DETERMINING A PARTICULAR SOLUTION FOR SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS THAT HAVE ON THE COLUMN OF FREE TERMS FUNCTIONS SUCH AS

$$e^{a \cdot x} \cdot P(x) \cdot \cos^n x \text{ or } e^{b \cdot x} \cdot Q(x) \cdot \sin^n x$$

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Abstract: The determination of a particular solution for the systems of linear differential equations with constant coefficients that have on the column of free terms functions such as $e^{a \cdot x} \cdot P(x) \cdot \cos^n x$ or/and $e^{b \cdot x} \cdot Q(x) \cdot \sin^n x$, $n \in N$, is based on the expansion of $\cos^n x$ and $\sin^n x$, and, on the other hand, on how a particular solution for a column of free terms with functions such as $e^{a \cdot x} \cdot P(x) \cdot \cos^n x$ or $e^{b \cdot x} \cdot Q(x) \cdot \sin^n x$, $n \in N$ looks like. We can also write the way a particular solution looks like when we have a combination of two or more functions on the form $e^{a \cdot x} \cdot P(x) \cdot \cos^n x$ or $e^{b \cdot x} \cdot Q(x) \cdot \sin^n x$, $n \in N$ in the column of free terms.

Keywords: systems of linear differential equations with constant coefficients, particular solution, general solution, column of free terms with functions such as $P(x) \cdot e^{a \cdot x} \cdot \cos^n x$ or/and $Q(x) \cdot e^{b \cdot x} \cdot \sin^n x$

Introduction

Based on the expansion of $\cos^n x$ and $\sin^n x$, in this study it is presented the way in which a particular solution for the systems of linear differential equations with constant coefficients that has on the column of free terms functions such as

$$e^{a \cdot x} \cdot P(x) \cdot \cos^n x \text{ or/and}$$

$e^{b \cdot x} \cdot Q(x) \cdot \sin^n x$, $n \in N$ can be directly constructed similarly as it was done in [1]. As we mentioned before, we will start from the expansion of $\cos^n x$ or $\sin^n x$, according to the relations below:

$$\cos^n x = \frac{1}{2^{n-1}} \cdot \left[\sum_{\substack{r=0 \\ 2 \cdot r < n}} C_n^r \cdot \cos(n - 2 \cdot r) \cdot x \right] + \frac{1}{2^n} \cdot C_n^{n/2}, n \text{ even}; \quad (1)$$

$$\cos^n x = \frac{1}{2^{n-1}} \cdot \left[\sum_{\substack{r=0 \\ 2 \cdot r < n}} C_n^r \cdot \cos(n - 2 \cdot r) \cdot x \right], n \text{ odd}; \quad (2)$$

$$\sin^n x = \frac{(-1)^{n/2}}{2^{n-1}} \cdot \left[\sum_{\substack{r=0 \\ 2 \cdot r < n}} (-1)^r \cdot C_n^r \cdot \cos(n-2 \cdot r) \cdot x \right] + \frac{1}{2^n} \cdot C_n^{n/2}, n \text{ even}; \quad (3)$$

$$\sin^n x = \frac{(-1)^{(n-1)/2}}{2^{n-1}} \cdot \left[\sum_{\substack{r=0 \\ 2 \cdot r < n}} (-1)^r \cdot C_n^r \cdot \sin(n-2 \cdot r) \cdot x \right], n \text{ odd}, \quad (4)$$

relations which, as we have also mentioned in the case of n -th order linear differential equations with constant coefficients, are proved using mathematical induction.

Problem description

Let us consider that we have a system of linear differential equations with constant coefficients, such as

$$\begin{cases} y_1'(x) = a_{11} \cdot y_1(x) + a_{12} \cdot y_2(x) + \dots + a_{1n} \cdot y_n(x) + f_1(x) \\ y_2'(x) = a_{21} \cdot y_1(x) + a_{22} \cdot y_2(x) + \dots + a_{2n} \cdot y_n(x) + f_2(x) \\ \vdots \\ y_n'(x) = a_{n1} \cdot y_1(x) + a_{n2} \cdot y_2(x) + \dots + a_{nn} \cdot y_n(x) + f_n(x) \end{cases}, \quad (5)$$

where $x \in I \subset \mathbf{R}$ is an interval of $\mathbf{R}$, and f_i , $i \in \overline{1, n}$, are functions that will be defined below.

The homogeneous system attached to the system (5) will be like

$$\begin{cases} y_1'(x) = a_{11} \cdot y_1(x) + a_{12} \cdot y_2(x) + \dots + a_{1n} \cdot y_n(x) \\ y_2'(x) = a_{21} \cdot y_1(x) + a_{22} \cdot y_2(x) + \dots + a_{2n} \cdot y_n(x) \\ \vdots \\ y_n'(x) = a_{n1} \cdot y_1(x) + a_{n2} \cdot y_2(x) + \dots + a_{nn} \cdot y_n(x) \end{cases}, \quad (6)$$

where $x \in I \subset \mathbf{R}$ is an interval of $\mathbf{R}$.

