DETERMINING A PARTICULAR SOLUTION FOR n -th ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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Abstract: For n -th order linear differential equations with constant coefficients, the problem to be solved is related to determining a particular solution, and then, with the general solution of n -th homogeneous linear differential equation with constant coefficients attached, to write the general solution of n -th linear differential equation with the given constant coefficients. In all the works that deal with this issue three situations are analyzed: the situation in which the free term is a polynomial $P(x)$, the situation in which the free term is like $P(x) \cdot e^{\alpha \cdot x}$ and lastly, the situation in which the free term is like $e^{\omega \cdot x} \cdot (P(x) \cdot \cos(\beta \cdot x) + Q(x) \cdot \sin(\beta \cdot x))$. In this study we aim to analyze if the free term is a combination of the three cases mentioned.

Keywords: n -th order linear differential equations with constant coefficients, n -th order homogeneous linear differential equations with constant coefficients, particular solution, general solution

1. Introduction

It is known that various processes and phenomena are modeled using differential equations. They generally have complicated forms that make it impossible to obtain a general explicit or even implicit solution, approximate solutions being satisfactory, which can be obtained with different numerical methods.

However, at least classes of equations to which this can be done should be known and they should be as general as possible.

The study is basically an observation of how you can get a particular solution to an n -th order linear differential equation with constant coefficients where the constant term is part of the more general class of

functions for which an explicit solution can be obtained.

Basically, the class of functions that may occur for the free term is a combination of the three known cases: the situation in which the free term is a polynomial $P(x)$, the situation in which the free term is like $P(x) \cdot e^{\alpha \cdot x}$ and lastly, the situation in which the free term is like $e^{\omega \cdot x} \cdot (P(x) \cdot \cos(\beta \cdot x) + Q(x) \cdot \sin(\beta \cdot x))$.

2. Description of the problem

We will consider that we have an n -th order linear differential equation with constant coefficients of the form

$$a_0 \cdot y^{(n)}(x) + a_1 \cdot y^{(n-1)}(x) + \dots + a_n \cdot y(x) = f(x), \quad x \in I \subset \mathbf{R} \text{ an interval of } \mathbf{R}. \quad (1)$$

The attached homogeneous n -th order linear differential equation with constant

coefficients will be like

$$a_0 \cdot y^{(n)}(x) + a_1 \cdot y^{(n-1)}(x) + \dots + a_n \cdot y(x) = 0, \quad x \in I \subset \mathbf{R} \text{ an interval of } \mathbf{R}. \quad (2)$$

We will suppose that $f(x)$ looks like

$$\begin{aligned} f(x) = & P(x) + P_1(x) \cdot e^{\alpha_1 \cdot x} + P_2(x) \cdot e^{\alpha_2 \cdot x} + \dots + P_n(x) \cdot e^{\alpha_n \cdot x} + \\ & + e^{\omega_1 \cdot x} \cdot (Q_{11}(x) \cdot \cos(\beta_1 \cdot x) + Q_{12}(x) \cdot \sin(\beta_1 \cdot x)) + \\ & + e^{\omega_2 \cdot x} \cdot (Q_{21}(x) \cdot \cos(\beta_2 \cdot x) + Q_{22}(x) \cdot \sin(\beta_2 \cdot x)) + \dots + \\ & + e^{\omega_m \cdot x} \cdot (Q_{m1}(x) \cdot \cos(\beta_m \cdot x) + Q_{m2}(x) \cdot \sin(\beta_m \cdot x)) + \\ & + Q_1(x) \cdot \cos(\gamma_1 \cdot x) + Q_2(x) \cdot \cos(\gamma_2 \cdot x) + \dots + Q_k(x) \cdot \cos(\gamma_k \cdot x) + \\ & + R_1(x) \cdot \sin(\delta_1 \cdot x) + R_2(x) \cdot \sin(\delta_2 \cdot x) + \dots + R_l(x) \cdot \sin(\delta_l \cdot x) + \\ & + e^{\varepsilon_1 \cdot x} S_1(x) \cdot \cos(\lambda_1 \cdot x) + e^{\varepsilon_2 \cdot x} S_2(x) \cdot \cos(\lambda_2 \cdot x) + \dots + e^{\varepsilon_r \cdot x} S_r(x) \cdot \cos(\lambda_r \cdot x) + \\ & + e^{\rho_1 \cdot x} T_1(x) \cdot \sin(\theta_1 \cdot x) + e^{\rho_2 \cdot x} T_2(x) \cdot \sin(\theta_2 \cdot x) + \dots + e^{\rho_s \cdot x} T_s(x) \cdot \sin(\theta_s \cdot x) \end{aligned} \quad (3)$$

