DOI 10.7603/s40956-015-0003-8

Moroccan J. Pure and Appl. Anal.(MJPAA) RESEARCH ARTICLE
Volume 1(1), 2015, Pages 38–50
ISSN: 2351-8227

Partial Sums of Certain Classes of Meromorphic Functions Related to the Hurwitz-Lerch Zeta Function

H. M. Srivastava[1,*], S. Gaboury[2] and F. Ghanim[3]

Abstract. In the present paper, we give sufficient conditions for a function f to be in the subclasses $\Sigma\mathcal{S}^*_{a,s}(A,B,\alpha,\beta)$ and $\Sigma\mathcal{K}_{a,s}(A,B,\alpha,\beta)$ of the class Σ of meromorphic functions which are analytic in the punctured unit disk $\mathbb{U}^*$. We further investigate the ratio of a function related to the Hurwitz-Lerch zeta function and its sequence of partial sums.

2010 Mathematics Subject Classification. 30C45, 11M35, 30C10.

Key words and phrases. Analytic functions; Partial sums; Meromorphic functions; Integral operators; Meromorphic starlike functions; Meromorphic convex functions; Meromorphic close-to-convex functions; Hurwitz-Lerch zeta function; Hadamard product.

1. Introduction and Preliminaries

Let Σ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the puntured unit disk

$$\mathbb{U}^* = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} = \mathbb{U} \setminus \{0\},$$

Received April 15, 2015 - Accepted June 25, 2015.

[1]Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada. E-Mail: harimsri@math.uvic.ca

[2]Department of Mathematics and Computer Science, University of Québec at Chicoutimi, Chicoutimi, Québec G7H 2B1, Canada. E-Mail: s1gabour@uqac.ca

[3]Department of Mathematics, College of Sciences, University of Sharjah, Sharjah, United Arab Emirates. E-Mail: fgahmed@sharjah.ac.ae

[*]Corresponding Author

Published online: 24 July 2015

$\mathbb{C}$ being (as usual) the set of complex numbers. We denote by $\Sigma\mathcal{S}^*(\beta)$ and $\Sigma\mathcal{K}(\beta)$ ($\beta \geqq 0$) the subclasses of Σ consisting of all meromorphic functions which are, respectively, starlike of order β and convex of order β in $\mathbb{U}^*$ (see also the recent works [19] and [20]).

For functions $f_j(z)$ $(j = 1, 2)$ defined by

$$f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k \qquad (j = 1, 2), \tag{2}$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k. \tag{3}$$

Let us consider the function $\widetilde{\phi}(\alpha, \beta; z)$ defined by

$$\widetilde{\phi}(\alpha, \beta; z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\alpha)_{k+1}}{(\beta)_{k+1}} a_k z^k \tag{4}$$

$$\left(\beta \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \alpha \in \mathbb{C}\right),$$

where

$$\mathbb{Z}_0^- = \{0, -1, -2, \cdots\} = \mathbb{Z}^- \cup \{0\}.$$

Here, and in the remainder of this paper, $(\lambda)_\kappa$ denotes the general Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_\kappa := \frac{\Gamma(\lambda + \kappa)}{\Gamma(\lambda)} = \begin{cases} \lambda(\lambda+1)\cdots(\lambda+n-1) & (\kappa = n \in \mathbb{N};\ \lambda \in \mathbb{C}) \\ 1 & (\kappa = 0;\ \lambda \in \mathbb{C} \setminus \{0\}), \end{cases} \tag{5}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [18, p. 21 *et seq.*]), $\mathbb{N}$ being the set of positive integers.

It is easy to see that, in the case when $a_k = 1$ $(k = 0, 1, 2, \cdots)$, the following relationship holds true between the function $\widetilde{\phi}(\alpha, \beta; z)$ and the Gaussian hypergeometric function [8]:

$$\widetilde{\phi}(\alpha, \beta; z) = \frac{1}{z}\, {}_2F_1(1, \alpha; \beta; z). \tag{6}$$