We will assume that

$$f_m(x) = \sum_{i=1}^{p_m} P_{mi}(x) \cdot e^{\alpha_{mi} \cdot x} \cdot \cos^{b_{mi}} x + \sum_{j=1}^{q_m} Q_{mj}(x) \cdot e^{\beta_{mj} \cdot x} \cdot \cos^{c_{mj}} x + \sum_{k=1}^{r_m} R_{mk}(x) \cdot e^{\gamma_{mk} \cdot x} \cdot \sin^{d_{mk}} x + \sum_{l=1}^{s_m} S_{ml}(x) \cdot e^{\delta_{ml} \cdot x} \cdot \sin^{g_{ml}} x, \quad (7)$$

where $m \in \overline{1, n}$; $p_m, q_m, r_m, s_m \in \mathbf{N}^*$, $(\forall) m \in \overline{1, n}$; $\alpha_{mi}, \beta_{mj}, \gamma_{mk}, \delta_{ml} \in \mathbf{R}$, $(\forall) m \in \overline{1, n}$, $(\forall) i \in \overline{1, p_m}$, $(\forall) j \in \overline{1, q_m}$, $(\forall) k \in \overline{1, r_m}$ and $(\forall) l \in \overline{1, s_m}$; $b_{mi}, c_{mj}, d_{mk}, g_{ml} \in \mathbf{N}^*$, b_{mi}, d_{mk} even numbers and c_{mj}, g_{ml} odd numbers, $(\forall) m \in \overline{1, n}$, $(\forall) i \in \overline{1, p_m}$, $(\forall) j \in \overline{1, q_m}$, $(\forall) k \in \overline{1, r_m}$ si $(\forall) l \in \overline{1, s_m}$; and $P_{mi}, Q_{mj}, R_{mk}, S_{ml}$ are polynomials,

$$(\forall) m \in \overline{1, n}, (\forall) i \in \overline{1, p_m}, (\forall) j \in \overline{1, q_m}, (\forall) k \in \overline{1, r_m} \text{ and } (\forall) l \in \overline{1, s_m}.$$

As we mentioned before, the particular solution of the system (5), in which the free terms $f_i(x)$, $i \in \overline{1, n}$ are functions of the type (7), will be directly constructed similarly as it was done in [1].

The form of the particular solution and the exemplification of its determination

The characteristic equation attached to the homogeneous system (6) is of the following form:

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ & \vdots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix} = 0. \quad (8)$$

We consider that the equation (8) has the following expansion:

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ & \vdots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix} = \begin{aligned} & (-1)^n \cdot (\lambda - u_1)^{n_1} \cdot (\lambda - u_2)^{n_2} \cdot \dots \cdot (\lambda - u_h)^{n_h} \cdot \\ & \cdot [(\lambda - (v_1 + w_1 \cdot i)) \cdot (\lambda - (v_1 - w_1 \cdot i))]^{m_1} \cdot \\ & \cdot [(\lambda - (v_2 + w_2 \cdot i)) \cdot (\lambda - (v_2 - w_2 \cdot i))]^{m_2} \cdot \dots \cdot \\ & \cdot [(\lambda - (v_t + w_t \cdot i)) \cdot (\lambda - (v_t - w_t \cdot i))]^{m_p} \end{aligned} \quad (9)$$

where $n_1 + n_2 + \dots + n_q + 2 \cdot m_1 + 2 \cdot m_2 + \dots + 2 \cdot m_p = n$.

If in the expression of the function $f_m(x)$ we use the relations (1)-(4), then $f_m(x)$ will be like

$$\begin{aligned} f_m(x) = & \sum_{i=1}^{p_m} P_{mi}(x) \cdot e^{\alpha_{mi} \cdot x} + \sum_{j=1}^{q_m} Q_{mj}(x) \cdot e^{\beta_{mj} \cdot x} \cdot \cos(2 \cdot j \cdot x) + \\ & + \sum_{k=1}^{r_m} R_{mk}(x) \cdot e^{\gamma_{mk} \cdot x} \cdot \cos[(2 \cdot k + 1) \cdot x] + \sum_{l=1}^{s_m} S_{ml}(x) \cdot e^{\delta_{ml} \cdot x} \cdot \sin[(2 \cdot l + 1) \cdot x] \end{aligned} \quad (10)$$

where, even if we used the same notations as in (7), they no longer have the meaning of that expression. Also, the powers β_{mj} , $(\forall) j \in \overline{1, q_m}$, are not necessarily distinct two by two and the same happens with the powers γ_{mk} , $(\forall) k \in \overline{1, r_m}$ si δ_{ml} , $(\forall) l \in \overline{1, s_m}$. Instead, we can consider the powers α_{mi} as distinct two by two, because if we happen

to have the same power we only add the polynomials with which they are multiplied $e^{\alpha_{mi} \cdot x}$.

