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Evaluation of Generalized Variance Functions in the Analysis of Complex Survey Data

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Two sets of diagnostics are presented to evaluate the properties of generalized variance functions (GVFs) for a given sample survey. The first set uses test statistics for the coefficients of multiple regression forms of GVF models. The second set uses smoothed estimators of the mean squared error (MSE) of GVF-based variance estimators. The smooth version of the MSE estimator can provide a useful measure of the performance of a GVF estimator, relative to the variance of a standard design-based variance estimator. Some of the proposed methods are applied to sample data from the Current Employment Statistics survey.

Key words: Complex sample design; degrees of freedom; design-based inference; model-based inference; quarterly census of employment and wages; superpopulation model; U.S. current employment statistics (CES) survey; variance estimator stability.

1. Introduction

In the analysis of sample survey data, statisticians generally prefer to use variance estimation and inference methods that account for the complex design used in the selection of sample units. However, in some cases (especially those involving relatively small domains or other specialized subpopulations), standard design-based variance estimators may be unstable. For such cases, some analysts prefer to use "generalized variance functions" estimators, in which one seeks to approximate the true design or design-model variance as a function of known predictors X.

For some background on generalized variance functions for survey data, see Johnson and King (1987), Valliant (1987) and the references cited therein. (Some of this literature discusses other reasons for use of GVFs, for example, simplicity of use for secondary data analysts. The remainder of this article will not consider these other reasons in further detail.) Much of the GVF literature has focused on the variances of point estimators of population proportions or population totals related to a binary outcome variable (see, e.g., Bureau of Labor Statistics 2006, pp. 189–193). The current article, however, considers the more complex setting in which the point estimator of interest depends primarily on survey variables that are not binary. For example, the Current Employment Statistics survey

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application in Subsection 2.1 and Section 5 depends on unit-level employment count reports that may range from one to tens of thousands.

Following the introduction of an illustrative example and a development of notation and prospective models in Section 2, this article develops two sets of diagnostic tools for GVFs. First, Section 3 presents design-based estimators of the variance-covariance matrix of the coefficient estimators for a GVF. The covariance-matrix estimators in turn lead to construction of test statistics and confidence sets for the GVF coefficients under standard large-sample conditions. Second, Section 4 develops diagnostics for the mean squared error of a GVF as an estimator of the true design variance of a given point estimator. An initial development reviews the relative magnitudes of error terms associated, respectively, with pure sampling variability of the design-based variance estimators; the deterministic lack of fit in the proposed GVF model; and the random equation error associated with the GVF model. Subsection 4.4 characterizes the unbiased MSE estimators of the GVF-based variance estimators in terms of the direct variance estimators. Subsection 4.5 fits models of these MSE estimators; produces a smooth version of the MSE estimators; and presents some simple methods of evaluating the relative magnitudes of the sampling error and equation error terms. Section 5 applies the proposed diagnostics to data from the U.S. Current Employment Statistics survey. Section 6 presents a simulation study that evaluates the properties of GVF coefficient estimators and of the related predictors of the true design variance. Section 7 summarizes the main ideas of this article and outlines some possible extensions. In addition, Table 1 provides a summary of the notation used in this article.

2. Illustrative Example, Background, Notation, and GVF Models

2.1. *Illustrative Example: Subpopulation Total Estimators for the U.S. Current Employment Statistics Survey*

The CES survey collects data monthly on employment, hours, and earnings from nonfarm establishments. Employment is the total number of persons employed full or part time in a nonfarm establishment during a specified month. One important feature of the CES survey is that complete universe employment counts of the previous year become available from the Unemployment Insurance (UI) tax records on a lagged basis (Butani et al. 1997). U.S. Bureau of Labor Statistics (2011, Ch. 2) describes the design features relevant to the analysis of the historical data considered in this article.

The CES sample design uses stratified sampling of UI accounts. UI account is a cluster that may contain a single or multiple establishment(s). An establishment is defined to be an economic unit, generally located at a single place, which is engaged predominantly in one type of economic activity. All establishments within a sampled UI account are included in the sample. When establishments are rotated into the sample, they are retained for two years or more. The strata are defined by state, industry, and the size class of UIs. The sample units in areas within each stratum are sorted in a way ensuring that the number of sampled units in each area is proportional to the area's size (i.e., proportional to the number of UIs in the frame for a given stratum).

Table 1. Description of notation

Notation	Description
b	index for elements of the coefficient vector γ
B	dimensionality of γ
C	dimensionality of ω
D	set of all j distinct domains
d_{jt}	degrees of freedom associated with the design-based distribution of V_{pjt}
$d*$	degrees of freedom associated with the superpopulation distribution of $(V^*_{pjt})^{-1}V_{pjt}$
d_w	degrees of freedom in the Wishart distribution for $\hat{V}\hat{w}(\hat{\gamma})$
h_f	smooth version of $E\{(V^*_{pjt} - V_{pjt})^2 \vert X_{jt}\}$
i	industry
j	domain
n_{jt}	number of responding sample UI accounts in domain j at time t
p	sample design
q_{jt}	equation error
r_{jt}	residuals with expectation $E(q^2_{jt} \vert X_{jt})$
$\hat{R}$	growth ratio estimate
$SE1$	square root of $(2\hat{V}^2_{pjt})/(d_{jt}+2)$
$SE2$	square root of $(2V^{*2}_{pjt})/d$
t	months from benchmark month
V_{pjt}	design variance of $\hat{\theta}_{jt}$
$\hat{V}_{pjt}$	variance estimator based on the design
V^*_{pjt}	variance estimator based on the model
X	vector of predictor variables for GVF model
y	unknown true employment total
Z	vector of predictor variables for the residual Models (21) through (24)
γ	variance function parameters in Model (f)
ε_{jt}	sampling error $\hat{V}_{pjt} - V_{pjt}$
η_{jt}	error term in Model (22)
θ_{jt}	finite population quantity
$\theta_{\xi jt}$	superpopulation analogue of θ_{jt}
$\hat{\theta}_{jt}$	point estimator of θ_{jt}
ξ	superpopulation index
$\hat{\sigma}^2_e$	residual mean squared error terms
ω	variance function parameters

For this article, the survey variable of main interest is y_{jtk}, defined to equal the total employment reported by establishment k within domain j for reference month t. The universe data, known as Quarterly Census of Employment and Wages (QCEW) data, are used annually to benchmark the CES sample estimates to these universe counts (Werking 1997). Specifically, let x_{j0} equal the known QCEW employment total within domain j for the benchmark month 0. In addition, let y_{jt} equal the unknown true employment total for domain j in month t. CES uses a "weighted link relative estimator" of y_{jt}, computed as the product,

$$\hat{y}_{jt} = x_{j0}\hat{R}_{jt},$$

where $\hat{R}_{jt}$ is an estimator of the relative employment growth that took place from benchmark month 0 to the current month t. Specifically,

$$\hat{R}_{jt} = \prod_{\tau=1}^{t} \hat{R}^{*}_{j\tau},$$

where $\hat{R}^{*}_{j\tau} = \left(\sum_{k \in s_{j\tau}} w_k y_{jk,\tau-1}\right)^{-1} \sum_{k \in s_{j\tau}} w_k y_{jk\tau}$, $s_{j\tau}$ is the matched sample of establishments in domain j that report positive employment in both months $\tau - 1$ and τ, and w_k is the sampling weight of establishment k. Note especially that $\hat{R}_{jt}$ equals the product of t separate estimators of one-month change. Consequently, under regularity conditions, one may anticipate that $\hat{R}_{jt}$ and $\hat{y}_{jt}$ may have design variances that are increasing functions of t. For more detailed information on the weighted link relative estimator, see BLS Handbook of Methods (2011) and Gershunskaya and Lahiri (2005). For data used in this article, the benchmark month ($t = 0$) is March 1999 and our sample data will lead to employment estimates for each month from January through December 2000 ($t = 10$ to $t = 21$).

