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What is Hilbert's 24th Problem?

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Abstract In 2000, a draft note of David Hilbert was found in his Nachlass concerning a 24th problem he had consider to include in the his famous problem list of the talk at the International Congress of Mathematicians in 1900 in Paris. This problem concerns *simplicity of proofs*. In this paper we review the (very few) traces of this problem which one can find in the work of Hilbert and his school, as well as modern research started on it after its publication. We stress, in particular, the mathematical nature of the problem.[1]

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1. Hilbert's 24th Problem

In 2000, Rüdiger Thiele [Thi03] found in a notebook of David Hilbert, kept in Hilbert's *Nachlass* at the University of Göttingen, a small note concerning a 24th problem he had considered for inclusion in his famous

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problem list for the International Congress of Mathematicians in Paris in 1900 [Hil01b, Hil01a]. The short paragraph reads in German as follows:

> "Als 24stes Problem in meinem Pariser Vortrag wollte ich die Frage stellen: Kriterien für die Einfachheit bez. Beweis der grössten Einfachheit von gewissen Beweisen führen. Ueberhaupt eine Theorie der Beweismethoden in der Mathematik entwickeln. Es kann doch bei gegebenen Voraussetzungen nur einen einfachsten Beweis geben. Überhaupt, wenn man für einen Satz 2 Beweise hat, so muss man nicht eher ruhen, als bis man sie beide aufeinander zurückgeführt hat oder genau erkannt hat, welche verschiedenen Voraussetzungen (und Hilfsmittel) bei den Beweisen benutzt werden: Wenn man 2 Wege hat, so muss man nicht bloss diese Wege gehen oder neue suchen, sondern dann das ganze zwischen den beiden Wegen liegende Gebiet erforschen. Ansätze, die Einfachheit der Beweise zu beurteilen, bieten meine Untersuchungen über Syzygien und Syzygien zwischen Syzygien. Die Benutzung oder Kenntnisse einer Syzygie vereinfacht den Beweis, dass eine gewisse Identität richtig ist, erheblich. Da jeder Process des Addierens Anwendung des commutativen Gesetzes der Addition ist — dies immer geometrischen Sätzen oder logischen Schlüssen entspricht, so kann man diese zählen und z. B. beim Beweis bestimmter Sätze in der Elementargeometrie (Pythagoras oder über merkwürdige Punkte im Dreieck) sehr wohl entscheiden, welches der einfachste Beweis ist."

An English translation was given by Thiele as follows:

> "The 24th problem in my Paris lecture was to be: Criteria of simplicity, or proof of the greatest simplicity of certain proofs. Develop a theory of the method of proof in mathematics in general. Under a given set of conditions there can be but one simplest proof. Quite generally, if there are two proofs for a theorem, you must keep going until you have derived each from the other, or until it becomes quite

> evident what variant conditions (and aids) have been used in the two proofs. Given two routes, it is not right to take either of these two or to look for a third; it is necessary to investigate the area lying between the two routes. Attempts at judging the simplicity of a proof are in my examination of syzygies and syzygies between syzygies. The use or the knowledge of a syzygy simplifies in an essential way a proof that a certain identity is true. Because any process of addition [is] an application of the commutative law of addition etc. [and because] this always corresponds to geometric theorems or logical conclusions, one can count these [processes], and, for instance, in proving certain theorems of elementary geometry (the Pythagoras theorem, [theorems] on remarkable points of triangles), one can very well decide which of the proofs is the simplest."

2. Hilbert's 24th Problem in the Work of Hilbert and the Hilbert School

Hilbert decided not to include this problem in the published list of the now famous 23 problems from the International Congress of Mathematicians in Paris in 1900. But we have no indications for his motives, and it would be pure speculation to discuss such possible motives.

In his work one can, however, find some, albeit very thin, threads of H24.

The published version[2] of his contribution to the International Congress of Mathematician contains some general considerations which address *simplicity* in relation to *rigor* [Hil01a]:

> "Besides it is an error to believe that rigor in the proof is the enemy of simplicity. On the contrary we find it confirmed by numerous examples that the rigorous method is at the same time the simpler and the more easily com-

2 It is known that in his talk he didn't read the full paper; in particular, he presented only 10 of the 23 problems explicitly, see [GG00].

> prehended. The very effort for rigor forces us to find out simpler methods of proof."

This claim is illustrated by referring to three examples concerning *the theory of algebraic curves*, *power series*, and *the calculus of variations*. In these examples he is far from any form of criteria for simplicity, but it is worth noting that they are all taken from core mathematics.

