of Inequalities and Applications* 2012, 2012:28.

- [16] M. A. Latif, S. Hussain and S. S. Dragomir, Refinements of Hermite-Hadamard type inequalities for co-ordinated quasi-convex functions, *International Journal of Mathematical Archive*-3(1), 2012, 161-171.
- [17] S.-L. Lyu, On the Hermite-Hadamard inequality for convex functions of two variable, *Numerical Algebra, Control and Optimization*, Volume 4, Number 1, March 2014.
- [18] M.E. Özdemir, E. Set and M.Z. Sarikaya, New some Hadamard's type inequalities for co-ordinated m -convex and (α, m) -convex functions, *Hacettepe Journal of Mathematics and Statistics* 40 (2), 219-229.
- [19] M.E. Özdemir, M. A. Latif and A. O. Akdemir, On some Hadamard-type inequalities for product of two s -convex functions on the co-ordinates, *Journal of Inequalities and Applications* 2012, 2012:21. doi:10.1186/1029-242X-2012-21.
- [20] M.E. Özdemir, A. O. Akdemir, Ağrı, C. Yıldız and Erzurum, On co-ordinated quasi-convex functions, *Czechoslovak Mathematical Journal*, 62 (137) (2012), 889-900.
- [21] C. M. E. Pearce and J. E. Pečarić, Inequalities for differentiable mappings with applications to special means and quadrature formula, *Appl. Math. Lett.* 13 (2000) 51-55.
- [22] J. E. Pečarić, F. Proschan and Y. L. Tong, *Convex Functions, Partial Ordering and Statistical Applications*, Academic Press, New York, 1991.
- [23] M.Z. Sarikaya, E. Set, M.E. Özdemir and S. S. Dragomir, New some Hadamard's type inequalities for co-ordinated convex functions, *Tamsui Oxford Journal of Information and Mathematical Sciences* 28(2) (2012) 137-152.