The primary CES design goal is to satisfy the precision requirements specified for the national estimates. However, there is strong substantive interest in finer domains which are defined by geographic characteristics and industrial classifications. For example, the data analyses in Section 5 focus on estimates of total employment for 430 domains defined by the intersection of metropolitan statistical area (MSA) with industry, for example, durable goods manufacturing in the St. Louis MSA or wholesale trade in the Charleston MSA. Within these domains, effective sample sizes become so small that the standard design-based estimators are not precise enough to satisfy the needs of prospective data users (Eltinge et al. 2001). It is necessary to have stable estimators of $V(\hat{y}_{jt})$ for the finer domains. Consequently, we considered the use of GVF methods to produce domain-level variance estimators that would be more stable than direct design-based variance estimators.

2.2. Background and Notation

Let θ_{jt} be a finite population mean or total for period t, and let $\theta_{\xi jt}$ be a superpopulation analogue of θ_{jt} where j is the domain index. For example, in the CES survey, domains are the combinations of industries and areas, and are generally studied for a sequence of months $t = 1, \ldots, T$. In addition, let $\hat{\theta}_{jt}$ be a point estimator of θ_{jt}; and define $V_{pjt} = V_p(\hat{\theta}_{jt})$ to be the design variance of $\hat{\theta}_{jt}$. Throughout this article, the subscript "p" denotes an expectation or variance evaluated with respect to the sample design. The GVF models the variance of a survey estimator, V_{pjt}, as a function of the parameter θ_{jt} and possibly other variables (Wolter 2007, sec. 7.2). A common specification is

$$V_{pjt} = f(X_{jt,}\gamma) + q_{jt}, \tag{1}$$

where X_{jt} is a vector of predictor variables potentially relevant to estimators of V_{pjt}, q_{jt} is a random univariate "equation error" with the mean 0, and γ is a vector of B-dimensional variance function parameters which we need to estimate. Note especially that q_{jt}

represents the deviation of the true design variance V_{pjt} from its modeled value $f(X_{jt,}\gamma)$. One generally would view the error term q_{jt} as arising from the superpopulation model that generated our finite population.

In some GVF applications, one may consider functions $f(\cdot)$ that depend on the domain-specific parameter θ_{jt} and may also consider cases for which some predictors X_{jt} are unknown and replaced by estimated terms, say $\hat{X}_{jt}$. However, these cases did not arise in the CES application considered here, so this article will limit its attention to forms of the Model (1) with known predictors X_{jt}.

In general, it is not possible to observe the true design variance V_{pjt}. Instead it is possible to compute an estimator $\hat{V}_{pjt} = \hat{V}_p(\hat{\theta}_{jt})$ based on, for example linearization or replication-based methods. Consequently, Model (1) must be supplemented with the decomposition

$$\hat{V}_{pjt} = V_{pjt} + \epsilon_{jt}, \tag{2}$$

where ϵ_{jt} is a random term that reflects sampling error in the estimator $\hat{V}_{pjt}$. Under the assumption that $\hat{V}_{pjt}$ is design unbiased for V_{pjt}, the error term ϵ_{jt} has design expectation equal to zero. The distinction between the equation error in Model (1) and the sampling error in Model (2) has been considered in other settings for analysis of experiments with replicates (e.g., Draper and Smith 1998, p. 47) and measurement error models (e.g., Fuller 1987).

Our CES applications will use a special form of Model (1) on the logarithmic scale,

$$\ln(V_{pjt}) = X_{jt}\gamma + q^*_{jt}, \tag{3}$$

where q^*_{jt} is a general error term with mean equal to zero; Appendix C provides some related details. A relatively simple form of Model (3) that incorporates factors related to domain size (x_{j0}), number of respondents (n_{jt}) and distance from benchmark month 0 to the reference period (t) is:

$$\ln(V_{pjt}) = \gamma_0 + \gamma_1 \ln(x_{j0}) + \gamma_2 \ln(n_{jt}) + \gamma_3 \ln(t) + q^*_{jt}. \tag{$f1$}$$

To estimate the parameters of Models (2) and (3), let D be the set of all J distinct domains (area-industry combinations) and for each $j \in D$, let D_{jt} be the set of responding sample establishments in domain j for month t. In addition, let $\mathbf{Y}_j$ be a $T \times 1$ vector with t-th element $\ln(\hat{V}_{jt})$ and define the $(J \cdot T) \times 1$ vector $\mathbf{Y} = (\mathbf{Y}'_1, \mathbf{Y}'_2, \ldots, \mathbf{Y}'_J)'$. Similarly, let $\mathbf{X}_j$ be a $T \times B$ matrix with t-th row $\mathbf{X}_j(t,:)$ equal to the predictors used for the specified GVF model. Also, define the $(J \cdot T) \times B$ matrix $\mathbf{X} = (\mathbf{X}'_1, \mathbf{X}'_2, \ldots, \mathbf{X}'_J)'$ and $B \times 1$ vector $\gamma = [\gamma_1, \ldots, \gamma_B]'$. For example, under the Model ($f1$), $\mathbf{X}_j(t,:) = [1, \ln(x_{j0}), \ln(n_{jt}), \ln(t)]$ and $\gamma = [\gamma_0, \gamma_1, \gamma_2, \gamma_3]'$. Then one may compute the ordinary least squares estimator of the coefficient vector in (3) as

$$\hat{\gamma} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}. \tag{4}$$

2.3. GVF Models

We used the logarithms of direct variance estimators $\hat{V}_{pjt}$ from the survey as the dependent variables in GVF models. The CES data we considered were from reference year 2000, and direct estimators, $\hat{V}_{pjt}$ of V_{pjt}, were computed from Fay's variant of the balanced half-sample replication method with adjustment term $K = 0.5$. For general background on balanced half-sample replication and Fay's method, see Wolter (2007, Ch. 3) and Judkins (1990). For sampling within a given industry, the CES uses eight size classes. For variance estimation, the CES combines the three largest size classes (6, 7 and 8). So there are six size-based variance strata within each area-industry domain.

We assume that $\hat{V}_{pjt}$ is a design-unbiased estimator for V_{pjt}, i.e., $E_p(\hat{V}_{pjt}) = V_{pjt}$. Let n_{jt} be the number of responding sample UI accounts within the domain j and month t. In this article, we consider only domains with at least twelve reporting UI accounts. There are 430 area-industry combinations in our CES data. Each area-industry combination has data from January to December of the year 2000. For the current analysis, we considered data from the six industries described in Table 2. For areas with a substantial amount of mining activity, CES produces separate employment estimates for the mining and construction industries respectively. For other areas, CES produces a single employment estimate for the combined mining and construction industries. For the 430 domains considered here, the mean number of reporting sample UI accounts was 475. For the CES application, this article will consider three special cases of Model (1) on a logarithmic scale. First, note that Model ($f1$) from Subsection 2.2 constrains both intercepts and slopes to be constant across industries and areas. A generalization that allows the intercepts to vary across industries is:

$$\ln(V_{jt}) = \gamma_{0i(j)} + \gamma_1 \ln(x_{j0}) + \gamma_2 \ln(n_{jt}) + \gamma_3 \ln(t) + q^*_{jt}, \qquad (f2)$$

where $i(j)$ represents the industry i that is represented in a specific domain j. A further generalization that allows all coefficients to vary across industries is:

$$\ln(V_{jt}) = \gamma_{0i(j)} + \gamma_{1i(j)} \ln(x_{j0}) + \gamma_{2i(j)} \ln(n_{jt}) + \gamma_{3i(j)} \ln(t) + q^*_{jt}. \qquad (f3)$$

Thus, Model ($f3$) allows interaction between the industry classification and the predictors x_{j0}, n_{jt} and t. Note that in the notation of the general expression (1), Models ($f1$) through ($f3$) involve only predictors X determined by the respondent count n_{jt}, the time lag t and the terms x_{j0}. In contrast with GVFs used for binary outcome variables (e.g., Johnson and King 1987), Models ($f1$) through ($f3$) do not use the population parameters θ_{jt} as scale factors. Instead, our models use the known benchmark total x_{j0} as the scale-factor predictor. Also, for each industry considered in Model ($f3$), we used data from twelve months and from two to 131 areas, as specified in Table 2. In addition, Wolter (2007, Sec. 7.3) and others have noted the importance of fitting GVF models for groups of

Table 2. Number of metropolitan areas (MSAs) and UIs in each industry

Industry description	MSAs	Sample UIs
1 Mining	2	549
2 Mining and construction	36	22,359
3 Construction	61	54,552
4 Durable manufacturing	131	76,150
5 Nondurable manufacturing	100	50,717
6 Wholesale trade	100	58,424

statistics $\hat{\theta}_{jt}$ for which a "common model" will hold. Model $(f1)$ uses a common model for all domains (j), while Model $(f3)$ has distinct coefficient vectors γ for each industry i. In other words, Model $(f1)$ uses a single large "group" while Model $(f3)$ allows each industry to be a separate group.