where

$$\begin{aligned} \beta_i &\neq \gamma_j, (\forall) i \in \overline{1, m} \text{ and } j \in \overline{1, k}, & \gamma_i &\neq \theta_j, (\forall) i \in \overline{1, k} \text{ and } j \in \overline{1, s}, \\ \beta_i &\neq \delta_j, (\forall) i \in \overline{1, m} \text{ and } j \in \overline{1, l}, & \delta_i &\neq \lambda_j, (\forall) i \in \overline{1, l} \text{ and } j \in \overline{1, r}, \\ \beta_i &\neq \lambda_j, (\forall) i \in \overline{1, m} \text{ and } j \in \overline{1, r}, & \delta_i &\neq \theta_j, (\forall) i \in \overline{1, l} \text{ and } j \in \overline{1, s}, \\ \beta_i &\neq \theta_j, (\forall) i \in \overline{1, m} \text{ and } j \in \overline{1, s}, & \lambda_i &\neq \theta_j, (\forall) i \in \overline{1, r} \text{ and } j \in \overline{1, s}, \\ \gamma_i &\neq \delta_j, (\forall) i \in \overline{1, k} \text{ and } j \in \overline{1, l}, & & \text{and } P, P_i, i \in \overline{1, n}, Q_{ij}, i \in \overline{1, m} \text{ and } j \in \overline{1, 2}, \\ \gamma_i &\neq \lambda_j, (\forall) i \in \overline{1, k} \text{ and } j \in \overline{1, r}, & & Q_i, i \in \overline{1, k}, S_i, i \in \overline{1, r}, T_i, i \in \overline{1, s}, \text{ are} \\ & & & \text{polynomials.} \end{aligned}$$

The free term $f(x)$ contains all possible combinations of the situations where the free term is a $P(x)$ polynomial, the situation in which the free term is like $P(x) \cdot e^{\alpha \cdot x}$ and eventually the situation in

which the free term is like $e^{\omega \cdot x} \cdot (P(x) \cdot \cos(\beta \cdot x) + Q(x) \cdot \sin(\beta \cdot x))$. The above described situation may consist in the following situation:

$$\begin{aligned} f(x) = & P_1(x) \cdot e^{\alpha_1 \cdot x} + P_2(x) \cdot e^{\alpha_2 \cdot x} + \dots + P_n(x) \cdot e^{\alpha_n \cdot x} + \\ & + e^{\gamma_1 \cdot x} \cdot (Q_1(x) \cdot \cos(\beta_1 \cdot x) + R_1(x) \cdot \sin(\beta_1 \cdot x)) + \\ & + e^{\gamma_2 \cdot x} \cdot (Q_2(x) \cdot \cos(\beta_2 \cdot x) + R_2(x) \cdot \sin(\beta_2 \cdot x)) + \dots + \\ & + e^{\gamma_m \cdot x} \cdot (Q_m(x) \cdot \cos(\beta_m \cdot x) + R_m(x) \cdot \sin(\beta_m \cdot x)) \end{aligned} \quad (4)$$

as we may consider that

$$\begin{aligned} P(x) &= e^{0 \cdot x} \cdot P(x); \\ Q_i(x) \cdot \cos(\gamma_i \cdot x) &= e^{0 \cdot x} \cdot (Q_i(x) \cdot \cos(\gamma_i \cdot x) + 0 \cdot \sin(\gamma_i \cdot x)), \quad i \in \overline{1, k}; \\ R_i(x) \cdot \sin(\delta_i \cdot x) &= e^{0 \cdot x} \cdot (0 \cdot \cos(\delta_i \cdot x) + R_i(x) \cdot \sin(\delta_i \cdot x)), \quad i \in \overline{1, l}; \end{aligned}$$