Very recently, Ghanim ([3]; see also [4]) made use of the Hadamard product for functions $f(z) \in \Sigma$ in order to introduce a new linear operator $L_a^s(\alpha, \beta)$ defined on Σ by

$$\begin{aligned} L_a^s(\alpha, \beta)(f)(z) &= \widetilde{\phi}(\alpha, \beta; z) * G_{s,a}(z) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s a_k z^k \quad \left(z \in \mathbb{U}^*\right), \end{aligned} \tag{7}$$

where

$$G_{s,a}(z) := (a+1)^s \left[\Phi(z,s,a) - a^{-s} + \frac{1}{z(a+1)^s}\right]$$
$$= \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{a+1}{a+k}\right)^s z^k \qquad \big(z \in \mathbb{U}^*\big) \tag{8}$$

and the function $\Phi(z,s,a)$ is the well-known Hurwitz-Lerch zeta function defined by (see, for example, [13, p. 121 *et seq.*]; see also [11], [12], [15], [16], [17] and [14, p. 194 *et seq.*])

$$\Phi(z,s,a) := \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s} \tag{9}$$

$$\big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \text{ when } |z|<1;\ \Re(s)>1 \text{ when } |z|=1\big).$$

Silverman [9] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. Also, Cho and Owa [2], Latha and Shivarudrappa [7], Ghanim and Darus [5] and Ibrahim and Darus [6] have investigated the ratio of a function $f \in \Sigma$ to its sequence of partial sums given by

$$f_m(z) = \frac{1}{z} + \sum_{k=1}^{m} a_k z^k.$$

Let the following classes:

$$\Sigma\mathcal{S}^*_{a,s}(A,B,\alpha,\beta), \quad \Sigma\mathcal{K}_{a,s}(A,B,\alpha,\beta) \qquad (-1 \leqq A < B \leqq 1,\ \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \alpha \in \mathbb{C})$$

and

$$\Sigma^*_{a,s}(A,B,\alpha,\beta) \qquad (-1 \leqq A < B \leqq 1,\ \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \alpha \in \mathbb{C})$$

be the subclasses of functions in Σ satisfying the conditions given by

$$-\left\{\frac{z\left(L_a^s(\alpha,\beta)f'(z)\right)}{L_a^s(\alpha,\beta)f(z)}\right\} \prec \frac{1+Az}{1+Bz} \tag{10}$$

$$\big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ z \in \mathbb{U}^*;\ s \in \mathbb{C} \text{ when } |z|<1;\ \Re(s)>1 \text{ when } |z|=1\big),$$

$$-\left\{\frac{z\left(L_a^s(\alpha,\beta)f''(z)\right)}{L_a^s(\alpha,\beta)f'(z)}\right\} \prec \frac{1+Az}{1+Bz} \tag{11}$$

$$\big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ z \in \mathbb{U}^*;\ s \in \mathbb{C} \text{ when } |z|<1;\ \Re(s)>1 \text{ when } |z|=1\big)$$

and

$$-z^2\left(L_a^s(\alpha,\beta)f'(z)\right) \prec \frac{1+Az}{1+Bz}, \tag{12}$$

$$\big(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ z \in \mathbb{U}^*;\ s \in \mathbb{C} \text{ when } |z|<1;\ \Re(s)>1 \text{ when } |z|=1\big),$$

respectively.

The classes $\Sigma\mathcal{S}^*_{a,0}(2\beta-1,1,\ \alpha,\alpha)$ and $\Sigma\mathcal{K}_{a,0}(2\beta-1,1,\ \alpha,\alpha)$ are, respectively, the well-known subclasses of Σ consisting the meromorphic starlike functions of order β, the meromorphic convex functions of order β and meromorphic close-to-convex functions of order β denoted by $\Sigma\mathcal{S}^*(\beta)$, $\Sigma\mathcal{K}(\beta)$ and $\Sigma^*(\beta)$, respectively.

In the present paper, we give sufficient conditions for a function f to be in the subclasses $\Sigma\mathcal{S}^*_{a,s}(A,B,\alpha,\beta)$ and $\Sigma\mathcal{K}_{a,s}(A,B,\alpha,\beta)$. We further investigate the ratio of a function of the form (1) related to the Hurwitz-Lerch zeta function and its sequence of partial sums when the coefficients are sufficiently small to satisfy the following conditions:

$$\sum_{k=1}^{\infty}\left[k(1+B)+(1+A)\right]\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s\right||a_n| \leqq B-A$$

and

$$\sum_{k=1}^{\infty}k\left[k(1+B)+(1+A)\right]\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s\right||a_n| \leqq B-A.$$

Also, we will study sharp lower bounds for

$$\Re\left\{\frac{L_a^s(\alpha,\beta)f(z)}{L_a^s(\alpha,\beta)f_m(z)}\right\},\quad \Re\left\{\frac{L_a^s(\alpha,\beta)f_m(z)}{L_a^s(\alpha,\beta)f(z)}\right\},\quad \Re\left\{\frac{L_a^s(\alpha,\beta)f'(z)}{L_a^s(\alpha,\beta)f'_m(z)}\right\}$$

and

$$\Re\left\{\frac{L_a^s(\alpha,\beta)f'_m(z)}{L_a^s(\alpha,\beta)f'(z)}\right\}.$$

Moreover, we will demonstrate an interesting property for the partial sums of certain integral operators in connection with functions defined by the linear operator $L_a^s(\alpha,\beta)(f)(z)$ given in the form (7).