3. Estimation and Inference for Coefficients in a GVF Model

3.1. Point Estimation Methods

For each of Models $(f1)$ through $(f3)$, we computed estimators $\hat{\gamma}$ of the coefficients γ through ordinary least squares (OLS) regression of $\ln(\hat{V}_{jt})$ on the corresponding vector of predictors. In principle, one could consider alternative coefficient estimators based on weighted least squares or generalized least squares approaches. However, the efficiency gains from these alternative approaches, if any, would depend on the covariance structure of the error terms; details will not be considered in the current article. See Valliant (1987) for a discussion of conditions under which weighted least squares estimation may be preferred to ordinary least squares estimation for GVFs.

3.2. Design-Based Variance Estimation for GVF Coefficients

We obtain an estimator $\hat{V}_p(\hat{\gamma})$ of the variance of the approximate distribution of $\hat{\gamma}$ from an extension of standard estimating equation approaches for complex-survey estimators (Binder 1983). Then the estimator $\hat{\gamma}$ in Expression (4) can be rewritten as the solution of the estimating equation,

$$
\begin{aligned}
0 &= \hat{\mathbf{w}}(\gamma) \\
&= \mathbf{X}'\mathbf{Y} - \mathbf{X}'\mathbf{X}\gamma \\
&= \sum_{j \in \mathcal{D}} \hat{\mathbf{w}}_{j\cdot}(\gamma),
\end{aligned}
$$

where $\hat{\mathbf{w}}_{j\cdot}(\gamma) = \mathbf{X}_j'(\mathbf{Y}_j - \mathbf{X}_j\gamma)$. In addition, let $\hat{\mathbf{w}}_{jb}(\gamma)$ be the b-th element of $\hat{\mathbf{w}}_{j\cdot}(\gamma)$ and let $\hat{\mathbf{w}}_{\cdot b}(\gamma)$ be the b-th element of $\hat{\mathbf{w}}(\gamma)$. The Taylor expansion of $\hat{\mathbf{w}}(\gamma)$ at $\gamma = \gamma*$, where $\gamma*$ is the population parameter value, leads to:

$$
\begin{aligned}
0 &= \hat{\mathbf{w}}(\hat{\gamma}) \\
&= \hat{\mathbf{w}}(\gamma*) + \hat{\mathbf{w}}^{(1)}(\gamma*)(\hat{\gamma} - \gamma*) + R,
\end{aligned}
$$

where $\hat{\mathbf{w}}^{(1)}(\gamma*) = \left.\frac{\partial \hat{\mathbf{w}}(\gamma)}{\partial \gamma}\right|_{\gamma=\gamma*}$ and R is a $B \times 1$ vector with b-th element equal to $2^{-1}(\hat{\gamma} - \gamma*)'\left(\left.\frac{\partial^2 \hat{\mathbf{w}}_{\cdot b}(\gamma)}{\partial \gamma \partial \gamma'}\right|_{\gamma=\gamma**}\right)(\gamma - \gamma*)$ for some $\gamma**$ with $|\gamma** - \gamma*| < |\hat{\gamma} - \gamma*|$. Thus,

$$
\hat{\mathbf{w}}(\gamma*) = -\hat{\mathbf{w}}^{(1)}(\gamma*)(\hat{\gamma} - \gamma*) - R. \tag{5}
$$

Under regularity conditions, the second term on the right-hand side of Expression (5) is of a smaller order of magnitude than the first term. Consequently, an estimator of the variance-covariance matrix of the approximate distribution of $\hat{\gamma}$ is

$$\hat{V}(\hat{\gamma}) = \left\{\hat{\mathbf{w}}^{(1)}(\hat{\gamma})\right\}^{-1}\hat{V}\{\hat{\mathbf{w}}(\hat{\gamma})\}\left[\left\{\hat{\mathbf{w}}^{(1)}(\hat{\gamma})\right\}\right]^{-1}, \tag{6}$$

where $\hat{\mathbf{w}}^{(1)}(\hat{\gamma}) = \frac{\partial \hat{\mathbf{w}}(\gamma)}{\partial \gamma}\Big|_{\gamma=\hat{\gamma}}$ and $\hat{V}\{\hat{\mathbf{w}}(\hat{\gamma})\}$ is an estimator of the variance of $\hat{\mathbf{w}}(\hat{\gamma})$, evaluated at the point $\gamma = \hat{\gamma}$.

3.3. *Application to the Current Employment Statistics Program*

Let T be the total number of months covered by the data; for the CES design, $T = 12$. Then

$$\hat{\mathbf{w}}^{(1)}(\hat{\gamma}) = -\sum_{j \in \mathcal{D}} \sum_{t=1}^{T} \boldsymbol{X}'_{jt} \boldsymbol{X}_{jt}.$$

In addition, under the CES design, selection of sample units is essentially independent across domains. However, due to the CES design and estimation methods, estimators within a domain may be strongly correlated across consecutive months. Consequently, an estimator for the middle term in Expression (6) is

$$\begin{aligned} \hat{V}\{\hat{\mathbf{w}}(\hat{\gamma})\} &= \hat{V}\left(\sum_{j \in D} \hat{\mathbf{w}}_j(\hat{\gamma})\right) \\ &= J^2 \hat{V}\left(J^{-1} \sum_{j \in D} \hat{\mathbf{w}}_j(\hat{\gamma})\right) \\ &= (J-1)^{-1} J \sum_{j \in \mathcal{D}} \left\{\hat{\mathbf{w}}_j(\hat{\gamma}) - \hat{\bar{\mathbf{w}}}(\hat{\gamma})\right\}\left\{\hat{\mathbf{w}}_j(\hat{\gamma}) - \hat{\bar{\mathbf{w}}}(\hat{\gamma})\right\}', \end{aligned} \tag{7}$$

where $\hat{\bar{\mathbf{w}}} = J^{-1} \sum_{j \in D} \hat{\mathbf{w}}_j(\hat{\gamma})$. Note that the final equality in Expression (7) uses the independence across domains j and accounts for correlation across periods t. Under regularity conditions (e.g., Korn and Graubard 1990) $d_w \hat{V}(\hat{\gamma})$ is distributed approximately as a Wishart random matrix on d_w degrees of freedom and matrix parameter $V(\hat{\gamma})$.