In order to be able to write a particular solution of the system we will add all the functions that appear in the column of free terms and we will obtain a function like the following

$$\begin{aligned} f(x) = \sum_{m=1}^n f_m(x) = & \sum_{i=1}^p P_i(x) \cdot e^{\alpha_i \cdot x} + \sum_{j=1}^q Q_j(x) \cdot e^{\beta_j \cdot x} \cdot \cos(2 \cdot j \cdot x) + \\ & + \sum_{k=1}^r R_k(x) \cdot e^{\gamma_k \cdot x} \cdot \cos[(2 \cdot k + 1) \cdot x] + \sum_{l=1}^s S_l(x) \cdot e^{\delta_l \cdot x} \cdot \sin[(2 \cdot l + 1) \cdot x] \end{aligned} \quad (11)$$

where, as previously stated, we can consider that the powers α_i are distinct two by two, because if we happen to have the same power $\alpha_i = \alpha_{mi}$ for several functions f_m we add only the polynomials with which

they are multiplied $e^{\alpha_i \cdot x}$. In contrast, the powers β_j , $(\forall) j \in \overline{1, q}$, are not necessarily distinct two by two, and the same happens with the powers γ_k , $(\forall) k \in \overline{1, r}$ and δ_l ,

$(\forall) l \in \overline{1, s}$. Moreover, we can consider, without restricting the generality, that we do not have terms such as $R_h(x) \cdot e^{h \cdot x} \cdot \cos[(2 \cdot h + 1) \cdot x]$ and $S_h(x) \cdot e^{h \cdot x} \cdot \sin[(2 \cdot h + 1) \cdot x]$ in the last two terms of the expression of $f(x)$, because the corresponding term generated in the particular solution, regardless of whether both terms appear or only one of them, is of

$$\begin{aligned} u_{i_1}, u_{i_2}, \dots, \alpha_{i_a} \in \{\alpha_i / i \in \overline{1, p}\}, v_{j_1} + w_{j_1} \cdot i, v_{j_2} + w_{j_2} \cdot i, \dots, \\ v_{j_b} + w_{j_b} \cdot i \in \{\beta_j + 2 \cdot j \cdot i / j \in \overline{1, q}\}, v_{k_1} + w_{k_1} \cdot i, v_{k_2} + w_{k_2} \cdot i, \dots, \\ v_{k_c} + w_{k_c} \cdot i \in \{\gamma_k + (2 \cdot k + 1) \cdot i / k \in \overline{1, r}\}, v_{l_1} + w_{l_1} \cdot i, v_{l_2} + w_{l_2} \cdot i, \dots, \\ v_{l_d} + w_{l_d} \cdot i \in \{\delta_l + (2 \cdot l + 1) \cdot i / l \in \overline{1, s}\}. \end{aligned}$$

Under these conditions, as in [1], a particular solution of the system (3), in which $f_m(x)$, $m \in \overline{1, n}$, are as (7) will be like

$$\begin{aligned} & \sum_{i \in \{i_1, i_2, \dots, i_a\}} P_i'(x) \cdot x^{n_i} \cdot e^{u_i \cdot x} + \sum_{i \in \overline{1, p} \setminus \{i_1, i_2, \dots, i_a\}} P_i'(x) \cdot e^{\alpha_i \cdot x} + \\ & + \sum_{j \in \{j_1, j_2, \dots, j_b\}} x^{m_j} \cdot e^{v_j \cdot x} \cdot [Q_j'(x) \cdot \cos(2 \cdot j \cdot x) + Q_j''(x) \cdot \sin(2 \cdot j \cdot x)] + \\ & + \sum_{j \in \overline{1, q} \setminus \{j_1, j_2, \dots, j_b\}} e^{\beta_j \cdot x} \cdot [Q_j'(x) \cdot \cos(2 \cdot j \cdot x) + Q_j''(x) \cdot \sin(2 \cdot j \cdot x)] + \\ y_{mp}(x) = & + \sum_{k \in \{k_1, k_2, \dots, k_c\}} x^{m_k} \cdot e^{v_k \cdot x} \cdot \{R_k'(x) \cdot \cos[(2 \cdot k + 1) \cdot x] + R_k''(x) \cdot \sin[(2 \cdot k + 1) \cdot x]\} +, m \in \overline{1, n} \quad (12) \\ & + \sum_{k \in \overline{1, r} \setminus \{k_1, k_2, \dots, k_c\}} e^{\gamma_k \cdot x} \cdot \{R_k'(x) \cdot \cos[(2 \cdot k + 1) \cdot x] + R_k''(x) \cdot \sin[(2 \cdot k + 1) \cdot x]\} + \\ & + \sum_{l \in \{l_1, l_2, \dots, l_d\}} x^{m_l} \cdot e^{v_l \cdot x} \cdot \{S_l'(x) \cdot \cos[(2 \cdot l + 1) \cdot x] + S_l''(x) \cdot \sin[(2 \cdot l + 1) \cdot x]\} + \\ & + \sum_{l \in \overline{1, s} \setminus \{l_1, l_2, \dots, l_d\}} e^{\delta_l \cdot x} \cdot \{S_l'(x) \cdot \cos[(2 \cdot l + 1) \cdot x] + S_l''(x) \cdot \sin[(2 \cdot l + 1) \cdot x]\} \end{aligned}$$