2. A Set of Basic Results

Theorem 1. *A function $f\in\Sigma$ is said to be a member of the class $S^*_{a,s}(A,B,\alpha,\beta)$ if it satisfies the following inequality*:

$$\sum_{k=1}^{\infty}\left[k(1+B)+(1+A)\right]\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s\right||a_n| \leqq B-A \tag{13}$$

$$\left(a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ s\in\mathbb{C};\ -1\leqq A<B\leqq 1;\ z\in\mathbb{U}^*\right).$$

Proof. It suffices to show that

$$\left|\frac{\frac{z(L_a^s(\alpha,\beta)f'(z))}{L_a^s(\alpha,\beta)f(z)}+1}{B\left(\frac{z(L_a^s(\alpha,\beta)f'(z))}{L_a^s(\alpha,\beta)f(z)}\right)+A}\right|<1$$

or, equivalently,

$$\left|\frac{z\left(L_a^s(\alpha,\beta)f'(z)\right)+L_a^s(\alpha,\beta)f(z)}{Bz\left(L_a^s(\alpha,\beta)f'(z)\right)+AL_a^s(\alpha,\beta)f(z)}\right|<1. \tag{14}$$

Rewriting the left part of (14) explicitly with the help of (7) yields

$$\left| \frac{z\left(L_a^s(\alpha,\beta)f'(z)\right) + L_a^s(\alpha,\beta)f(z)}{Bz\left(L_a^s(\alpha,\beta)f'(z)\right) + AL_a^s(\alpha,\beta)f(z)} \right|$$

$$= \left| \frac{\sum\limits_{k=1}^{\infty} (k+1) \frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s a_k z^k}{(B-A) - \sum\limits_{k=1}^{\infty} (kB+A) \frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s a_k z^k} \right|$$

$$\leqq \frac{\sum\limits_{k=1}^{\infty} (k+1) \left| \frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s \right| |a_k|}{(B-A) - \sum\limits_{k=1}^{\infty} (kB+A) \left| \frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s \right| |a_k|}. \tag{15}$$

The last expression in (15) is bounded by 1 if

$$\sum_{k=1}^{\infty} (k+1) \left| \frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s \right| |a_k| \leqq (B-A) - \sum_{k=1}^{\infty} (kB+A) \left| \frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s \right| |a_k|,$$

which is equivalent to (13). □

In the same way, we can prove Theorem 2 given below.

Theorem 2. *A function $f \in \Sigma$ is said to be a member of the class $\Sigma\mathcal{K}_{a,s}(A,B,\alpha,\beta)$ if it satisfies the following inequality*:

$$\sum_{k=1}^{\infty} k\left(k\left(1+B\right)+\left(1+A\right)\right) \left| \frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s \right| |a_k| \leqq (B-A) \tag{16}$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C};\ -1 \leqq A < B \leqq 1;\ z \in \mathbb{U}^*\right).$$

3. Main Results

Theorem 3. *If f of the form* (1) *satisfies* (13), *then*

$$\Re\left\{ \frac{L_a^s(\alpha,\beta)f(z)}{L_a^s(\alpha,\beta)f_m(z)} \right\} \geqq \frac{2(m+1+A)}{2m+2+A+B} \left| \frac{(\beta)_{m+1}}{(\alpha)_{m+1}} \left(\frac{a+m}{a+1}\right)^s \right|. \tag{17}$$

The result is sharp for every $m \in \mathbb{N}$, with extremal function given by

$$f(z) = \frac{1}{z} + \frac{(B-A)}{2m+2+A+B} \left| \frac{(\beta)_{m+1}}{(\alpha)_{m+1}} \left(\frac{a+m}{a+1}\right)^s \right| z^{m+1} \tag{18}$$