4. Comparison of the Direct and GVF Methods in Prediction of the True Variance

4.1. *Decomposition of Differences of $\hat{V}_{pjt} - V^*_{pjt}$*

Once we have selected and estimated a specific GVF Model (f), it is useful to evaluate the properties of the resulting predictors of V_{pjt}. Suppose that a model-fitting method (e.g., ordinary least squares, perhaps on a transformed scale; or nonlinear least squares) leads to the coefficient point estimator $\hat{\gamma}$, and define the resulting variance terms,

$$V^*_{pjt} \stackrel{def}{=} f(X_{jt}, \hat{\gamma}). \tag{8}$$

Appendix C presents two options for specific ways in which to incorporate parameter estimators into the adjusted predictors V^*_{pjt}. The data analysis for this article will use a fairly conservative predictor V^*_{pjt}. Note that V^*_{pjt} is based on the general model (1) given on the original variance scale. Under the variance function model (1), error model (2) and the definition of V^*_{pjt} in Expression (8), $\hat{V}_{pjt} - V_{pjt} = \epsilon_{jt}$, and $V^*_{pjt} - V_{pjt} = f(X_{jt}, \hat{\gamma}) - \left\{f(X_{jt}, \gamma) + q_{jt} + E(q_{jt}) - E(q_{jt})\right\}$. Consequently, we may decompose the difference $\hat{V}_{pjt} - V^*_{pjt}$ as

$$\begin{aligned} \hat{V}_{pjt} - V^*_{pjt} &= (\hat{V}_{pjt} - V_{pjt}) - (V^*_{pjt} - V_{pjt}) \\ &= \epsilon_{jt} + \{q_{jt} - E(q_{jt})\} + E(q_{jt}) - \{f(X_{jt}, \hat{\gamma}) - f(X_{jt}, \gamma)\}. \end{aligned} \tag{9}$$

In Equation (9), ϵ_{jt} is a pure estimation error in the original $\hat{V}_{pjt}$ estimates with $E(\epsilon_{jt}) = 0$; $\{q_{jt} - E(q_{jt})\}$ is a random equation error; and $E(q_{jt})$ represents the deterministic lack-of-fit in our model attributable, for example, to omitted regressors or a misspecified functional form. The last term in Equation (9), $\{f(X_{jt}, \hat{\gamma}) - f(X_{jt}, \gamma)\}$, is a parameter estimation error attributable to the errors $\hat{\gamma} - \gamma$.

Exploratory analysis of the adequacy of our estimated values, V^*_{pjt}, may focus on the magnitude of the prediction errors $\left(V^*_{pjt} - V_{pjt}\right)$, relative to the errors $\left(\hat{V}_{pjt} - V_{pjt}\right)$, in the original estimators $\hat{V}_{pjt}$. If $E\left(V^*_{pjt} - V_{pjt}\right)^2$ is smaller than the variance of $\hat{V}_{pjt}$, then we would prefer V^*_{pjt}. In addition,

$$\delta(X_{jt}, \gamma) \stackrel{def}{=} E[\{f(X_{jt}, \hat{\gamma}) - V_{pjt}\}^2 | X_{jt}, \gamma]$$

may vary across values of X_{jt} with $\delta(X_{jt}, \gamma) << V_p(\hat{V}_{pjt} - V_{pjt})$ only in some cases. In this case, we might prefer V^*_{pjt} for some, but not all values of X_{jt}.

4.2. Properties of the Direct Estimator $\hat{V}_{pjt}$

We evaluate error sizes in terms of conditional expected squared error. In keeping with standard evaluation of design-based variance estimators, assume that for positive d_{jt},

$$E_p(\hat{V}_{pjt} | V_{pjt}) = V_{pjt}, \quad V_p(\hat{V}_{pjt} | V_{pjt}) = \frac{2V^2_{pjt}}{d_{jt}}. \tag{10}$$

The moment properties (10) would hold if $V^{-1}_{pjt} d_{jt} \hat{V}_{pjt}$ followed a $\chi^2(d_{jt})$ distribution. However, the current article will assume that $\hat{V}_{pjt}$ follows a lognormal distribution that in general would allow somewhat greater modeling flexibility; see Appendix B for related comments. Note that

$$\begin{aligned} E_p(\hat{V}^2_{pjt} | V_{pjt}) &= \{E_p(\hat{V}_{pjt}) | V_{pjt}\}^2 + V_p(\hat{V}_{pjt} | V_{pjt}) \\ &= V^2_{pjt} + \frac{2V^2_{pjt}}{d_{jt}} \\ &= d^{-1}_{jt}(d_{jt} + 2) V^2_{pjt}. \end{aligned} \tag{11}$$

Consequently from (11), an unbiased estimator of $V_p(\hat{V}_{pjt} | V_{pjt})$ is:

$$\hat{V}_p(\hat{V}_{pjt} | V_{pjt}) = (d_{jt} + 2)^{-1} 2\hat{V}^2_{pjt}. \tag{12}$$

Six employment size classes were used for stratification for our CES survey example, so the data analysis in Section 5 will use $d_{jt} = 6$. In addition, sample sizes within employment class generally were large enough for each t that stratum-level sample means were considered to follow an approximate normal distribution.

4.3. Properties of the GVF Estimator V^*_{pjt}

Now consider the properties of V^*_{pjt}, and the conditions under which V^*_{pjt} may have a smaller mean squared error than $\hat{V}_{pjt}$. In the general case,

$$V_{pjt} - V^*_{pjt} = q_{jt} - \{f(X_{jt}, \hat{\gamma}) - f(X_{jt}, \gamma)\}. \tag{13}$$

To simplify the discussion, assume that the product $(J \cdot T)$ is increasing without bound. This would occur with, for example, increases in the number of geographical areas or the number of time periods. For example, the CES application uses data from 430 area-industry combinations and 12 time periods, so the product $J \cdot T$ is relatively large. Then, under mild regularity conditions on the function $f(\cdot)$,

$$E[\{f(X_{jt}, \hat{\gamma}) - f(X_{jt}, \gamma)\}^2 | X_{jt}] = \mathbf{O}_p\{(J \cdot T)^{-1}\}, \tag{14}$$

while the domain-specific term $E\left(q^2_{jt} | X_{jt}\right)$ does not necessarily decrease as the product $(J \cdot T)$ increases. For example, result (14) generally holds for each of Models $(f1)$-$(f3)$ because these models do not include terms θ_{jt}; include only known predictors X_{jt}; and involve errors $\hat{\gamma} - \gamma$ that are $\mathbf{O}_p\{(J \cdot T)^{-1/2}\}$. Under result (14) and additional technical conditions,

$$E\{(V^*_{pjt} - V_{pjt})^2 | X_{jt}\} = E(q^2_{jt}) + \mathbf{O}_p\{(J \cdot T)^{-1}\} \tag{15}$$

and the leading term $E(q^2_{jt})$ will generally be of larger magnitude than the $\mathbf{O}_p\{(J \cdot T)^{-1}\}$ term associated with the error $f(X_{jt}, \hat{\gamma}) - f(X_{jt}, \gamma)$. Consequently, our task of evaluation of the approximate mean squared error of V^*_{pjt} simplifies to an evaluation of the expected square of q_{jt}.

4.4. Diagnostics for Comparison of $\hat{V}_{pjt}$ And V^*_{pjt}

We do not observe q_{jt} directly, but we can estimate its expected square through the following steps. First, note from Expression (9) that

$$\hat{V}_{pjt} - V^*_{pjt} = \epsilon_{jt} + q_{jt} - \{f(X_{jt}, \hat{\gamma}) - f(X_{jt}, \gamma)\}$$

and so

$$\begin{aligned}\left(\hat{V}_{pjt} - V^*_{pjt}\right)^2 = {} & \epsilon^2_{jt} + q^2_{jt} \\ & + \{f(X_{jt}, \hat{\gamma}) - f(X_{jt}, \gamma)\}^2 \\ & + 2q_{jt}\left\{f(X_{jt}, \hat{\gamma}) - f(X_{jt}, \gamma)\right\} \\ & + 2\epsilon_{jt}\left[q_{jt} - \{f(X_{jt}, \hat{\gamma}) - f(X_{jt}, \gamma)\}\right].\end{aligned} \tag{16}$$