$$e^{\varepsilon_i \cdot x} \cdot S_i(x) \cdot \cos(\lambda_i \cdot x) = e^{\varepsilon_i \cdot x} \cdot (S_i(x) \cdot \cos(\lambda_i \cdot x) + 0 \cdot \sin(\gamma_i \cdot x)), \quad i \in \overline{1, r};$$

$$e^{\rho_i \cdot x} \cdot T_i(x) \cdot \sin(\theta_i \cdot x) = e^{\rho_i \cdot x} \cdot (0 \cdot \cos(\theta_i \cdot x) + T_i(x) \cdot \sin(\theta_i \cdot x)), \quad i \in \overline{1, s}.$$

The particular solution of n -th order linear differential equation with constant coefficients (1), in which the free term $f(x)$ is like (3) is constructed by combining the way in which particular solutions are written in the three cases mentioned before.

3. The form of the particular solution and exemplification

As we have seen in the previous section, without restricting the generality degree, it

$$\begin{aligned} a_0 \cdot \lambda^n + a_1 \cdot \lambda^{n-1} + \dots + a_{n-1} \cdot \lambda + a_n = a_0 \cdot (\lambda - a_1)^{n_1} \cdot (\lambda - a_2)^{n_2} \cdot \dots \cdot (\lambda - a_q)^{n_q} \cdot \\ \cdot [(\lambda - (b_1 + c_1 \cdot i)) \cdot (\lambda - (b_1 - c_1 \cdot i))]^{m_1} \cdot [(\lambda - (b_2 + c_2 \cdot i)) \cdot (\lambda - (b_2 - c_2 \cdot i))]^{m_2} \cdot \dots \cdot \\ \cdot [(\lambda - (b_p + c_p \cdot i)) \cdot (\lambda - (b_p - c_p \cdot i))]^{m_p} \end{aligned} \quad (6)$$

where $n_1 + n_2 + \dots + n_q + 2 \cdot m_1 + 2 \cdot m_2 + \dots + 2 \cdot m_p = n$. In other words, we assume that the characteristic equation (6) attached to the differential equation (2) has q real and particular solutions with multiplicity orders $n_1, n_2, \dots$, respectively n_q and $2 \cdot p$ particular conjugate complex solutions with multiplicity orders $m_1, m_2, \dots$, respectively m_p .

We also assume that $\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r}$ and

$$\begin{aligned} y_p(x) = + \sum_{i \in \overline{1, n} \setminus \{i_1, i_2, \dots, i_r\}} S_i(x) \cdot e^{\alpha_i \cdot x} + \sum_{j \in \{i_1, i_2, \dots, i_r\}} S_j(x) \cdot x^{n_j} \cdot e^{\alpha_j \cdot x} + \\ \sum_{j \in \overline{1, m} \setminus \{j_1, j_2, \dots, j_s\}} e^{\gamma_j \cdot x} \cdot [U_j(x) \cdot \cos(\beta_j \cdot x) + V_j(x) \cdot \sin(\beta_j \cdot x)] + \\ + \sum_{k \in \{j_1, j_2, \dots, j_s\}} x^{m_k} \cdot e^{\gamma_k \cdot x} \cdot [U_k(x) \cdot \cos(\beta_k \cdot x) + V_k(x) \cdot \sin(\beta_k \cdot x)] \end{aligned} \quad (7)$$

where $S_i, i \in \overline{1, n}$ and $U_j, V_j, j \in \overline{1, m}$ are polynomials of degree $S_i = \text{degree } P_i, i \in \overline{1, n}$ and degree $U_j = \text{degree } V_j = \max \{\text{degree } Q_j, \text{degree } R_j\}, j \in \overline{1, m}$, and n_j is the multiplicity order of the radical α_{i_j} ,

is sufficient to rest upon a free term such as (4).