where $\text{degree } P_i' = \text{degree } P_i$, $(\forall) i \in \overline{1, p}$, $\text{degree } Q_j' = \text{degree } Q_j'' = \text{degree } Q_j$, $(\forall) j \in \overline{1, q}$, $\text{degree } R_k' = \text{degree } R_k'' = \text{degree } R_k$, $(\forall) k \in \overline{1, r}$, $\text{degree } S_l' = \text{degree } S_l'' = \text{degree } S_l$, $(\forall) l \in \overline{1, s}$.

The construction of the solution (12), takes into account the form of the functions $f_m(x)$,

$$\begin{cases} y_1'(x) = -y_2(x) + 17 \cdot y_3(x) + 2 \cdot e^x \cdot \cos^2 x \\ y_2'(x) = +13 \cdot y_3(x) + (8 \cdot x + 4) \cdot e^{2 \cdot x} \cdot \sin^3 x \\ y_3'(x) = -y_1(x) + 5 \cdot y_3(x) + (2 \cdot x + 4) \cdot e^x \cdot \sin^2 x \end{cases} \quad (13)$$

the form $R_h'(x) \cdot e^{h \cdot x} \cdot \cos[(2 \cdot h + 1) \cdot x] + S_h'(x) \cdot e^{h \cdot x} \cdot \sin[(2 \cdot h + 1) \cdot x]$, only if both terms appear we take polynomials R_h' si S_h' , with $\text{degree } R_h' = \text{degree } S_h' = \max \{\text{degree } R_h, \text{degree } S_h\}$.

We also assume that some of the solutions of the characteristic equation (9) are in the following situations:

$m \in \overline{1, n}$ and the form of the characteristic equation (9).

We will further exemplify the way in which the method described above must be applied. Let us consider the system of linear differential equations with constant coefficients

$$\begin{cases} y_1'(x) = & -y_2(x) + 17 \cdot y_3(x) + e^x + e^x \cdot \cos(2 \cdot x) \\ y_2'(x) = & +13 \cdot y_3(x) + (6 \cdot x + 3) \cdot e^{2 \cdot x} \cdot \sin x - (2 \cdot x + 1) \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x) \\ y_3'(x) = -y_1(x) & + 5 \cdot y_3(x) + (x + 2) \cdot e^x - (x + 2) \cdot e^x \cdot \cos(2 \cdot x) \end{cases} \quad (14)$$

$$\begin{cases} y_1'(x) = & -y_2(x) + 17 \cdot y_3(x) \\ y_2'(x) = & +13 \cdot y_3(x) \\ y_3'(x) = -y_1(x) & + 5 \cdot y_3(x) \end{cases} \quad (15)$$

We will first determine the solutions of the characteristic equation attached to the homogeneous system (15):

$$\begin{vmatrix} -\lambda & -1 & 17 \\ 0 & -\lambda & 13 \\ -1 & 0 & 5-\lambda \end{vmatrix} = \lambda^2 \cdot (5-\lambda) + 13 - 17 \cdot \lambda = 0. \quad (16)$$

The solutions of this equation are

$$\lambda_1 = 1 \text{ and } \lambda_2, \lambda_3 = 2 \pm 3 \cdot i.$$

Given the form of free terms and the root of the characteristic equation (16), we will try to find a particular solution like

