A new Ostrowski type inequality for functions whose first derivatives are of bounded variation

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ABSTRACT. In this paper, we establish some new Ostrowski type integral inequalities for mappings whose first derivatives are of bounded variation and quadrature formula is provided.

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1. Introduction

Let $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e. $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$. Then, we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty, \quad (1)$$

for all $x \in [a, b]$ [15]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality*.

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Definition 1.1. Let $P : a = x_0 < x_1 < \dots < x_n = b$ be any partition of $[a, b]$ and let $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$. Then f is said to be of bounded variation if the sum

$$\sum_{i=1}^n |\Delta f(x_i)|$$

is bounded for all such partitions.

Let f be of bounded variation on $[a, b]$, and $\sum \Delta f(P)$ denotes the sum $\sum_{i=1}^n |\Delta f(x_i)|$ corresponding to the partition P of $[a, b]$. The number

$$\bigvee_a^b(f) := \sup \left\{ \sum \Delta f(P) : P \in P([a, b]) \right\},$$

is called the total variation of f on $[a, b]$. Here $P([a, b])$ denotes the family of partitions of $[a, b]$.

In [9], Dragomir proved the following Ostrowski type inequalities related functions of bounded variation:

Theorem 1.1. Let $f : [a, b] \rightarrow \mathbb{R}$ be a mapping of bounded variation on $[a, b]$. Then

$$\left| \int_a^b f(t) dt - (b-a) f(x) \right| \leq \left[\frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b(f)$$

holds for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

Recently, many authors have studied on Ostrowski type inequality for function of bounded variation, please refer to references [1],[2],[4],[6]-[12].

In [14], Liu proved following Ostrowski type inequalities for the functions with first derivatives of bounded variation:

Theorem 1.2. Let $f : [a, b] \rightarrow \mathbb{R}$ be such that f' is a continuous function of bounded variation on $[a, b]$. Then for any $x \in [a, b]$ and $\theta \in [0, 1]$ we have

$$\begin{aligned} & \left| \int_a^b f(t) dt - (b-a) \left[(1-\theta) f(x) + \theta \frac{f(a)+f(b)}{2} - (1-\theta) \left(x - \frac{a+b}{2} \right) f'(x) \right] \right| \\ & \leq \frac{1}{16} [4(x-b)^2 - 4\theta(b-a)(b-x) + \theta^2(b-a)^2] \\ & \quad [4(x-b)^2 - 4\theta(b-a)(b-x) - \theta^2(b-a)^2] \bigvee_a^b(f') \end{aligned}$$

for $a \leq x \leq \frac{a+b}{2}$ and

$$\left| \int_a^b f(t) dt - (b-a) \left[(1-\theta) f(x) + \theta \frac{f(a)+f(b)}{2} - (1-\theta) \left(x - \frac{a+b}{2} \right) f'(x) \right] \right|$$

$$\leq \frac{1}{16} [4(x-a)^2 - 4\theta(b-a)(x-a) + \theta^2(b-a)^2]$$

$$[4(x-a)^2 - 4\theta(b-a)(x-a) - \theta^2(b-a)^2] \bigvee_a^b(f')$$

for $\frac{a+b}{2} \leq x \leq b$.

In [5], Budak and Sarikaya obtained following weighted Ostrowski type inequality for the mappings with first derivatives of bounded variation:

Theorem 1.3. *Let $w : [a, b] \rightarrow \mathbb{R}$ be nonnegative and continuous and let $f : [a, b] \rightarrow \mathbb{R}$ be differentiable mapping on $[a, b]$. If f' is of bounded variation on $[a, b]$, then we have the weighted inequality*

$$\left| \left(\int_a^b (u-x) w(u) du \right) f'(x) + \left(\int_a^b w(u) du \right) f(x) - \int_a^b w(t) f(t) dt \right|$$

$$\leq \left(\int_a^x (u-x) w(u) du \right) \bigvee_a^x(f') + \left(\int_x^b (u-x) w(u) du \right) \bigvee_x^b(f')$$

for any $x \in [a, b]$, where $\bigvee_c^d(f')$ denotes the total variation of f' on $[c, d]$.

The aim of this paper is to obtain a new bound for the statement

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} [f(x) + f(a+b-x)] + \frac{1}{2} \left(x - \frac{3a+b}{4} \right) [f'(x) - f'(a+b-x)] \right|.$$

In [13], Liu establish some bounds for this statement using the absolutely continuous function. In this study, we obtain anew bound using the functions whose first derivatives are of bounded variation.