$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C};\ -1 \leqq A < B \leqq 1;\ z \in \mathbb{U}^*\right).$$

Proof. Consider

$$\frac{2m+2+A+B}{B-A}\left\{\frac{L_a^s(\alpha,\beta)f(z)}{L_a^s(\alpha,\beta)f_m(z)}-\frac{2m+2+2A}{2m+2+A+B}\right\}$$

$$=\frac{1+\sum\limits_{k=1}^{m}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s a_k z^{k+1}+\frac{2m+2+A+B}{B-A}\sum\limits_{k=m+1}^{\infty}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s a_k z^{k+1}}{1+\sum\limits_{k=1}^{m}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s a_k z^{k+1}}$$

$$=\frac{1+A(z)}{1+B(z)}.$$

Set

$$\frac{1+A(z)}{1+B(z)}=\frac{1+w(z)}{1-w(z)},$$

so that

$$w(z)=\frac{A(z)-B(z)}{2+A(z)+B(z)}.$$

Then

$$w(z)=\frac{\frac{2m+2+A+B}{B-A}\sum\limits_{k=m+1}^{\infty}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s a_k z^{k+1}}{2+2\sum\limits_{k=1}^{m}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s a_k z^{k+1}+\frac{2m+2+A+B}{B-A}\sum\limits_{k=m+1}^{\infty}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s a_k z^{k+1}}$$

and

$$|w(z)|\leq\frac{\frac{2m+2+A+B}{B-A}\sum\limits_{k=m+1}^{\infty}\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s\right||a_k|}{2-2\sum\limits_{k=1}^{m}\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s\right||a_k|-\frac{2m+2+A+B}{B-A}\sum\limits_{k=m+1}^{\infty}\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s\right||a_k|}.$$

Now, clearly, $|w(z)|\leqq 1$ if and only if

$$2\left(\frac{2m+2+A+B}{B-A}\right)\sum_{k=m+1}^{\infty}\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s\right||a_k|$$

$$\leqq 2-2\sum_{k=1}^{m}\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s\right||a_k|,$$

which is equivalent to

$$\sum_{k=1}^{m}\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s\right||a_k|$$

$$+\left(\frac{2m+2+A+B}{B-A}\right)\sum_{k=m+1}^{\infty}\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s\right||a_k|\leqq 1. \tag{19}$$

It suffices to show that the left-hand side of (19) above is bounded by

$$\sum_{k=1}^{\infty}\left[\frac{k\,(1+B)+(1+A)}{B-A}\right]\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^{s}\right|\left|a_k\right|,$$

which is equivalent to

$$\sum_{k=1}^{m}\left[\frac{k\,(1+B)+2A-B+1}{B-A}\right]\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^{s}\right|\left|a_k\right|$$

$$+\sum_{k=m+1}^{\infty}\left[\frac{(k-1)\,(1+B)-2m}{B-A}\right]\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^{s}\right|\left|a_k\right| \geqq 0.$$

To see that the function f given by (18) gives the sharp result, we observe, for

$$z = r\mathrm{e}^{\frac{i\pi}{(m+2)}},$$

that

$$\begin{aligned}\frac{L_a^s(\alpha,\beta)f(z)}{L_a^s(\alpha,\beta)f_m(z)} &= 1+\frac{(B-A)}{2m+2+A+B}\left|\frac{(\beta)_{m+1}}{(\alpha)_{m+1}}\left(\frac{a+m}{a+1}\right)^{s}\right| z^{m+2}\\ &\to 1-\frac{(B-A)}{2m+2+A+B}\left|\frac{(\beta)_{m+1}}{(\alpha)_{m+1}}\left(\frac{a+m}{a+1}\right)^{s}\right|\\ &= \frac{2(m+1+A)}{(2m+2+A+B)}\left|\frac{(\beta)_{m+1}}{(\alpha)_{m+1}}\left(\frac{a+m}{a+1}\right)^{s}\right| \qquad (20)\end{aligned}$$

when $r \to 1$. Hence, the proof of Theorem 3 is complete. □

Theorem 4. *If f of the form* (1) *satisfies* (16), *then*

$$\Re\left\{\frac{L_a^s(\alpha,\beta)f(z)}{L_a^s(\alpha,\beta)f_m(z)}\right\} \geqq$$

$$\frac{(2m+2)\,(m+1)+m\,(A+B)+2A}{(m+1)\,(2m+2+A+B)}\left|\frac{(\beta)_{m+1}}{(\alpha)_{m+1}}\left(\frac{a+m}{a+1}\right)^{s}\right|. \qquad (21)$$