$$\begin{cases} y_{1p}(x) = (A_1 \cdot x + B_1) \cdot e^x + (C_1 \cdot x + D_1) \cdot x \cdot e^x + (E_1 \cdot x + F_1) \cdot e^x \cdot \cos(2 \cdot x) + \\ \quad + (G_1 \cdot x + H_1) \cdot e^x \cdot \sin(2 \cdot x) + (I_1 \cdot x + J_1) \cdot e^{2 \cdot x} \cdot \cos x + (K_1 \cdot x + L_1) \cdot e^{2 \cdot x} \cdot \sin x + \\ \quad + (M_1 \cdot x + N_1) \cdot e^{2 \cdot x} \cdot \cos(3 \cdot x) + (O_1 \cdot x + P_1) \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x) + \\ \quad + (R_1 \cdot x + S_1) \cdot x \cdot e^{2 \cdot x} \cdot \cos(3 \cdot x) + (T_1 \cdot x + U_1) \cdot x \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x) \\ y_{2p}(x) = (A_2 \cdot x + B_2) \cdot e^x + (C_2 \cdot x + D_2) \cdot x \cdot e^x + (E_2 \cdot x + F_2) \cdot e^x \cdot \cos(2 \cdot x) + \\ \quad + (G_2 \cdot x + H_2) \cdot e^x \cdot \sin(2 \cdot x) + (I_2 \cdot x + J_2) \cdot e^{2 \cdot x} \cdot \cos x + (K_2 \cdot x + L_2) \cdot e^{2 \cdot x} \cdot \sin x + \\ \quad + (M_2 \cdot x + N_2) \cdot e^{2 \cdot x} \cdot \cos(3 \cdot x) + (O_2 \cdot x + P_2) \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x) + \\ \quad + (R_2 \cdot x + S_2) \cdot x \cdot e^{2 \cdot x} \cdot \cos(3 \cdot x) + (T_2 \cdot x + U_2) \cdot x \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x) \\ y_{3p}(x) = (A_3 \cdot x + B_3) \cdot e^x + (C_3 \cdot x + D_3) \cdot x \cdot e^x + (E_3 \cdot x + F_3) \cdot e^x \cdot \cos(2 \cdot x) + \\ \quad + (G_3 \cdot x + H_3) \cdot e^x \cdot \sin(2 \cdot x) + (I_3 \cdot x + J_3) \cdot e^{2 \cdot x} \cdot \cos x + (K_3 \cdot x + L_3) \cdot e^{2 \cdot x} \cdot \sin x + \\ \quad + (M_3 \cdot x + N_3) \cdot e^{2 \cdot x} \cdot \cos(3 \cdot x) + (O_3 \cdot x + P_3) \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x) + \\ \quad + (R_3 \cdot x + S_3) \cdot x \cdot e^{2 \cdot x} \cdot \cos(3 \cdot x) + (T_3 \cdot x + U_3) \cdot x \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x) \end{cases}.$$

The derivatives of the components of the particular solution created previously are:

$$\begin{aligned}
 y'_{1p}(x) &= [C_1 \cdot x^2 + (A_1 + 2 \cdot C_1 + D_1) \cdot x + A_1 + B_1 + D_1] \cdot e^x + \\
 &\quad + [(E_1 + 2 \cdot G_1) \cdot x + E_1 + F_1 + 2 \cdot H_1] \cdot e^x \cdot \cos(2 \cdot x) + \\
 &\quad + [(-2 \cdot E_1 + G_1) \cdot x - 2 \cdot F_1 + G_1 + H_1] \cdot e^x \cdot \sin(2 \cdot x) + \\
 &\quad + [(2 \cdot I_1 + K_1) \cdot x + I_1 + 2 \cdot J_1 + L_1] \cdot e^{2 \cdot x} \cdot \cos x + \\