Under condition (14), the conditional expectation $E\left(\{f(X_{jt}, \hat{\gamma}) - f(X_{jt}, \gamma)\}^2 | X_{jt}\right)$ is small relative to $E(q^2_{jt} | X_{jt})$. Under additional mild conditions, the conditional expectations $E\left[2q_{jt}\left\{f(X_{jt}, \hat{\gamma}) - f(X_{jt}, \gamma)\right\} \middle| X_{jt}\right]$ and $E\left(2\epsilon_{jt}\left[q_{jt} - \{f(X_{jt}, \hat{\gamma}) - f(X_{jt}, \gamma)\}\right] \middle| X_{jt}\right)$ are small relative to $E\left(q_{jt}^2|X_{jt}\right)$, so

$$E\left\{\left(\hat{V}_{pjt}-V_{pjt}^*\right)^2|X_{jt}\right\} \doteq V_p(\hat{V}_{pjt}|V_{pjt})+E\left(q_{jt}^2|X_{jt}\right). \tag{17}$$

Expressions (9) and (16) lead to two general conclusions regarding diagnostics for V_{pjt}^*. First, due to distinctions between $V(\epsilon_{jt})$ and $E(q_{jt}^2)$, care is required in the interpretation of standard regression diagnostics when applied to GVF models like the general model (1), or the specific models ($f1$) through ($f3$). For example, the customary regression mean squared error, $\hat{\sigma}_e^2$, is an estimator of the sum $V(\epsilon_{jt})+E(q_{jt}^2)$. In addition, under regularity conditions, the customary squared coefficient of variation, R^2, satisfies the approximate relationship,

$$1-R^2 \doteq \left\{V(\epsilon_{jt})+E(q_{jt}^2)+(J-1)^{-1}\sum_{j=1}^{J}\left\{f(X_{jt},\hat{\gamma})-\bar{\hat{V}}\right\}^2\right\}^{-1} \left\{V(\epsilon_{jt})+E(q_{jt}^2)\right\}, \tag{18}$$

where $\bar{\hat{V}}=J^{-1}\sum_{j=1}^{J}\hat{V}_{jt}$. Under an ideal fit for Model (1), $E(q_{jt}^2)$ would be approximately equal to zero, but $1-R^2$ would not necessarily be close to zero, due to the presence of $V(\epsilon_{jt})$ in the numerator of Expression (18). Thus, relatively small values of R^2 by themselves do not necessarily indicate a poor fit for GVF Model (1). Similar comments apply to other regression goodness-of-fit diagnostics used for GVF models.

Second, to address these limitations, it is useful to consider estimators of $E(q_{jt}^2|X_{jt})$ and related diagnostics that adjust for the effects of $V(\epsilon_{jt})$. In particular, Expression (12) is an unbiased estimator of the first term on the right-hand side of Expression (17). Consequently, we may define a direct estimator of $E(q_{jt}^2|X_{jt})$ to be

$$r_{jt} \stackrel{def}{=} \left(\hat{V}_{pjt}-V_{pjt}^*\right)^2-(d_{jt}+2)^{-1}2\hat{V}_{pjt}^2. \tag{19}$$

Note that r_{jt} is a random variable with properties that depend on the distributions of both the equation error term q_{jt} and the sampling error term ϵ_{jt}. For example, if $E(q_{jt}^2|X_{jt})=0$, then the leading terms of a Taylor expansion of r_{jt} would have an expectation equal to zero. Similarly, if $E(q_{jt}^2|X_{jt})$ is not large relative to $E(\epsilon_{jt}^2|X_{jt})$, then there is a substantial probability that a given value of r_{jt} is less than zero. These results are similar to properties of unadjusted estimators of "between group" variance terms in standard variance component models. For example, for the data analysis detailed in Section 5, approximately 36% of the r_{jt} values were less than zero.

Consequently, in assessment of $E(q_{jt}^2|X_{jt})$, use of smoothed versions of r_{jt} would generally be preferred. For example, one could extend the standard variance-component literature on "restricted maximum likelihood" (REML) estimation (e.g., Patterson and Thompson 1971; Corbeil and Searle 1976; and Harville 1977). However, a detailed extension of REML methods to the current setting is beyond the scope of the current work. Instead, the next subsection presents a relatively simple regression approach to estimation of $E(q_{jt}^2|X_{jt})$.

4.5. Model Fitting for Conditional Expected Squared Equation Error

In general, one may consider a model

$$
\begin{aligned}
E(q_{jt}^2|X) &= Z_{jt}\omega + \eta_{jt} \\
&= \sum_{c=1}^{C} Z_{cjt}\omega_c + \eta_{jt}
\end{aligned} \tag{20}
$$

for the conditional expectation of q_{jt}^2, where $Z_{jt} = (Z_{1jt}, \ldots, Z_{Cjt})$ is a $1 \times C$ vector of predictors (generally functions of θ_{jt}, X_{jt} and γ); $\omega = (\omega_1, \ldots, \omega_C)'$ is a $C \times 1$ column of fixed regression coefficients; and η_{jt} is a random error term arising from the underlying superpopulation model. Let $\mathbf{r}_j$ be a $T \times 1$ vector with t-th element r_{jt} and define the $(J \cdot T) \times 1$ vector $\mathbf{r} = (\mathbf{r}_1', \mathbf{r}_2', \ldots, \mathbf{r}_J')'$. Similarly, let $\mathbf{Z}_j$ be a $T \times C$ matrix with t-th row $\mathbf{Z}_j(t,:)$ equal to the predictors used for the specified model. Also, define the $(J \cdot T) \times C$ matrix $\mathbf{Z} = (\mathbf{Z}_1', \mathbf{Z}_2', \ldots, \mathbf{Z}_J')'$. Define $\hat{\omega} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{r}$. See Appendix A for development of the variance estimators and inferential statistics for $\hat{\omega}$. Finally, define an estimator of $E(\mathbf{r}|\mathbf{Z})$ by

$$
\hat{h}_f = \mathbf{Z}\hat{\omega}. \tag{21}
$$

For example, in keeping with the general approach to error analysis in variance function models (e.g., Davidian et al. 1988), a quadratic function version of Model (20) is

$$
V(q_{jt}|\theta_{jt}, X_{jt}, \gamma) = \omega_0 + \omega_1 f(X_{jt}, \gamma) + \omega_2 \left\{f(X_{jt}, \gamma)\right\}^2 + \eta_{jt}, \tag{22}
$$

where $E(\eta_{jt}) = 0$. Under approximation (15) and Model (22), the relative variance of the prediction error $V_{pjt} - V_{pjt}^*$ is

$$
\begin{aligned}
&RelVar\left(V_{pjt} - V_{pjt}^*|X_{jt}\right) \\
&\doteq \left\{f(X_{jt}, \gamma)\right\}^{-2} V\left(V_{pjt} - V_{pjt}^*\right) \\
&\doteq \left\{f(X_{jt}, \gamma)\right\}^{-2}\omega_0 + \left\{f(X_{jt}, \gamma)\right\}^{-1}\omega_1 + \omega_2 + \left\{f(X_{jt}, \gamma)\right\}^{-2}\eta_{jt}.
\end{aligned} \tag{23}
$$

When condition (14) does not hold, one could consider an expansion of Model (22) to account for predictors of the additional components of $RelVar\left(V_{pjt} - V_{pjt}^*|X_{jt}\right)$. For a given function $f(X_{jt}, \gamma)$, we may consider a model to produce a smooth version, $h_f(X_{jt}, \omega)$, of the conditional expectation, $E\{(V_{pjt}^* - V_{pjt})^2|X_{jt}\}$, such that:

$$
E\left\{\left(V_{pjt}^* - V_{pjt}\right)^2|X_{jt}\right\} = h_f(X_{jt}, \omega) + \eta_{jt}.
$$

For example, Expression (22) leads to

$$
r_{jt} = \omega_0 + \omega_1 V_{pjt}^* + \omega_2 V_{pjt}^{*2} + a_{jt}, \tag{24}
$$

where we substitute the observed values V_{pjt}^* for the unknown quantities $f(X_{jt}, \gamma)$, and a_{jt} is a remainder term. In addition, it is of interest to consider the reduced form of Model (24) in which $\omega_0 = 0 = \omega_1$:

$$\bar{V}^{*-2} r_{jt} = \bar{V}^{*-2} V_{pjt}^{*2} \omega_2 + \bar{V}^{*-2} a_{jt}, \tag{25}$$

where $\bar{V}^* = J^{-1}\sum_{j=1}^{J} V_{pjt}^*$. For example, under Model (24), $\mathbf{Z}_j(t, \cdot) = [1, V_{pjt}^*, V_{pjt}^{*2}]$ and $\omega = [\omega_0, \omega_1, \omega_2]'$ where j is the number of domains, and $C = 3$ is the number of coefficients in (24). Similarly, for Model (25), $\mathbf{Z}_j(t, \cdot) = \left[V_{pjt}^{*2}\right]$ and $C = 1$.