The result is sharp for every $m \in \mathbb{N}$, with the extremal function given by

$$f\,(z) = \frac{1}{z}+\frac{(B-A)}{2m+2+A+B}\left|\frac{(\beta)_{m+1}}{(\alpha)_{m+1}}\left(\frac{a+m}{a+1}\right)^{s}\right| z^{m+1}, \qquad (22)$$

$$\left(a \in \mathbb{C}\setminus\mathbb{Z}_0^-;\ s \in \mathbb{C};\ -1 \leqq A < B \leqq 1;\ z \in \mathbb{U}^*\right).$$

Proof. We write

$$\frac{(m+1)(2m+2+A+B)}{B-A}\left\{\frac{L_a^s(\alpha,\beta)f(z)}{L_a^s(\alpha,\beta)f_m(z)}-\frac{(2m+2)(m+1)+(m(A+B)+2A)}{(m+1)(2m+2+A+B)}\right\}$$

$$=\frac{1+\sum\limits_{k=1}^{m}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^{s}a_k z^{k+1}+\frac{(m+1)(2m+2+A+B)}{B-A}\sum\limits_{k=m+1}^{\infty}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^{s}a_k z^{k+1}}{1+\sum\limits_{k=1}^{m}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^{s}a_k z^{k+1}}$$

$$= \frac{1+A(z)}{1+B(z)}.$$

Set

$$\frac{1+A(z)}{1+B(z)} = \frac{1+w(z)}{1-w(z)},$$

so that

$$w(z) = \frac{A(z)-B(z)}{2+A(z)+B(z)}.$$

Then

$$w(z) \leqq \frac{\frac{(m+1)(2m+2+A+B)}{B-A} \sum\limits_{k=m+1}^{\infty} \frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s a_k z^{k+1}}{2+2\sum\limits_{k=1}^{m} \frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s a_k z^{k+1} + \frac{(m+1)(2m+2+A+B)}{B-A} \sum\limits_{k=m+1}^{\infty} \frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s a_k z^{k+1}}.$$

Now

$$|w(z)|$$

$$\leqq \frac{\frac{(m+1)(2m+2+A+B)}{B-A} \sum\limits_{k=m+1}^{\infty} \left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s\right| |a_k|}{2-2\sum\limits_{k=1}^{m} \left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s\right| |a_k| - \frac{(m+1)(2m+2+A+B)}{B-A} \sum\limits_{k=m+1}^{\infty} \left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s\right| |a_k|}$$

if

$$\sum_{k=1}^{m} \left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s\right| |a_k|$$

$$+\frac{(m+1)(2m+2+A+B)}{B-A} \sum_{k=m+1}^{\infty} \left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s\right| |a_k| \leq 1. \tag{23}$$

The left-hand side of (23) above is bounded by

$$\sum_{k=1}^{\infty} \left(\frac{2m+2+A+B}{B-A}\right) \left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s\right| |a_k|,$$

if

$$\frac{1}{B-A} \left(\sum_{k=1}^{m} \left[k\left(k\left(1+B\right)+\left(1+A\right)\right)-\left(B-A\right)\right] \left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s\right| |a_k| +\right.$$

$$\left.\sum_{k=m+1}^{\infty} \left[k\left(k\left(1+B\right)+\left(1+A\right)\right)-\left(m+1\right)\left(2m+2+A+B\right)\right] \left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s\right| |a_k|\right)$$

$$\geqq 0.$$

□

Theorem 5.

(I) *If f of the form* (1) *satisfies the condition* (13), *then*

$$\Re\left\{\frac{L_a^s(\alpha,\beta)f_m(z)}{L_a^s(\alpha,\beta)f(z)}\right\} \geqq \frac{(2m+2+A+B)}{2\,(m+1+B)}\left|\frac{(\alpha)_{m+1}}{(\beta)_{m+1}}\left(\frac{a+1}{a+m}\right)^s\right|; \tag{24}$$

(II) *If f of the form* (1) *satisfies condition* (16), *then*

$$\Re\left\{\frac{L_a^s(\alpha,\beta)f_m(z)}{L_a^s(\alpha,\beta)f(z)}\right\}$$

$$\geqq \frac{(m+1)\,(2m+2+A+B)}{2(m+1)\,(m+1+B)-m\,(m+1+A)}\left|\frac{(\alpha)_{m+1}}{(\beta)_{m+1}}\left(\frac{a+1}{a+m}\right)^s\right| \tag{25}$$

$$\left(a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ s\in\mathbb{C};\ -1\leqq A<B\leqq 1;\ z\in\mathbb{U}^*\right).$$