 &\quad + [(-I_1 + 2 \cdot K_1) \cdot x - J_1 + K_1 + 2 \cdot L_1] \cdot e^{2 \cdot x} \cdot \sin x + \\
 &\quad + [(2 \cdot R_1 + 3 \cdot T_1) \cdot x^2 + (2 \cdot M_1 + 3 \cdot O_1 + 2 \cdot R_1 + 2 \cdot S_1 + 3 \cdot U_1) \cdot x + M_1 + \\
 &\quad + 2 \cdot N_1 + 3 \cdot P_1 + S_1] \cdot e^{2 \cdot x} \cdot \cos(3 \cdot x) + [(-3 \cdot R_1 + 2 \cdot T_1) \cdot x^2 + \\
 &\quad + (-3 \cdot M_1 + 2 \cdot O_1 - 3 \cdot S_1 + 2 \cdot T_1 + 2 \cdot U_1) \cdot x - 3 \cdot N_1 + O_1 + 2 \cdot P_1 + U_1] \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x) \\
 y'_{2p}(x) &= [C_2 \cdot x^2 + (A_2 + 2 \cdot C_2 + D_2) \cdot x + A_2 + B_2 + D_2] \cdot e^x + \\
 &\quad + [(E_2 + 2 \cdot G_2) \cdot x + E_2 + F_2 + 2 \cdot H_2] \cdot e^x \cdot \cos(2 \cdot x) + \\
 &\quad + [(-2 \cdot E_2 + G_2) \cdot x - 2 \cdot F_2 + G_2 + H_2] \cdot e^x \cdot \sin(2 \cdot x) + \\
 &\quad + [(2 \cdot I_2 + K_2) \cdot x + I_2 + 2 \cdot J_2 + L_2] \cdot e^{2 \cdot x} \cdot \cos x + \\
 &\quad + [(-I_2 + 2 \cdot K_2) \cdot x - J_2 + K_2 + 2 \cdot L_2] \cdot e^{2 \cdot x} \cdot \sin x + \\
 &\quad + [(2 \cdot R_2 + 3 \cdot T_2) \cdot x^2 + (2 \cdot M_2 + 3 \cdot O_2 + 2 \cdot R_2 + 2 \cdot S_2 + 3 \cdot U_2) \cdot x + M_2 + \\
 &\quad + 2 \cdot N_2 + 3 \cdot P_2 + S_2] \cdot e^{2 \cdot x} \cdot \cos(3 \cdot x) + [(-3 \cdot R_2 + 2 \cdot T_2) \cdot x^2 + \\
 &\quad + (-3 \cdot M_2 + 2 \cdot O_2 - 3 \cdot S_2 + 2 \cdot T_2 + 2 \cdot U_2) \cdot x - 3 \cdot N_2 + O_2 + 2 \cdot P_2 + U_2] \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x) \\
 y'_{3p}(x) &= [C_3 \cdot x^2 + (A_3 + 2 \cdot C_3 + D_3) \cdot x + A_3 + B_3 + D_3] \cdot e^x + \\
 &\quad + [(E_3 + 2 \cdot G_3) \cdot x + E_3 + F_3 + 2 \cdot H_3] \cdot e^x \cdot \cos(2 \cdot x) + \\
 &\quad + [(-2 \cdot E_3 + G_3) \cdot x - 2 \cdot F_3 + G_3 + H_3] \cdot e^x \cdot \sin(2 \cdot x) + \\
 &\quad + [(2 \cdot I_3 + K_3) \cdot x + I_3 + 2 \cdot J_3 + L_3] \cdot e^{2 \cdot x} \cdot \cos x + \\
 &\quad + [(-I_3 + 2 \cdot K_3) \cdot x - J_3 + K_3 + 2 \cdot L_3] \cdot e^{2 \cdot x} \cdot \sin x + \\
 &\quad + [(2 \cdot R_3 + 3 \cdot T_3) \cdot x^2 + (2 \cdot M_3 + 3 \cdot O_3 + 2 \cdot R_3 + 2 \cdot S_3 + 3 \cdot U_3) \cdot x + M_3 + \\
 &\quad + 2 \cdot N_3 + 3 \cdot P_3 + S_3] \cdot e^{2 \cdot x} \cdot \cos(3 \cdot x) + [(-3 \cdot R_3 + 2 \cdot T_3) \cdot x^2 + \\
 &\quad + (-3 \cdot M_3 + 2 \cdot O_3 - 3 \cdot S_3 + 2 \cdot T_3 + 2 \cdot U_3) \cdot x - 3 \cdot N_3 + O_3 + 2 \cdot P_3 + U_3] \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x)
 \end{aligned}$$