4.6. A Degrees-of-Freedom Interpretation of Prediction Error Properties

Application of the ideas in Appendix B indicate that under Model (22), the term

$$\{f(X_{jt}, \gamma)\}^{-1} d_{jt}^* V_{pjt} \tag{26}$$

has the same first and second moments as a $\chi^2_{d_{jt}^*}$ random variable where

$$\begin{aligned} d_{jt}^* &= 2\left\{RelVar(V_{pjt} - V_{pjt}^*)\right\}^{-1} \\ &\doteq \left[\{f(X_{jt}, \gamma)\}^{-2}\omega_0 + \{f(X_{jt}, \gamma)\}^{-1}\omega_1 + \omega_2 + \{f(X_{jt}, \gamma)\}^{-2}\eta_{jt}\right]^{-1} 2. \end{aligned} \tag{27}$$

In addition, under Model (24), results presented in Appendix B indicate that $2\left(V_{pjt}^{*-2}\hat{h}_f\right)^{-1}$ is an estimator of Expression (27) provided the error difference $V_{pjt}^{*-2} a_{jt} - V_{pjt}^{*-2}\eta_{jt}$ is small. Thus, the degrees-of-freedom attributable to the error term q_{jt} may in general depend on the function $f(X_{jt}, \gamma)$ and thus vary across domains.

However, under the reduced Model (25), if the remainder term a_{jt} is small, then

$$d_{jt}^* \doteq \omega_2^{-1} 2, \tag{28}$$

that is, the degrees-of-freedom term d_{jt}^* is approximately constant and can be estimated on the basis of the estimated coefficient ω_2 from the reduced Model (25).

5. Data Analysis

5.1. Estimation for GVF Model Coefficients

For the CES example introduced in Section 2, Tables 3 through 5 report coefficient estimates, standard errors and inferential statistics for Models $(f1)$ through $(f3)$ respectively. The reported standard errors equal the square root of the variance estimates

Table 3. Coefficient estimates and inferential statistics for Model $(f1)$

	Intercept γ_0	$\ln(x_{j0})$ γ_1	$\ln(n_{jt})$ γ_2	$\ln(t)$ γ_3	R^2	$\hat{\sigma}_e^2$
EST.	−1.43	1.16	0.22	1.17	0.52	1.31
s.e.	0.66	0.09	0.12	0.07		
t_γ	−2.17	12.77	1.78	16.72		
meff	5.45	9.87	10.63	1.02		

Table 4. Coefficient estimates and inferential statistics for Model ($f2$)

	Intercept						$\ln(x_{j0})$	$\ln(n_{jt})$	$\ln(t)$		
	γ_{01}	γ_{02}	γ_{03}	γ_{04}	γ_{05}	γ_{06}	γ_1	γ_2	γ_3	R^2	$\hat{\sigma}_e^2$
EST.	−3.98	−3.28	−3.44	−4.85	−4.89	−4.26	1.70	−0.57	1.25	0.61	1.06
s.e.	1.08	0.65	0.66	0.72	0.73	0.71	0.11	0.15	0.07		
t_γ	−3.68	−5.03	−5.23	−6.76	−6.74	−5.99	16.06	−3.86	17.93		
meff	9.53	6.22	6.13	6.72	6.00	6.81	11.77	12.19	1.26		

computed from Expression (6). In addition, the design-based test statistic for the coefficient γ_b is:

$$t_{\gamma_b} = \left\{\hat{V}_p(\hat{\gamma}_b)\right\}^{-1/2}\hat{\gamma}_b.$$

Recall that Model ($f1$) has coefficients that are constant across all industries, Model ($f2$) allows different intercept terms across industries and Model ($f3$) allows all coefficients to vary across industries. Also, note that Models ($f1$) through ($f3$) all include both $\ln(x_{j0})$ and $\ln(n_{jt})$. In general, subpopulations with a larger benchmark employment, x_{j0}, will tend to receive larger initial sample sizes and thus also have larger numbers of respondents, n_{jt}, in month t. Consequently, $\ln(x_{j0})$ and $\ln(n_{jt})$ will tend to be positively correlated across our 430 domains j. However, inclusion of both predictors allowed us to account for the effects of the changes in numbers of respondents across months. In Table 3, the positive coefficient on $\ln(n_{jt})$ is an outcome of this positive association between $\ln(x_{j0})$ and $\ln(n_{jt})$. On the other hand, after incorporation of industry-specific intercept terms in Models ($f2$) and ($f3$), the estimated coefficients for $\ln(n_{jt})$ are negative.

In addition, the final rows of Tables 3 through 5 present "misspecification effect" ratios for each of the estimated coefficients. In a slight extension of the ideas in Skinner (1986), define the misspecification effect ratio for the coefficient estimator $\hat{\gamma}_b$ as:

$$\text{meff}_{mb} = \left[\frac{se_{f_m,complex}(\hat{\gamma}_b)}{se_{f_m,direct}(\hat{\gamma}_b)}\right]^2, \tag{29}$$

where $se_{f_m,complex}(\hat{\gamma}_b)$ is the estimated standard error of the ordinary least squares coefficient estimator $\hat{\gamma}_b$ computed with Expression (6) for model f_m; and $se_{f_m,direct}(\hat{\gamma}_b)$ is the corresponding standard error obtained directly from ordinary least squares results, without any adjustment for the correlation across $\hat{V}_{pjt}$ terms induced by the CES design and estimation methods. For cases in which meff_{mb} is greater than one, direct use of unadjusted errors from ordinary least squares regression output will lead to confidence intervals for $\hat{\gamma}_b$ that are too narrow and that have coverage rates below their nominal levels. As one would expect in the analysis of data with relatively strong correlation over time, Table 3 reports misspecification effect ratios that are substantially greater than one for the coefficients γ_0, γ_1 and γ_2. For γ_3 (the coefficient of the $\ln(t)$ predictor), the misspecification effect ratio is close to one. Tables 4 and 5 display qualitatively similar patterns for their misspecification effect ratios, with the exception of the coefficients for Industry 1. This industry had data for only two MSAs, while Industries 2 through 6 had data for 36, 61, 131, 100 and 100 MSAs, respectively.