Equalities hold true in (I) and (II) for the functions given by (18) *and* (22), *respectively.*

Proof. The proof of (II) is similar to the proof of (I). Hence, we will only give the proof of (I). We have

$$\frac{2\,(m+1+B)}{B-A}\left\{\frac{L_a^s(\alpha,\beta)f_m(z)}{L_a^s(\alpha,\beta)f(z)}-\frac{(2m+2+A+B)}{2\,(m+1+B)}\right\}$$

$$=\frac{1+\sum\limits_{k=1}^{m}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s a_k z^{k+1}+\left(\frac{2m+2+A+B}{B-A}\right)\sum\limits_{k=m+1}^{\infty}\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s\right| a_k z^{k+1}}{1+\sum\limits_{k=1}^{m}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s a_k z^{k+1}}$$

$$=\frac{1+A(z)}{1+B(z)}.$$

Set

$$\frac{1+A(z)}{1+B(z)}=\frac{1+w(z)}{1-w(z)},$$

so that

$$w(z)=\frac{A(z)-B(z)}{2+A(z)+B(z)}.$$

Then

$$|w(z)|\leqq\frac{\left(\frac{m+1+B}{B-A}\right)\sum\limits_{k=m+1}^{\infty}\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s\right||a_k|}{2-2\sum\limits_{k=1}^{m}\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s\right||a_k|-\left(\frac{m+1+A}{B-A}\right)\sum\limits_{k=m+1}^{\infty}\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s\right||a_k|}.$$

This last inequality is equivalent to

$$\sum_{k=1}^{m}\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^{s}\right||a_k|$$
$$+\left(\frac{2m+2+A+B}{B-A}\right)\sum_{k=m+1}^{\infty}\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^{s}\right||a_k| \leqq 1. \tag{26}$$

Since the left-hand side of (26), above, is bounded by

$$\sum_{k=1}^{\infty}\left[\frac{k(1+B)+(1+A)}{B-A}\right]\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^{s}\right||a_k|,$$

the proof is thus completed. □

Theorem 6. *If f of the form* (1) *satisfies condition* (13) *with $A=-B$, then*

$$\text{(I)}\qquad \Re\left\{\frac{L_a^s(\alpha,\beta)f'(z)}{L_a^s(\alpha,\beta)f_m'(z)}\right\}\geqq 0$$

and

$$\text{(II)}\qquad \Re\left\{\frac{L_a^s(\alpha,\beta)f_m'(z)}{L_a^s(\alpha,\beta)f'(z)}\right\}\geqq \frac{1}{2}\left|\frac{(\alpha)_{m+1}}{(\beta)_{m+1}}\left(\frac{a+1}{a+m}\right)^{s}\right|$$

$$\left(a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ s\in\mathbb{C};\ -1\leqq A<B\leqq 1;\ z\in\mathbb{U}^*\right).$$

In both cases (*I*) *and* (*II*), *the extremal function is given by* (18) *with $A=-B$.*

Proof. Using the same technique as in the proof of Theorem 5 combined with Part (I) of Theorem 3, the proof of Theorem 6 follows easily. Hence, we will not go through the details. □

Theorem 7. *If f of the form* (1) *satisfies condition* (16), *then*

$$\text{(I)}\qquad \Re\left\{\frac{L_a^s(\alpha,\beta)f'(z)}{L_a^s(\alpha,\beta)f_m'(z)}\right\}\geqq \frac{2(m+1+A)}{(2m+2+A+B)}\left|\frac{(\alpha)_{m+1}}{(\beta)_{m+1}}\left(\frac{a+1}{a+m}\right)^{s}\right|$$

and

$$\text{(II)}\qquad \Re\left\{\frac{L_a^s(\alpha,\beta)f_m'(z)}{L_a^s(\alpha,\beta)f'(z)}\right\}\geqq \frac{(2m+2+A+B)}{2(m+1+B)}\left|\frac{(\alpha)_{m+1}}{(\beta)_{m+1}}\left(\frac{a+1}{a+m}\right)^{s}\right|,$$

$$\left(a\in\mathbb{C}\setminus\mathbb{Z}_0^-;\ s\in\mathbb{C};\ -1\leqq A<B\leqq 1;\ z\in\mathbb{U}^*\right).$$

In both cases (*I*) *and* (*II*), *the extremal function is given by* (22).