By introducing the functions y_{1p} , y_{2p} and y_{3p} , as well as their derivatives, in the system (14) we obtain, by identifying the coefficients, the following system of linear equations:

$$\left\{ \begin{array}{l} C_1 + C_2 - 17 \cdot C_3 = 0 \\ A_1 + D_1 + A_2 + D_2 - 17 \cdot (A_3 + D_3) + 2 \cdot C_1 = 0 \\ A_1 + D_1 + B_1 + B_2 - 17 \cdot B_3 = 1 \\ C_2 - 13 \cdot C_3 = 0 \\ A_2 + D_2 - 13 \cdot (A_3 + D_3) + 2 \cdot C_2 = 0 \\ A_2 + D_2 + B_2 - 13 \cdot B_3 = 0 \\ C_1 - 4 \cdot C_3 = 0 \\ A_1 + D_1 - 4 \cdot (A_3 + D_3) + 2 \cdot C_3 = 1 \\ A_3 + D_3 + B_1 - 4 \cdot B_3 = 2 \\ E_1 + E_2 - 17 \cdot E_3 + 2 \cdot G_1 = 0 \\ E_1 + F_1 + F_2 - 17 \cdot F_3 + 2 \cdot H_1 = 1 \\ -2 \cdot E_1 + G_1 + G_2 - 17 \cdot G_3 = 0 \\ -2 \cdot F_1 + G_1 + H_1 + H_2 - 17 \cdot H_3 = 0 \\ E_2 - 13 \cdot E_3 + 2 \cdot G_2 = 0 \\ E_2 + F_2 - 13 \cdot F_3 + 2 \cdot H_2 = 0 \\ -2 \cdot E_2 + G_2 - 13 \cdot G_3 = 0 \\ -2 \cdot F_2 + G_2 + H_2 - 13 \cdot H_3 = 0 \\ E_1 - 4 \cdot E_3 + 2 \cdot G_3 = -1 \\ E_3 + F_1 - 4 \cdot F_3 + 2 \cdot H_3 = -2 \\ -2 \cdot E_3 + G_1 - 4 \cdot G_3 = 0 \\ -2 \cdot F_3 + G_3 + H_1 - 4 \cdot H_3 = 0 \\ 2 \cdot I_1 + I_2 - 17 \cdot I_3 + K_1 = 0 \\ I_1 + 2 \cdot J_1 + J_2 - 17 \cdot J_3 + L_1 = 0 \\ -I_1 + 2 \cdot K_1 + K_2 - 17 \cdot K_3 = 0 \\ -J_1 + K_1 + 2 \cdot L_1 + L_2 - 17 \cdot L_3 = 0 \\ 2 \cdot I_2 - 13 \cdot I_3 + K_2 = 0 \\ I_2 + 2 \cdot J_2 - 13 \cdot J_3 + L_2 = 0 \\ -I_2 + 2 \cdot K_2 - 13 \cdot K_3 = 6 \\ -J_2 + K_2 + 2 \cdot L_2 - 13 \cdot L_3 = 3 \\ I_1 - 3 \cdot I_3 + K_3 = 0 \\ I_3 + J_1 - 3 \cdot J_3 + L_3 = 0 \\ -I_3 + K_1 - 3 \cdot K_3 = 0 \\ -J_3 + K_1 + L_1 - 3 \cdot L_3 = 0 \\ 2 \cdot R_1 + R_2 - 17 \cdot R_3 + 3 \cdot T_1 = 0 \\ 2 \cdot (M_1 + S_1) + M_2 + S_2 - 17 \cdot (M_3 + S_3) + 3 \cdot (O_1 + U_1) + 2 \cdot R_1 = 0 \\ M_1 + S_1 + 2 \cdot N_1 + N_2 - 17 \cdot N_3 + 3 \cdot P_1 = 0 \\ -3 \cdot R_1 + 2 \cdot T_1 + T_2 - 17 \cdot T_3 = 0 \\ -3 \cdot (M_1 + S_1) + 2 \cdot (O_1 + U_1) + O_2 + U_2 - 17 \cdot (O_3 + U_3) + 2 \cdot T_1 = 0 \\ -3 \cdot N_1 + O_1 + U_1 + 2 \cdot P_1 + P_2 - 17 \cdot P_3 = 0 \\ 2 \cdot R_2 - 13 \cdot R_3 + 3 \cdot T_2 = 0 \\ 2 \cdot (M_2 + S_2) - 13 \cdot (M_3 + S_3) + 3 \cdot (O_2 + U_2) + 2 \cdot R_2 = 0 \\ M_2 + S_2 + 2 \cdot N_2 - 13 \cdot N_3 + 3 \cdot P_2 = 0 \\ -3 \cdot R_2 + 2 \cdot T_2 - 13 \cdot T_3 = 0 \\ -3 \cdot (M_2 + S_2) + 2 \cdot (O_2 + U_2) - 13 \cdot (O_3 + U_3) + 2 \cdot T_2 = -2 \\ -3 \cdot N_2 + O_2 + U_2 + 2 \cdot P_2 - 13 \cdot P_3 = -1 \\ R_1 - 3 \cdot R_3 + 3 \cdot T_3 = 0 \\ M_1 + S_1 - 3 \cdot (M_3 + S_3) + 3 \cdot (O_3 + U_3) + 2 \cdot R_3 = 0 \\ M_3 + S_3 + N_1 - 3 \cdot N_3 + 3 \cdot P_3 = 0 \\ -3 \cdot R_3 + T_1 - 3 \cdot T_3 = 0 \\ -3 \cdot (M_3 + S_3) + O_1 + U_1 - 3 \cdot (O_3 + U_3) + 2 \cdot T_3 = 0 \\ -3 \cdot N_3 + O_3 + U_3 + P_1 - 3 \cdot P_3 = 0 \end{array} \right.$$

[illegible]

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$$\begin{cases}
A_1 + D_1 = 109/50 \\
A_2 + D_2 = 143/50 \\
A_3 + D_3 = 8/25 \\
B_1 = 42/25 \\
B_2 = -143/50 \\
B_3 = 0 \\
C_1 = 1/5 \\
C_2 = 13/20 \\
C_3 = 1/20
\end{cases}
\begin{cases}
E_1 = -55/26 \\
E_2 = -2 \\
E_3 = -3/26 \\
F_1 = -2187/338 \\
F_2 = -627/104 \\
F_3 = -935/1352 \\
G_1 = 14/13 \\
G_2 = 1/4 \\
G_3 = 17/52 \\
H_1 = 326/169 \\
H_2 = -25/52 \\
H_3 = 615/676
\end{cases}
\begin{cases}
I_1 = -3/2 \\
I_2 = -33/8 \\
I_3 = -3/8 \\
J_1 = 21/32 \\
J_2 = 291/128 \\
J_3 = 9/128 \\
K_1 = 3/4 \\
K_2 = 27/8 \\
K_3 = 3/8 \\
L_1 = -57/64 \\
L_2 = 63/128 \\
L_3 = -9/128
\end{cases}
\begin{cases}
M_1 + S_1 = 11/150 \\
M_2 + S_2 = 46/75 \\
M_3 + S_3 = 3/50 \\
N_1 = 101/450 \\
N_2 = 209/675 \\
N_3 = 64/675 \\
O_1 + U_1 = 23/150 \\
O_2 + U_2 = -16/225 \\
O_3 + U_3 = 11/450 \\
P_1 = 13/50 \\
P_2 = 0 \\
P_3 = 0 \\
R_1 = -1/10 \\
R_2 = -7/60 \\
R_3 = 1/60 \\
T_1 = 1/5 \\
T_2 = 3/20 \\
T_3 = 1/20
\end{cases}$$