2. Main Results

We give a new Ostrowski type inequality for mappings with first derivatives of bounded variation:

Theorem 2.1. *Let $f : [a, b] \rightarrow \mathbb{R}$ be such that f' is a continuous function of bounded variation on $[a, b]$. Then we have the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} [f(x) + f(a+b-x)] \right| \tag{2}$$

$$\begin{aligned}
& + \frac{1}{2} \left(x - \frac{3a+b}{4} \right) [f'(x) - f'(a+b-x)] \Big| \\
& \leq \frac{1}{16} \left[\frac{5(x-a)^2 - 2(x-a)(b-x) + (b-x)^2}{b-a} + 4 \left| x - \frac{3a+b}{4} \right| \right] \bigvee_a^b(f')
\end{aligned}$$

for any $x \in [a, \frac{a+b}{2}]$, where $\bigvee_c^d(f')$ denotes the total variation of f' on $[c, d]$.

Proof. Define the mapping $p(x, t)$ by,

$$p(x, t) = \begin{cases} (t-a)^2 & t \in [a, x] \\ \left(t - \frac{a+b}{2}\right)^2 & t \in (x, a+b-x] \\ (t-b)^2 & t \in (a+b-x, b]. \end{cases} \quad (3)$$

From (4), we have

$$\begin{aligned}
& \int_a^b p(x, t) df'(t) \\
& = \int_a^x (t-a)^2 df'(t) + \int_x^{a+b-x} \left(t - \frac{a+b}{2}\right)^2 df'(t) + \int_{a+b-x}^b (t-b)^2 df'(t) \\
& = I_1 + I_2 + I_3.
\end{aligned} \quad (4)$$

Integrating by parts, we get

$$\begin{aligned}
I_1 & = \int_a^x (t-a)^2 df'(t) \\
& = (x-a)^2 f'(x) - 2 \int_a^x (t-a) f'(t) dt \\
& = (x-a)^2 f'(x) - 2(x-a) f(x) + 2 \int_a^x f(t) dt.
\end{aligned} \quad (5)$$

And similarly we have

$$I_2 = \int_x^{a+b-x} \left(t - \frac{a+b}{2}\right)^2 df'(t) \quad (6)$$

$$\begin{aligned}
&= \left(x - \frac{a+b}{2}\right)^2 [f'(a+b-x) - f'(x)] \\
&\quad + 2 \left(x - \frac{a+b}{2}\right) [f(x) + f(a+b-x)] + 2 \int_x^{a+b-x} f(t) dt
\end{aligned}$$

and

$$\begin{aligned}
I_3 &= \int_{a+b-x}^b (t-b)^2 df'(t) \\
&= -(x-a)^2 f'(a+b-x) - 2(x-a) f(a+b-x) + 2 \int_{a+b-x}^b f(t) dt.
\end{aligned} \tag{7}$$

Adding the equalities (5)-(7) in (4) and multiplying by $\frac{1}{2(b-a)}$, we have the equality

$$\begin{aligned}
&\frac{1}{2(b-a)} \int_a^b p(x, t) df'(t) \\
&= \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} [f(x) + f(a+b-x)] \\
&\quad + \frac{1}{2} \left(x - \frac{3a+b}{4}\right) [f'(x) - f'(a+b-x)]
\end{aligned} \tag{8}$$

It is well known [3, see p. 159] that if $g, f : [a, b] \rightarrow \mathbb{R}$ are such that g is continuous on $[a, b]$ and f is of bounded variation on $[a, b]$, then $\int_a^b g(t) df(t)$ exist and [3, see p. 177]

$$\left| \int_a^b g(t) df(t) \right| \leq \sup_{t \in [a, b]} |g(t)| \bigvee_a^b(f). \tag{9}$$

Taking modulus in (8) and by using (9), we have

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} [f(x) + f(a+b-x)] \right. \\
&\quad \left. + \frac{1}{2} \left(x - \frac{3a+b}{4}\right) [f'(x) - f'(a+b-x)] \right|
\end{aligned}$$

$$\begin{aligned}
&= \left| \int_a^b p(x, t) df'(t) \right| \\
&\leq \frac{1}{2(b-a)} \left[\left| \int_a^x (t-a)^2 df'(t) \right| \right. \\
&\quad \left. + \left| \int_x^{a+b-x} \left(t - \frac{a+b}{2} \right)^2 df'(t) \right| + \left| \int_{a+b-x}^b (t-b)^2 df'(t) \right| \right] \\
&\leq \frac{1}{2(b-a)} \left[\sup_{t \in [a, x]} (t-a)^2 \bigvee_a^x(f') \right. \\
&\quad \left. + \sup_{t \in [x, a+b-x]} \left(t - \frac{a+b}{2} \right)^2 \bigvee_x^{a+b-x}(f') + \sup_{t \in [a+b-x, b]} (t-b)^2 \bigvee_{a+b-x}^b(f') \right] \\
&\leq \frac{1}{2(b-a)} \left[(x-a)^2 \bigvee_a^x(f') + \left(\frac{a+b}{2} - x \right)^2 \bigvee_x^{a+b-x}(f') + (x-a)^2 \bigvee_{a+b-x}^b(f') \right] \\
&= \frac{1}{2(b-a)} \max \left\{ (x-a)^2, \left(\frac{a+b}{2} - x \right)^2 \right\} \bigvee_a^b(f').
\end{aligned}$$