Let there be

$$a_0 \cdot \lambda^n + a_1 \cdot \lambda^{n-1} + \dots + a_{n-1} \cdot \lambda + a_n = 0 \quad (5)$$

the characteristic equation attached to the homogenous n -th order linear differential equation with constant coefficients (2).

We will assume that this has the following expansion:

$\gamma_{j_1} + \beta_{j_1} \cdot i, \gamma_{j_2} + \beta_{j_2} \cdot i, \dots, \gamma_{j_s} + \beta_{j_s} \cdot i$ are among the solutions of the characteristic equation (5) which has the expansion (6).

Under these circumstances, the particular solution of the n -th order linear differential equation with constant coefficients (1), where the free term $f(x)$ is of the form (3) will be like

$j \in \overline{1, r}$ and m_k is the multiplicity order of the radical $\gamma_{j_k} + \beta_{j_k} \cdot i, k \in \overline{1, s}$.

The form of the solution (7) considers, as we have already mentioned in the previous paragraph, the form of the two specific

situations given that we can consider case 1 as a specific case of situation 2.

Further on, we will illustrate the way in which the above described method should be applied.

$$y^{(9)} + y^{(8)} + 2 \cdot y^{(7)} + y^{(5)} - 3y^{(3)} - 2y^{(2)} = e^{2 \cdot x} + 2x + 1 + e^x \cdot \cos 2x + (x + 2) \cdot \cos x + \sin x.$$

We will determine a particular solution of this equation with the method described above.

We will first determine the solutions of the attached characteristic equation:

Let there be the order 9 linear differential equation

$$\lambda^9 + \lambda^8 + 2 \cdot \lambda^7 + \lambda^5 - 3 \cdot \lambda^3 - 2 \cdot \lambda^2 = 0. \quad (8)$$

The solutions of this equation are

$\lambda_1 = \lambda_2 = 0$; $\lambda_3 = 1$; $\lambda_4 = \lambda_5$, $\lambda_6 = \lambda_7 = \pm i$ and $\lambda_8, \lambda_9 = 1 \pm i$, meaning that the expansion takes place

$$\lambda^9 + \lambda^8 + 2 \cdot \lambda^7 + \lambda^5 - 3 \cdot \lambda^3 - 2 \cdot \lambda^2 = \lambda^2 \cdot (\lambda - 1) \cdot (\lambda^2 + 1)^2 \cdot (\lambda^2 + 2 \cdot \lambda + 2).$$

We will write the free term $f(x)$ as

$$f(x) = e^{2 \cdot x} \cdot 1 + e^0 \cdot (2 \cdot x + 1) + e^x \cdot (1 \cdot \cos 2x + 0 \cdot \sin 2x) + e^0 \cdot [(x + 2) \cdot \cos x + 1 \cdot \sin x],$$

to write easier the form that the particular solution should have.

Considering that 0 and i are order 2 solutions of the characteristic equation, and

2 and $1 + 2i$ are not solutions of the characteristic equation, it results that a particular solution of the given equation is like

$$y_p(x) = A \cdot e^{2 \cdot x} + x^2 \cdot (B \cdot x + C) + e^x \cdot (D \cdot \cos 2x + E \cdot \sin 2x) + x^2 \cdot [(F \cdot x + G) \cdot \cos x + (H \cdot x + I) \cdot \sin x],$$

where $A, B, C, D, E, F, G, H, I$ are real constant which are about to be determined by identification after introducing the

derivatives of y_p until order 9 in the given equation.
We have:

$$y_p'(x) = 2A \cdot e^{2 \cdot x} + 3B \cdot x^2 + 2C \cdot x + e^x \cdot [(D + 2E) \cdot \cos 2x + (-2D + E) \cdot \sin 2x] + [Hx^3 + (3F + I)x^2 + 2Gx] \cdot \cos x + [-Fx^3 + (-G + 3H)x^2 + 2Ix] \cdot \sin x,$$