Table 5. Coefficient estimates and inferential statistics for Model ($f3$)

	Intercept						$\ln(x_{j0})$						R^2	$\hat{\sigma}_e^2$
	γ_{01}	γ_{02}	γ_{03}	γ_{04}	γ_{05}	γ_{06}	γ_{11}	γ_{12}	γ_{13}	γ_{14}	γ_{15}	γ_{16}		
est.	8.04	−3.77	−2.24	−4.30	−2.13	−7.86	0.36	2.00	1.62	1.57	1.32	2.11	0.62	1.04
s.e.	0.46	1.46	1.03	2.28	1.63	1.03	0.15	0.25	0.14	0.36	0.23	0.14		
t_γ	17.50	−2.58	−2.19	−1.89	−1.31	−7.62	2.44	8.03	11.29	4.39	5.66	14.88		
meff	0.02	3.04	2.85	17.57	7.21	3.40	0.05	6.52	5.34	29.05	10.24	5.55		
	$\ln(n_{jt})$						$\ln(t)$							
	γ_{21}	γ_{22}	γ_{23}	γ_{24}	γ_{25}	γ_{26}	γ_{31}	γ_{32}	γ_{33}	γ_{34}	γ_{35}	γ_{36}		
est.	−0.18	−0.97	−0.44	−0.37	−0.13	−1.10	1.42	0.93	0.95	1.31	1.11	1.76		
s.e.	0.45	0.44	0.20	0.46	0.29	0.19	0.22	0.18	0.13	0.13	0.15	0.15		
t_γ	−0.40	−2.22	−2.22	−0.81	−0.43	−5.86	6.47	5.22	7.34	9.85	7.60	11.85		
meff	0.06	7.98	4.34	30.02	9.93	5.22	0.06	0.69	0.62	1.43	1.30	1.32		

Table 6. Wald test of $\omega_0 = \omega_1 = 0$ for Model (24). (Reference value: 6.00 at $\alpha = 0.05$)

First phase model	$f1$	$f2$	$f3$
Test statistics	3.22	2.63	3.13

In applying the residual-analysis methods developed in Section 4 and Appendix C, we used the estimators

$$V^*_{pjt,f_m} = \exp\left(X_{j,f_m}\hat{\gamma}_{f_m} + 2^{-1}\hat{\sigma}^2_{e,f_m}\right)$$

where X_{j,f_m} and $\hat{\gamma}_{f_m}$ are respectively the vectors of predictor variables and ordinary least squares coefficient estimators for a given model m, with each of Models ($f1$) through ($f3$) considered separately. In addition, $\hat{\sigma}^2_{e,f_m}$ is the residual mean squared error from the ordinary least squares regression fit for model m. See Karlberg (2000) for related comments.

5.2. Goodness-of-Fit Measures for the GVF Models

To evaluate the goodness-of-fit of our GVF models, note first that Tables 3, 4 and 5 present the aggregate measures R^2 equal to 0.52, 0.61 and 0.62 for Models ($f1$) through ($f3$), respectively; and the corresponding residual mean squared error terms $\hat{\sigma}^2_e$ are 1.31, 1.06 and 1.04, respectively. Thus, in a summary evaluation of fit across all domains, Model ($f2$) is somewhat better than ($f1$), but ($f3$) is only marginally better than Model ($f2$). In keeping with the comments following Expression (17), interpretation of R^2 and $\hat{\sigma}^2_e$ values warrants careful consideration of the effect of $V(\epsilon^*_{jt})$. Specifically, applications of the residual-analysis methods from Section 4 indicate several important ways in which Model ($f3$) may provide a better fit than Models ($f1$) or ($f2$) for the CES data.

First, for each of Models ($f1$) through ($f3$), Table 6 reports the results of standard Wald test statistics for the null hypothesis $H_0 : \omega_0 = 0 = \omega_1$:

$$W = \left(\hat{\omega}_0, \hat{\omega}_1\right)\left[\hat{V}\{(\hat{\omega}_0, \hat{\omega}_1)'\}\right]^{-1}(\hat{\omega}_0, \hat{\omega}_1)',$$

where $\hat{\omega} = \begin{bmatrix}\hat{\omega}_0, & \hat{\omega}_1, & \hat{\omega}_2\end{bmatrix}'$ is computed through an ordinary least squares fit to Model (24) with $\hat{V}(\hat{\omega}_0, \hat{\omega}_1)$ computed as shown in Appendix A. In addition, $\hat{\omega}$ and $\hat{V}(\hat{\omega})$ are based on data from a total of 430 area-industry combinations. Application of the quadratic form ideas reviewed in Appendix A, with $d = 430 - 1 = 429$ and $p = 2$, indicates that $(W/429)\{(429 - 2 + 1)/2\}$ has approximately a noncentral F distribution with 2 and $429 - 2 + 1 = 428$ degrees of freedom and with noncentrality parameter $W_0 = (\omega_0, \omega_1)\left[V\{\hat{\omega}_0, \hat{\omega}_1)'\}\right]^{-1}(\omega_0, \omega_1)'$. In our example, all test statistics from Models

Table 7. Degrees of Freedom ($d*$) among Models (f) given Model (24) with $\omega_0 = \omega_1 = 0$

Model	$f1$	$f2$	$f3$
ω_2	0.484	0.216	0.004
$se(\omega_2)$	(0.053)	(0.048)	(0.001)
$d*$	4.13	9.25	468.77

$(f1)$ to $(f3)$ were smaller than the reference value, $F_{\{2,428\}}(2)(429)/428 = 6.00$, at $\alpha = 0.05$.

Table 7 reports the estimates $\hat{\omega}_2$ and their standard errors computed under the reduced form of Model (25) with the constraints $\omega_0 = 0 = \omega_1$. Note that Model $(f3)$ has large estimated values for $d* = \hat{\omega}_2^{-1}2$, while Models $(f1)$ and $(f2)$ have much smaller estimated values for $d*$.

Second, we computed the terms r_{jt} from Expression (19) for each of Models $(f1)$ through $(f3)$ respectively. Figure 1 presents a plot of the resulting r_{jt} against the corresponding predicted values $\ln(V^*_{pjt})$ for Model $(f3)$. The grey circles display the plot of r_{jt}, an approximately unbiased estimator of the mean squared error of V^*_{pjt}, against $\ln(V^*_{pjt})$; and the solid black circles display the values of $\hat{h}_{f_3}$, the smoothed version of r_{jt} based on Expression (21) computed for the reduced Model (25). Figure 1 also includes results from a nonparametric regression method known as locally weighted regression (loess) with a span of 0.1. For general background on loess methods, see Cleveland and Grosse (1991). Note that the loess-smoothed estimates are relatively close to the corresponding values of $\hat{h}_{f_3}$ in Figure 1.

Similar plots were produced for Models $(f1)$ and $(f2)$ but are not shown in the article. For the relatively simple Model $(f1)$, the resulting plot indicates that $\hat{h}_{f_1}$, the estimator of $E(q^2_{jt}|X_{jt})$, is relatively large for large values of $\ln(V^*_{jt})$, reflecting a potential lack of fit for Model $(f1)$ in this upper range. For $(f2)$, which is a more refined model than $(f1)$, the corresponding values of $\hat{h}_{f_2}$ are not as large as $\hat{h}_{f_1}$ for high values of $\ln(V^*_{jt})$, indicating a somewhat better fit of $(f2)$. In addition, for cases with positive values of r_{jt}, we plotted

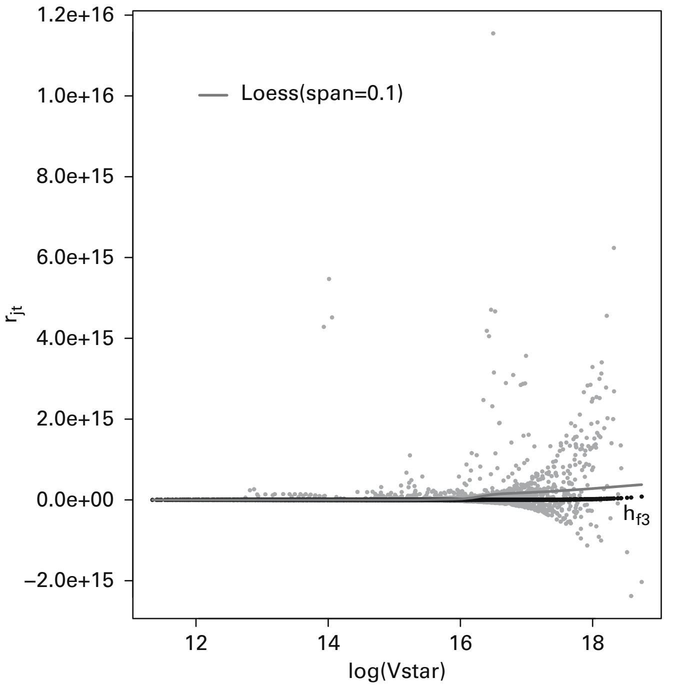

Fig. 1. Three overlaid plots of estimates of $E\left(q^2_{jt}|X_{jt}\right)$ against $\ln\left(V^*_{pjt}\right)$ based on Model $(f3)$. The grey circles present r_{jt} based on Expression (19). The grey line presents loess-smoothed values of r_{jt} with span $= 0.1$. The solid black circles present values of h_{f3} computed from the reduced Model (25)