After this, there is only one single instance where we found simplicity mentioned in David Hilbert's work, but at a very prominent place. In September 1917 Hilbert gave a talk at the Swiss Mathematical Society with the title *Axiomatisches Denken* (Axiomatic Thinking) [Hil18, Hil70]. This talk marked the "return" of Hilbert to foundational questions after his research in this direction was put aside after 1904; it was also the occasion where he invited Paul Bernays to return to Göttingen to assist him in this enterprise. The relevant paragraph of his lecture reads as follows [Hil70]:

> "By closer inspection, we realize soon that the question of consistency for the whole numbers and sets is not a isolated one, but it belongs to a wide range of most difficult epistemological questions of specific mathematical coloring: I mention, to characterize briefly this area of questions, the problem of the principle *solvability of any mathematical question*, the problem of a posteriori *controllability* of the result of a mathematical investigation, further the question of a *criterion for the simplicity* of mathematical proofs, the question of the relation between *contentness* and *formalism* in mathematics and logic, and finally the problem of the *decidability* of a mathematical question by a finite number of operations."

A "*criterion for the simplicity* of mathematical proofs" is mentioned here only alongside several other questions. But one may note the slight switch from the plural "criteria" in his notes of 1900 to the singular "criterion" in 1917. Albeit this explicit mentioning, the question of simplicity was never taken up by the Hilbert school. We only reencounter the issue when Bernays wrote 50 years later an encyclopedia entry for “David Hilbert” [Ber67, p. 500]. Obviously with the 1917 talk at hand, we writes:

> “Hilbert’s return to the problem of the foundations of arithmetic was announced by his delivery at Zurich in 1917 of the lecture “Axiomatisches Denken.” In the latter part of this lecture he pointed out several epistemological questions which, as he said, are connected with that of the consistency of number theory and set theory: the problem of the solubility in principle of every mathematical question; that of finding a standard of simplicity for mathematical proofs; that of the relation of contents and formalism in mathematics; and that of the decidability of a mathematical question by a finite procedure.”

This paragraph is of interest not because of the listed questions, but because the one which is not listed: the “problem of posteriori *controllability* of the result of a mathematical investigation” was left out by Bernays. A possible explanation would be that, by 1967, this question appeared to be solved in a way that not even the question would make much sense any longer. In return, simplicity was, obviously, even 50 years later an open question (as it is today).

Finally, we like to mention Saunders MacLane’s PhD dissertation [Mac34b][3] written in Göttingen in 1934. Its title “Abgekürzte Beweise im Logikkalkul” (Abbreviated Proofs in Logic Calculus) sounds like an echo of H24. The content, however, does not address simplicity directly, but just brevity of (encoding of) proofs. Officially, MacLane was supervised by Hermann Weyl (Hilbert’s successor in Göttingen), but one can safely assume that the de-facto supervisor was Paul Bernays, who could still have had reminiscences of H24 in mind. But there is no written evidence to support such a link.

3 See also [Mac34a].

3. Hilbert's 24th Problem in the Modern Literature

After the publication of H24 by Rüdiger Thiele [Thi03], various scholars took up the challenge of simplicity in one or the other form. This starts with Thiele himself who published a joint paper with Wos on H24 in relation to *automated reasoning* [TW02]. Wos and Pieper took the study of H24 in the context of automated reasoning further, linking it with elegance [WP03].

In a similar direction, H24 was taken up by investigating the logical structure of proofs with respect to criteria for simplicity; see, for instance, Hughes [Hug06a, Hug06b], Strassburger [Str05, Str06] and the book of Negri and von Plato [NvP14], among others. They link H24 mainly with structural properties of formal calculi relating it with modern developments in Gentzen-style proof theory or linear logic, invoking category theory, or combinatorial proofs. Strassburger even relates H24 to the question of *equality of proofs*, a question which, as such, was not addressed by Hilbert.

One may note that these studies concentrate mainly on aspects of simplicity within the *logical* representation of proofs. When Hilbert referred to the *counting* of proof processes, such a counting can, indeed, be best performed in logical calculi. But one should take into account that, by the time Hilbert took his notes on H24, logical calculi were still in development, and it was only in the 1920s that proof theory, as we know it today, was developed, in fact, mainly by the Hilbert school in Göttingen. Taking Hilbert's note on H24 as a whole into account, it is obvious that he thought more about simplicity in terms of mathematical concepts (as in the case of "syzygies between syzygies") rather than simplicity in terms of logical bookkeeping. Besides this, it is rather questionable whether length would even be a good measure for simplicity. For the very example of *Pythagoras Theorem*, mentioned by Hilbert, Loomis collected more than 350 different proofs [Loo68], and it is more than doubtful that one would like to pick the shortest of all these proofs as the simplest.