$$y_p''(x) = 4A \cdot e^{2 \cdot x} + 6B \cdot x + 2C + e^x \cdot [(-3D + 4E) \cdot \cos 2x + (-4D - 3E) \cdot \sin 2x] + [-Fx^3 + (-G + 6H)x^2 + (6F + 4I)x + 2G] \cdot \cos x + [-Hx^3 + (-6F - I)x^2 + (-4G + 6H)x + 2I] \cdot \sin x,$$

$$y_p'''(x) = 8A \cdot e^{2 \cdot x} + 6B + e^x \cdot [(-11D - 4E) \cdot \cos 2x + (2D - 3E) \cdot \sin 2x] + [-Hx^3 + (-9F - I)x^2 + (-6G + 18H)x + 6F + 6I] \cdot \cos x + [Fx^3 + (G - 9H)x^2 + (-18F - 6I)x - 6G + 6H] \cdot \sin x,$$

$$y_p^{(4)}(x) = 16A \cdot e^{2 \cdot x} + e^x \cdot [(-7D - 24E) \cdot \cos 2x + (24D - 7E) \cdot \sin 2x] + [Fx^3 + (G - 12H)x^2 + (-36F - 8I)x - 12G + 24H] \cdot \cos x + [Hx^3 + (12F + I)x^2 + (8G - 36H)x - 24F - 12I] \cdot \sin x,$$

$$y_p^{(5)}(x) = 32A \cdot e^{2 \cdot x} + e^x \cdot [(41D - 38E) \cdot \cos 2x + (38D + 41E) \cdot \sin 2x] + [Hx^3 + (15F + I)x^2 + (10G - 60H)x - 60F - 20I] \cdot \cos x + [-Fx^3 + (-G + 15H)x^2 + (60F + 10I)x + 20G - 60H] \cdot \sin x,$$

$$y_p^{(6)}(x) = 64A \cdot e^{2 \cdot x} + e^x \cdot [(117D + 44E) \cdot \cos 2x + (-44D + 117E) \cdot \sin 2x] + [-Fx^3 + (-G + 18H)x^2 + (90F + 12I)x + 30G - 120H] \cdot \cos x + [-Hx^3 + (-18F - I)x^2 + (-12G + 90H)x + 120F + 30I] \cdot \sin x,$$

$$y_p^{(7)}(x) = 128 \cdot A \cdot e^{2 \cdot x} + e^x \cdot [(29 \cdot D + 278 \cdot E) \cdot \cos 2 \cdot x + (-278 \cdot D + 29 \cdot E) \cdot \sin 2 \cdot x] + [-H \cdot x^3 + (-21 \cdot F - I) \cdot x^2 + (-14 \cdot G + 126 \cdot H) \cdot x + 210 \cdot F + 42 \cdot I] \cdot \cos x + [F \cdot x^3 + (G - 21 \cdot H) \cdot x^2 + (-126 \cdot F - 14 \cdot I) \cdot x - 42 \cdot G + 210 \cdot H] \cdot \sin x,$$

$$y_p^{(8)}(x) = 256 \cdot A \cdot e^{2 \cdot x} + e^x \cdot [(-527 \cdot D + 336 \cdot E) \cdot \cos 2 \cdot x + (-336 \cdot D - 527 \cdot E) \cdot \sin 2 \cdot x] + [F \cdot x^3 + (G - 24 \cdot H) \cdot x^2 + (-168 \cdot F - 16 \cdot I) \cdot x - 56 \cdot G + 336 \cdot H] \cdot \cos x + [H \cdot x^3 + (24 \cdot F + I) \cdot x^2 + (16 \cdot G - 168 \cdot H) \cdot x - 336 \cdot F - 56 \cdot I] \cdot \sin x,$$

$$y_p^{(9)}(x) = 512 \cdot A \cdot e^{2 \cdot x} + e^x \cdot [(-1199 \cdot D - 718 \cdot E) \cdot \cos 2 \cdot x + (718 \cdot D - 1199 \cdot E) \cdot \sin 2 \cdot x] + [H \cdot x^3 + (27 \cdot F + I) \cdot x^2 + (18 \cdot G - 216 \cdot H) \cdot x - 504 \cdot F - 72 \cdot I] \cdot \cos x + [-F \cdot x^3 + (-G + 27 \cdot H) \cdot x^2 + (216 \cdot F + 18 \cdot I) \cdot x + 72 \cdot G - 504 \cdot H] \cdot \sin x,$$