That is,

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} [f(x) + f(a+b-x)] \right. \\
&\quad \left. + \frac{1}{2} \left(x - \frac{3a+b}{4} \right) [f'(x) - f'(a+b-x)] \right| \\
&\leq \frac{1}{2(b-a)} \max \left\{ (x-a)^2, \left(\frac{a+b}{2} - x \right)^2 \right\} \bigvee_a^b(f').
\end{aligned} \tag{10}$$

If we use the fact that $\max\{x, y\} = \frac{x+y+|x-y|}{2}$, then we obtain the desired result. $\square$

Remark 2.1. If we choose $x = \frac{a+b}{2}$ in (2), then we have the inequality

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \frac{b-a}{8} \bigvee_a^b(f') \tag{11}$$

which was given by Liu in [14].

Corollary 2.1. *Under assumption of Theorem 2.1 with $x = \frac{3a+b}{4}$, then we have the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2} \left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \right] \right| \leq \frac{b-a}{32} \mathcal{V}_a^b(f'). \quad (12)$$

Corollary 2.2. *Under assumption of Theorem 2.1 with $x = a$, then we have the inequality*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2} [f(a) + f(b)] + \frac{b-a}{8} [f'(b) - f'(a)] \right| \\ & \leq \frac{b-a}{8} \mathcal{V}_a^b(f'). \end{aligned} \quad (13)$$

This inequality is similar to the third inequality in Remark 2 of [14]. However, the constant in [14, Remark 2] is $\frac{1}{4}$. Thus, the inequality 13 is sharper.

Under assumption of of Theorem 2.1, we have the following corollaries:

Corollary 2.3. *Let $f \in C^2[a, b]$. Then we have the inequality*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t)dt - \frac{1}{2} [f(x) + f(a+b-x)] \right. \\ & \quad \left. + \frac{1}{2} \left(x - \frac{3a+b}{4} \right) [f'(x) - f'(a+b-x)] \right| \\ & \leq \frac{1}{16} \left[\frac{5(x-a)^2 - 2(x-a)(b-x) + (b-x)^2}{b-a} + 4 \left| x - \frac{3a+b}{4} \right| \right] \|f''\|_{[a,b],1} \end{aligned} \quad (14)$$

for all $x \in [a, \frac{a+b}{2}]$, where $\|\cdot\|_{[c,d],1}$ is the L_1 -norm, namely

$$\|f''\|_{[c,d],1} = \int_c^d |f''(t)| dt.$$

Remark 2.2. *If we choose and $x = \frac{a+b}{2}$ in (14), then we have the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{8} \|f''\|_{[a,b],1}$$

which was given by Liu in [13].

Corollary 2.4. *Let $f' : [a, b] \rightarrow \mathbb{R}$ be a Lipschitzian mapping with the constants $L > 0$. Then, we have the inequality*

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(t) dt - \frac{1}{2} [f(x) + f(a+b-x)] \right. \\ & \quad \left. + \frac{1}{2} \left(x - \frac{3a+b}{4} \right) [f'(x) - f'(a+b-x)] \right| \\ & \leq \frac{L}{16} \left[5(x-a)^2 - 2(x-a)(b-x) + (b-x)^2 + 4(b-a) \left| x - \frac{3a+b}{4} \right| \right] \end{aligned} \quad (15)$$

for all $x \in [a, \frac{a+b}{2}]$.

Proof. As f' is L-Lipschitzian on $[a, b]$, it is also of bounded variation. If $P([a, b])$ denotes the family of divisions on $[a, b]$, then

$$\begin{aligned} \bigvee_a^b(f') &= \sup_{P \in P([a, b])} \sum_{i=0}^{n-1} |f'(x_{i+1}) - f'(x_i)| \\ &\leq L \sup_{P \in P([a, b])} \sum_{i=0}^{n-1} |x_{i+1} - x_i| \\ &= L(b-a) \end{aligned}$$

and the required result is proved. $\square$

Remark 2.3. *If we choose and $x = \frac{a+b}{2}$ in (15), then we have the inequality*

$$\left| \frac{1}{b-a} \int_a^b f(t) dt - f\left(\frac{a+b}{2}\right) \right| \leq \frac{L(b-a)}{8}. \quad (16)$$

This result is similar to the special case of Theorem 6 of [14]. But, the constant $\frac{1}{24}$ in [14, Remarks 5 and 9] is sharper than the constant in this result.