In conclusion, a particular solution of the given system (14) is

$$\begin{cases}
y_{1p}(x) = \left(\frac{1}{5} \cdot x^2 + \frac{109}{50} \cdot x + \frac{42}{25} \right) \cdot e^x + \left(-\frac{55}{26} \cdot x - \frac{2187}{338} \right) \cdot e^x \cdot \cos(2 \cdot x) + \\
+ \left(\frac{14}{13} \cdot x + \frac{326}{169} \right) \cdot e^x \cdot \sin(2 \cdot x) + \left(-\frac{3}{2} \cdot x + \frac{21}{32} \right) \cdot e^{2 \cdot x} \cdot \cos x + \left(\frac{3}{4} \cdot x - \frac{57}{64} \right) \cdot e^{2 \cdot x} \cdot \sin x + \\
+ \left(-\frac{1}{10} \cdot x^2 + \frac{11}{150} \cdot x + \frac{101}{450} \right) \cdot e^{2 \cdot x} \cdot \cos(3 \cdot x) + \left(\frac{1}{5} \cdot x^2 + \frac{23}{150} \cdot x + \frac{13}{50} \right) \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x) \\
y_{2p}(x) = \left(\frac{13}{20} \cdot x^2 + \frac{143}{50} \cdot x - \frac{143}{50} \right) \cdot e^x + \left(-2 \cdot x - \frac{627}{104} \right) \cdot e^x \cdot \cos(2 \cdot x) + \\
+ \left(\frac{1}{4} \cdot x - \frac{25}{52} \right) \cdot e^x \cdot \sin(2 \cdot x) + \left(-\frac{33}{8} \cdot x + \frac{291}{128} \right) \cdot e^{2 \cdot x} \cdot \cos x + \left(\frac{27}{8} \cdot x + \frac{63}{128} \right) \cdot e^{2 \cdot x} \cdot \sin x + \\
+ \left(-\frac{7}{60} \cdot x^2 + \frac{46}{75} \cdot x + \frac{209}{675} \right) \cdot e^{2 \cdot x} \cdot \cos(3 \cdot x) + \left(\frac{3}{20} \cdot x^2 - \frac{16}{225} \cdot x \right) \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x) \\
y_{3p}(x) = \left(\frac{1}{20} \cdot x^2 + \frac{8}{25} \cdot x \right) \cdot e^x + \left(-\frac{3}{26} \cdot x - \frac{935}{1352} \right) \cdot e^x \cdot \cos(2 \cdot x) + \\
+ \left(\frac{17}{52} \cdot x + \frac{615}{676} \right) \cdot e^x \cdot \sin(2 \cdot x) + \left(-\frac{3}{8} \cdot x + \frac{9}{128} \right) \cdot e^{2 \cdot x} \cdot \cos x + \left(\frac{3}{8} \cdot x - \frac{9}{128} \right) \cdot e^{2 \cdot x} \cdot \sin x + \\
+ \left(\frac{1}{60} \cdot x^2 + \frac{3}{50} \cdot x + \frac{64}{675} \right) \cdot e^{2 \cdot x} \cdot \cos(3 \cdot x) + \left(\frac{1}{20} \cdot x^2 + \frac{11}{450} \cdot x \right) \cdot e^{2 \cdot x} \cdot \sin(3 \cdot x)
\end{cases}$$

After determining the particular solution, the general solution of the given system can be written.

Conclusions

This study comes to complete the range of free terms for which we can determine a particular solution in the case of a system of linear differential equations with constant

coefficients. In this case, to the expression of free terms presented in [1], terms such as $P(x) \cdot e^{a \cdot x} \cdot \cos^n x$ or/and $Q(x) \cdot e^{b \cdot x} \cdot \sin^m x$ can be added, the particular solution being built the same as in [1], taking into account the type of functions that occur in the column of free terms.

In the case of systems of linear differential equations with constant coefficients, too, for the construction of the particular solution, we must know which of the values generated by the terms that appear in the expression of the function $f(x)$, which brings together all functions in the column of free terms, are among the solutions of the

characteristic equation and what order of multiplicity they have.

Also, in this case, determining the solution of the system of linear equations resulting from the identification of coefficients, as can be seen in the example presented is not simple, and the use of a computer application is essential.

References

- [1] **Vasile Căruțașu**, *Determining a Particular Solution for the Systems of Linear Differential Equations with Constant Coefficients*, Proceedings of the 23rd International Conference „The Knowledge-Based Organization” of „Nicolae Bălcescu” Land Forces Academy, Sibiu, 15-17 June, 2017, pag. 30-36, ISSN 1843-6722