By introducing these derivatives in the given equation, we get:

$$y_p^{(9)} + y_p^{(8)} + 2 \cdot y_p^{(7)} + y_p^{(5)} - 3 \cdot y_p^{(3)} - 2 \cdot y_p^{(2)} = 1000 \cdot A \cdot e^{2 \cdot x} - (12 \cdot B \cdot x + 4 \cdot C) + e^x \cdot [(-1600 \cdot D + 200 \cdot E) \cdot \cos 2 \cdot x + (-200 \cdot D - 1600 \cdot E) \cdot \sin 2 \cdot x] + [(-72 \cdot F - 24 \cdot H) \cdot x - 24 \cdot G + 264 \cdot H - 144 \cdot F - 8 \cdot I] \cdot \cos x + [(-72 \cdot H + 24 \cdot F) \cdot x - 24 \cdot I - 264 \cdot F - 144 \cdot H + 8 \cdot G] \cdot \sin x = e^{2 \cdot x} + 2 \cdot x + 1 + e^x \cdot \cos 2 \cdot x + (x + 2) \cdot \cos x + \sin x.$$

By identifying the coefficients, we get the following system of linear equations:

$$\begin{cases} 1000 \cdot A = 1 \\ -12 \cdot B = 2 \\ -4 \cdot C = 1 \\ -1600 \cdot D + 200 \cdot E = 1 \\ -200 \cdot D - 1600 \cdot E = 0 \\ -72 \cdot F - 24 \cdot H = 1 \\ -144 \cdot F - 24 \cdot G + 264 \cdot H - 8 \cdot I = 2 \\ 24 \cdot F - 72 \cdot H = 0 \\ -264 \cdot F + 8 \cdot G - 144 \cdot H - 24 \cdot I = 1 \end{cases}.$$

The solution of the above system is:

$$\begin{cases} A = 1/1000 \\ B = -1/6 \\ C = -1/4 \\ D = -8/13000 \\ E = 1/13000 \\ F = -1/80 \\ H = -1/240 \\ G = -34/400 \\ I = 37/400 \end{cases}$$

In conclusion, a particular solution of the given equation is

$$y_p(x) = \frac{1}{1000} \cdot e^{2 \cdot x} - \frac{1}{6} \cdot x^3 - \frac{1}{4} \cdot x^2 + e^x \cdot \left(-\frac{8}{13000} \cdot \cos 2 \cdot x + \frac{1}{13000} \cdot \sin 2 \cdot x \right) +$$

$$+(-\frac{1}{80} \cdot x^3 - \frac{34}{400} \cdot x^2) \cdot \cos x + (-\frac{1}{240} \cdot x^3 + \frac{37}{400} \cdot x^2) \cdot \sin x.$$

After determining the particular solution, we may write the general solution of the given equation.

Conclusions and debates

The determination of a particular solution for an n -th order linear differential equation with constant coefficients as (1), having the free term $f(x)$ as (3) can be done generally, regardless of the form of the characteristic equation (5).

It is enough to check which of the real and complex values that occur in the expression of the free term $f(x)$ ($\alpha_i, i \in \overline{1, n}$ and $\gamma_j + i \cdot \beta_j, j \in \overline{1, b}$ in the expression (4) are solutions of the

characteristic equation (5), and, by derivation, or with Horner's algorithm, which their multiplicity order is.

It is more difficult to determine the general solution for a differential equation like (1), having the free term $f(x)$ like (3), because it depends on the possibility to determine all solutions of the characteristic equation (5), which is an n order algebraic equation. For the example of the previous section, we should have checked which of the values 0, 2, $1+2i$ and i are solutions of the characteristic equation (8), and then, for those checking the characteristic equation, to see, by derivation or Horner's algorithm, which the multiplicity order is.