In another direction H24 was studied by considering variations of (non-logical) axiom systems (e.g., for geometry, for algebra, etc.) which were compared with respect to simplicity of proofs. Here we may refer to the discussion of Pambuccian and Alama [Pam88, Pam11, Ala14] which gives progressively improved axiomatizations for (hyperbolic) Geometry which allow to identify various criteria of simplicity for the chosen axioms. Here the emphasis is on mathematical content (semantically, in the form of axioms), with little concern for formal calculi and notions defined therein (e.g., proof length). But it also focuses more on the axiom systems rather than the proofs themselves.

Arana [Ara17][4] addressed the question how simplicity might be related to *purity*, a topic which was also already addressed by Hilbert, most notable at the end of this famous book *Grundlagen der Geometrie* [Hil10, p. 131]. It might not come to a surprise that there seems to be, in general, a trade-off between simplicity and purity of mathematical proofs.

Finally, one may ask whether simplicity can play a role to clarify the notion of explanation in mathematics (see, for instance, [Man01, Man08] or [Lan17, Part III]), even if such a clarification was—to our knowledge—not envisaged by Hilbert himself.

4. Mathematics and Proof Theory

4.1 Hilbert's 24th Problem as a Problem in (and about) Mathematics

As already mentioned we see H24, in the first place, as a problem concerning mathematics. That means, the criteria for simplicity have to take into account the mathematical concepts involved in a proof rather than (only) the logical structure of a proof. It is true that the logical structure might play a basic role and that without first studying simplicity in logical terms one hardly could reach criteria with respect to the mathematical concepts. Still, the purely logical investigations can only be a first step.

The simplified claim "there can be but one simplest proof" will surely be rejected by a vast majority of scholars; but Hilbert was careful enough to qualify it by adding "under a given set of conditions". It might still be doubtful whether "under a given set of conditions", indeed, "there can be but one simplest proof". But Hilbert made at least clear, that we should have to identify these "given set of conditions" for different proofs (of

4 This paper appeared in a recent book dedicated to *Simplicity: Ideals of Practice in Mathematics and the Arts* [KO17] which approaches H24 from a very broad perspective.

the same theorem). At least, the identification of different conditions can be seen as the result of an investigation of "the area lying between the two routes." And in modern terms, such conditions may correspond simply to different axiomatizations (non-logical axioms) of a certain field of mathematics.

A full appreciation of H24 will, first of all, require to take Hilbert's own suggestion into account to look to his "examination of syzygies and syzygies between syzygies". Thiele already managed to identify a passage in Hilbert's unpublished lecture notes which deal with this question [Hil97, lectures XXXII–XXXIX]. But today, we have a rather long list of well-studied examples of theorems with different proofs at hand; while in many instances, it is easy to judge whether one proof is simpler than another (or on which different conditions they depend), for now, it doesn't look like that these examples would give rise to general criteria. A standard reference for such examples are the *Proofs from THE BOOK* collected by Aigner and Ziegler [AZ09]. This book doesn't aim for simplicity in the proofs but rather for elegance. But this collection of examples should be the first benchmark for any theory of simplicity of mathematical proof.

4.2 Hilbert's 24th Problem and Proof Theory

By focusing on simplicity one might overlook a striking aim Hilbert was put down in the second sentence: "Develop a theory of the method of proof in mathematics in general." It is mentioned here, apparently, only as base for a later development of criteria of simplicity. But one may ask how it relates to Hilbert's *proof theory* developed in the 1920s. Usually, Hilbert's 2nd problem is taken as the starting point to develop proof theory. But one may note that such a theory was not mentioned in his exposition of the second problem of his problem list concerning the *consistency of the arithmetical axioms*.[5] In this exposition he only addresses axiom systems — which, as such, were known since Euclid's times — but no formal apparatus to derive formulas. A rudimentary example for what could be a formal theory of proofs is given in his talk for International Congress of Mathematicians in 1904 in Heidelberg [Hil05]. But it requires

5 It is worth noting that, by that time, the real numbers were subsumed under arithmetic, at least by Hilbert.

a lot of good will and a clear idea of modern proof theory to see here germs of proof theory as we know it today. In this respect "a theory of the method of proof in mathematics in general", as addressed in H24, is much closer to proof theory than anything mentioned in relation to Hilbert's 2nd problem. This connection of proof theory to H24 even vindicates the modern studies of structural proof theory for H24. But one should be careful: Hilbert is, in H24, not just speaking about a "theory of proof" but a "theory of *method* of proof".[6] Such a theory is still today a desideratum.

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6 The composed German term "Beweismethoden" makes this even clearer.

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