Proof. It is well known that

$$f\in\Sigma\mathcal{K}_{a,s}(A,B,\alpha,\beta)\Leftrightarrow -zf'\in\Sigma\mathcal{S}_{a,s}^{*}(A,B,\alpha,\beta).$$

In particular, $f(z)$ satisfies the condition (16) if and only if $-zf'(z)$ satisfies condition (13). Thus (I) is an immediate consequence of Theorem 3 and (II) follows directly from (I) of Theorem 7. □

4. A Family of Integral Operators

For a function $f \in L_a^s(\alpha,\, \beta)\, f(z)$, we define the integral operator $F(z)$ as follows

$$F(z) = \frac{1}{z^2}\int_0^z tf(t)dt = \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(\alpha)_{k+1}}{(k+2)(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s a_k z^k, \qquad (z \in \mathbb{U}^*).$$

The mth partial sum $F_m(z)$ of the function $F(z)$ is given by

$$F_m(z) = \frac{1}{z} + \sum_{k=1}^{m} \frac{(\alpha)_{k+1}}{(k+2)(\beta)_{k+1}} \left(\frac{a+1}{a+k}\right)^s a_k z^k, \qquad (z \in \mathbb{U}^*).$$

The following lemmas will be required for the proof of Theorem 8 given below.

Lemma 1. *For $0 \leqq \theta \leqq \pi$,*

$$\frac{1}{2} + \sum_{k=1}^{m} \frac{\cos(k\theta)}{k+1} \geqq 0.$$

Lemma 2. *Let P be analytic in $\mathbb{U}$ with $P(0) = 1$ and $\Re\{P(z)\} > \frac{1}{2}$ be in U. For any function Q which is analytic in $\mathbb{U}$, the function $P * Q$ takes values in the convex hull of the image on U under Q.*

Lemma 1 is due to Rogosinski and Szego [21] and Lemma 2 is a well known result (c.f. [[1] and [10]]) that can be derived from the Herglotz representation for P.

After giving the above lemmas, we can proceed to the proof of our last result.

Theorem 8. *If $f \in \Sigma\mathcal{K}_{a,s}(A, B, \alpha, \beta)$, then $F_m \in \Sigma\mathcal{K}_{a,s}(A, B, \alpha, \beta)$.*

Proof. Let f be of the form (7) and belongs to the class $\Sigma\mathcal{K}_{a,s}(A, B, \alpha, \beta)$. Then we have

$$\Re\left(1 - \frac{1}{B-A}\sum_{k=1}^{\infty} k\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s a_k z^{k+1}\right) > \frac{1}{2} \qquad (z \in \mathbb{U}^*). \tag{27}$$

Applying the convolution properties of power series to $F'_m(z)$, we may write

$$\begin{aligned} -z^2 F'_m(z) &= 1 - \sum_{k=1}^{m} \frac{k}{k+2}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s a_k z^{k+1} \\ &= \left(1 + \frac{1}{B-A}\sum_{k=1}^{\infty}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s a_k z^{k+1}\right) \\ &\quad * \left(1 + (B-A)\sum_{k=1}^{m+1}\frac{1}{k+1}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^s z^{k+1}\right). \end{aligned} \tag{28}$$

Putting

$$z = re^{i\theta} \qquad (0 \leqq r < 1;\ 0 \leqq |\theta| \leqq \pi)$$

and making use of the *Minimum Modulus Principle* for harmonic functions in conjunction with Lemma 1, we obtain

$$\begin{aligned}
&\Re\left(1+(B-A)\sum_{k=1}^{m+1}\frac{1}{k+1}\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^{s}z^{k+1}\right)\\
&\geqq 1-(B-A)\sum_{k=1}^{m+1}\frac{r^k\cos(k\theta)}{k+1}\left|\frac{(\alpha)_{k+1}}{(\beta)_{k+1}}\left(\frac{a+1}{a+k}\right)^{s}\right|\\
&> 1-(B-A)\sum_{k=1}^{m+1}\frac{r^k\cos(k\theta)}{k+1}\\
&\geqq 1-\left(\frac{B-A}{2}\right). \qquad (29)
\end{aligned}$$