3. Application to Quadrature Formula

Let us consider the arbitrary division $I_n : a = x_0 < x_1 < \dots < x_n = b$ with $h_i := x_{i+1} - x_i$ and $v(h) := \max \{h_i \mid i = 0, \dots, n-1\}$. Then the following Theorem holds:

Theorem 3.1. *Let $f : Q \rightarrow \mathbb{R}$ is of bounded variation on Q and $\xi_i \in [x_i, x_{i+1}]$ ($i = 0, \dots, n-1$). Then we have the quadrature formula:*

$$\int_a^b f(t) dt = \frac{1}{2} \sum_{i=0}^{n-1} [f(\xi_i) + f(x_i + x_{i+1} - \xi_i)] h_i$$

$$-\frac{1}{2} \sum_{i=0}^{n-1} \left(\xi_i - \frac{3x_i + x_{i+1}}{4} \right) [f(\xi_i) - f(x_i + x_{i+1} - \xi_i)] h_i + R(I_n, f, \xi).$$

The remainder term $R(I_n, f, \xi)$ satisfies

$$\begin{aligned} & |R(I_n, f, \xi)| \\ & \leq \frac{1}{2} \max_{i \in \{0, \dots, n-1\}} \left\{ \max \left\{ (\xi_i - x_i)^2, \left(\frac{x_i + x_{i+1}}{2} - \xi_i \right)^2 \right\} \right\} \bigvee_a^b (f'). \end{aligned} \quad (17)$$

Proof. Applying the inequality (10) (which is equal to Theorem 2.1) for interval $[x_i, x_{i+1}]$, we have

$$\begin{aligned} & \left| \int_{x_i}^{x_{i+1}} f(t) dt - \frac{1}{2} [f(\xi_i) + f(x_i + x_{i+1} - \xi_i)] h_i \right. \\ & \quad \left. \left(\xi_i - \frac{3x_i + x_{i+1}}{4} \right) [f(\xi_i) - f(x_i + x_{i+1} - \xi_i)] h_i \right| \\ & \leq \frac{1}{2} \max \left\{ (\xi_i - x_i)^2, \left(\frac{x_i + x_{i+1}}{2} - \xi_i \right)^2 \right\} \bigvee_{x_i}^{x_{i+1}} (f') \end{aligned} \quad (18)$$

Summing the inequality (18) over i from 0 to $n-1$, then we have

$$\begin{aligned} & |R(I_n, f, \xi)| \\ & \leq \frac{1}{2} \sum_{i=0}^{n-1} \max \left\{ (\xi_i - x_i)^2, \left(\frac{x_i + x_{i+1}}{2} - \xi_i \right)^2 \right\} \bigvee_{x_i}^{x_{i+1}} (f') \\ & \leq \frac{1}{2} \max_{i \in \{0, \dots, n-1\}} \left\{ \max \left\{ (\xi_i - x_i)^2, \left(\frac{x_i + x_{i+1}}{2} - \xi_i \right)^2 \right\} \right\} \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}} (f') \\ & \leq \frac{1}{2} \max_{i \in \{0, \dots, n-1\}} \left\{ \max \left\{ (\xi_i - x_i)^2, \left(\frac{x_i + x_{i+1}}{2} - \xi_i \right)^2 \right\} \right\} \bigvee_a^b (f'). \end{aligned}$$

This completes the proof of the Theorem. $\square$

Remark 3.1. If we choose $\xi_i = \frac{x_i + x_{i+1}}{2}$ ($i = 0, \dots, n-1$) in Theorem 3.1, then we have the midpoint rule

$$\int_a^b f(t)dt = \sum_{i=0}^{n-1} f(\xi_i)h_i + R_M(I_n, f)$$

and the remainder term satisfies

$$|R_M(I_n, f)| \leq \frac{(v(h))^2}{8} \bigvee_a^b(f')$$

which was given by Budak and Sarikaya in [5].

Remark 3.2. If we choose $\xi_i = \frac{3x_i + x_{i+1}}{4}$ ($i = 0, \dots, n-1$) in Theorem 3.1, then we have

$$\int_a^b f(t)dt = \frac{1}{2} \sum_{i=0}^{n-1} [f(\xi_i) + f(x_i + x_{i+1} - \xi_i)] h_i + R(I_n, f)$$

and the remainder term $R(I_n, f)$ satisfies

$$|R(I_n, f)| \leq \frac{(v(h))^2}{32} \bigvee_a^b(f').$$

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