Finally, in view of (27), (28), (29) and Lemma 2, we deduce that

$$-\Re\left\{z^2f'(z)\right\} > 1-\left(\frac{B-A}{2}\right) \qquad (0 \leqq A+B<2;\ z\in\mathbb{U}^*),$$

which completes the proof of Theorem 8. □

5. Concluding Remarks and Observations

In the present investigation, we have derived sufficient conditions for a function f to be in the above-defined subclasses $\Sigma\mathcal{S}^*_{a,s}(A,B,\alpha,\beta)$ and $\Sigma\mathcal{K}_{a,s}(A,B,\alpha,\beta)$ of the class Σ of meromprphic functions which are analytic in the puntured unit disk $\mathbb{U}^*$. We have further investigated the ratio of a function related to the Hurwitz-Lerch zeta function $\Phi(z,s,a)$ and its sequence of partial sums. The various results which we have presented here would extend and improve several earlier studies on the subject of this paper.

References

[1] A. W. Goodman, *Univalent Functions*, Vol. **I**, Mariner Publishing Company, Tampa, Florida, 1983.

[2] N. E. Cho and S. Owa, On partial sums of certain meromorphic functions, *J. Inequal. Pure Appl. Math.* **5** (2) (2004), Article 30, 1–7.

[3] F. Ghanim, A study of a certain subclass of Hurwitz-Lerch zeta function related to a linear operator, *Abstr. Appl. Anal.* **2013** (2013), Article ID 763756, 1–7.

[4] F. Ghanim and M. Darus, New result of analytic functions related to Hurwitz-Zeta function, *The Sci. World J.* **2013** (2013), Article ID 475643, 1–5.

[5] F. Ghanim and M. Darus, Partial sums of certain new subclasses for meromorphic functions, *Far East J. Math. Sci.* **55** (2011), 181–195.

[6] R. W. Ibrahim and M. Darus, Partial sums for certain classes of meromorphic functions, *Tamkang J. Math.* **41** (2010), 39–49.

[7] S. Latha and L. Shivarudrappa, Partial sum of meromorphic functions, *J. Inequal. Pure Appl. Math.* **7** (4) (2006), Article 140, 1–8.

[8] E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.

[9] H. Silverman, Partial sums of starlike and convex functions, *J. Math. Anal. Appl.* **209** (1997), 221–227.

[10] R. Singh and S. Singh, Convolution properties of a class of starlike fuctions, *Proc. Amer. Math. Soc.* **106** (1989), 145–152.

[11] H. M. Srivastava, Some formulas for the Bernoulli and Euler polynomials at rational arguments, *Math. Proc. Cambridge Philos. Soc.* **129** (2000), 77–84.

[12] H. M. Srivastava, Some generalizations and basic (or q-) extensions of the Bernoulli, Euler and Genocchi polynomials, *Appl. Math. Inform. Sci.* **5** (2011), 390–444.

[13] H. M. Srivastava and J. Choi, *Series Associated with Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.

[14] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.

[15] H. M. Srivastava, S. Gaboury and F. Ghanim, Certain subclasses of meromorphically univalent functions defined by a linear operator associated with the λ-generalized Hurwitz-Lerch zeta function, *Integral Transforms Spec. Funct.* **26** (2015), 258–272.

[16] H. M. Srivastava, S. Gaboury and F. Ghanim, Some further properties of a linear operator associated with the λ-generalized Hurwitz-Lerch zeta function related to the class of meromorphically univalent functions, *Appl. Math. Comput.* **259** (2015), 1019–1029.

[17] H. M. Srivastava, S. Gaboury and F. Ghanim, A study of meromorphically univalent functions defined by a linear operator associated with the λ-generalized Hurwitz-Lerch zeta function, Preprint (2015), 1–12.

[18] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwoord Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[19] S.-M. Yuan, Z.-M. Liu and H. M. Srivastava, Some inclusion relationships and integral-preserving properties of certain subclasses of meromorphic functions associated with a family of integral operators, *J. Math. Anal. Appl.* **337** (2008), 505–515.

[20] Z.-G. Wang, H. M. Srivastava and S.-M. Yuan, Some basic properties of certain subclasses of meromorphically starlike functions, *J. Inequal. Appl.* **2014** (2014), Article ID 2014:29, 1–12.

[21] W. Rogosinski and G. Szegö, Über die abschimlte von potenzreihen die in ernein kreise be schranket bleiben, *Math. Zeitschr.* **28** (1928), 73–94.