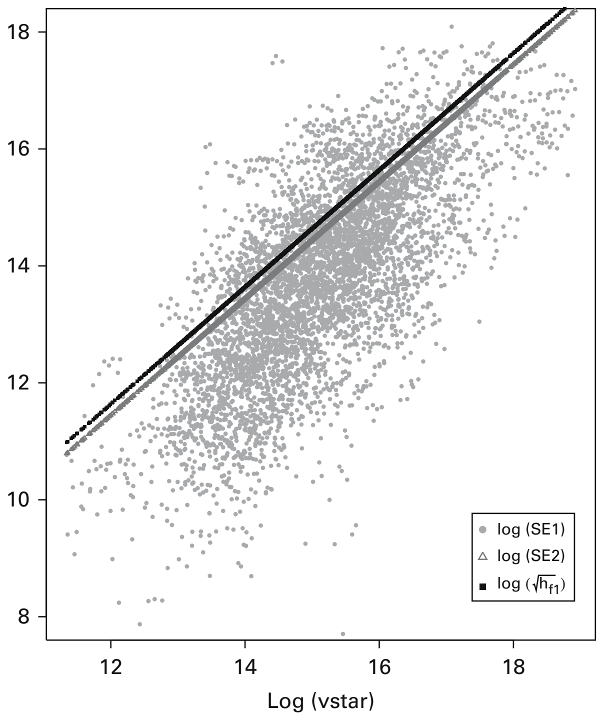

Fig. 2. Plot of ln (SE 1) (grey circles), ln (SE 2) (grey triangles) and ln ($\sqrt{h_{f1}}$) (black squares) against ln $\left(V^*_{pjt}\right)$ for the reduced form (25) of the regression model for the error terms r_{jt}. Here, SE 2, $\sqrt{h_{f1}}$ and V^*_{pjt} are all based on Model (f 1)

points of $\ln(r_{jt})$ against $\ln(V^*_{pjt})$ for Model ($f3$) (again not included here). A loess-smoothed line (span $=$ 0.1) drawn through the plotted points was roughly consistent with a linear relationship between $\ln(r_{jt})$ and $\ln(V^*_{pjt})$. Furthermore, for all values (j,t), the computed values $\hat{h}_{f_1}$, $\hat{h}_{f_2}$ and $\hat{h}_{f_3}$ were all greater than zero, thus addressing the negative individual values of r_{jt} noted in Subsection 4.4.

Figure 2 plots three measures of uncertainty in prediction of the true design variance V_{pjt}. The first measure, $SE1$, equals the square root of $\left(2\hat{V}^2_{pjt}\right)/(d+2)$, which is an unbiased direct estimator of the variance of the prediction error $\hat{V}_{pjt} - V_{pjt}$ under the moment condition (10). The second measure, $SE2$, equals the square root of $\left(2V^{*2}_{pjt}\right)/d$, where V^*_{pjt} is computed under Model ($f1$). Under Model ($f1$) and condition (10), $\left(2V^{*2}_{pjt}\right)/d$ is approximately unbiased for the variance of the prediction error $\hat{V}_{pjt} - V_{pjt}$. Thus $SE2$ may be considered as a smoothed version of $SE1$. The third measure, $\sqrt{h_{f_1}}$, is an estimator of the standard deviation of the equation error term q_{jt} under Model ($f1$) and the conditions outlined in Section 4. In Figure 2, the curve for $\ln(\sqrt{h_{f_1}})$ falls slightly above the curve for ln ($SE2$), which indicates that under the relatively simple Model ($f1$), use of the GVF will lead to an estimated standard error for prediction of V_{pjt} that is slightly larger than the standard error of $\hat{V}_{pjt}$ as a predictor of V_{pjt}. Figures 3 and 4 present the corresponding plots of ln ($SE1$), ln ($SE2$) and ln ($\sqrt{h_f}$) against ln (V^*_{pjt}) for Models ($f2$) and ($f3$), respectively. Note that in Figure 3, the curve for ln ($\sqrt{h_{f_2}}$) is slightly below the curve for ln ($SE2$), while in Figure 4, ln ($\sqrt{h_{f_3}}$) is substantially below ln ($SE2$).

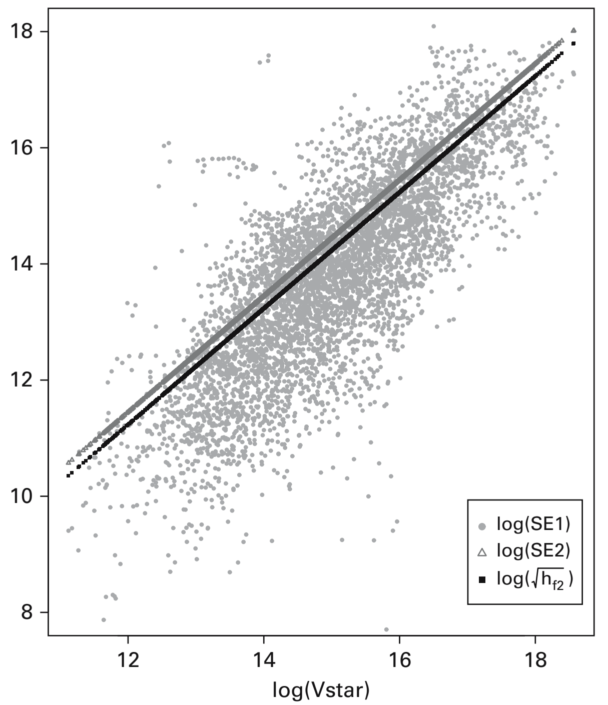

Fig. 3. Plot of ln (SE 1) (grey circles), ln (SE 2) (grey triangles) and ln ($\sqrt{h_{f2}}$) (black squares) against ln $\left(V^*_{pjt}\right)$ for the reduced form (25) of the regression model for the error terms r_{jt}. Here, SE 2, $\sqrt{h_{f2}}$ and V^*_{pjt} are all based on Model ($f2$)

Figure 5 displays plots of $\sqrt{h_f}$ against $\ln(V^*_{pjt})$ where both $\sqrt{h_f}$ and $\ln(V^*_{pjt})$ are computed separately for each of Models ($f1$) through ($f3$). For relatively large values of $\ln(V^*_{pjt})$, the curve for ($f3$) is substantially below the curves for ($f1$) and ($f2$). Thus, Figures 2 through 5 indicate that for prediction of the true variances V_{pjt}, under the specified conditions, use of Model ($f3$) is substantially better than use of either Models ($f1$) or ($f2$), or use of the directly computed terms $\hat{V}_{pjt}$. Finally, note that all figures present data for the same area-industry-month combinations from the calendar year 2000. Consequently, some common outlier patterns appear in several of the figures. For example, Figure 1 displays three large positive outliers corresponding to $\ln(V^*_{pjt})$ values approximately equal to 14.5. These three points represent three consecutive months for one specific area-industry combination. Similar three-point outlier patterns for the same area-industry combinations appear in Figures 2 through 4.

6. A Simulation Study

6.1. Design of the Study

To evaluate the properties of $\hat{\gamma}$ and V^*_{pjt}, we carried out a simulation study based on the following variables produced for each of $R = 1{,}000$ replicates.

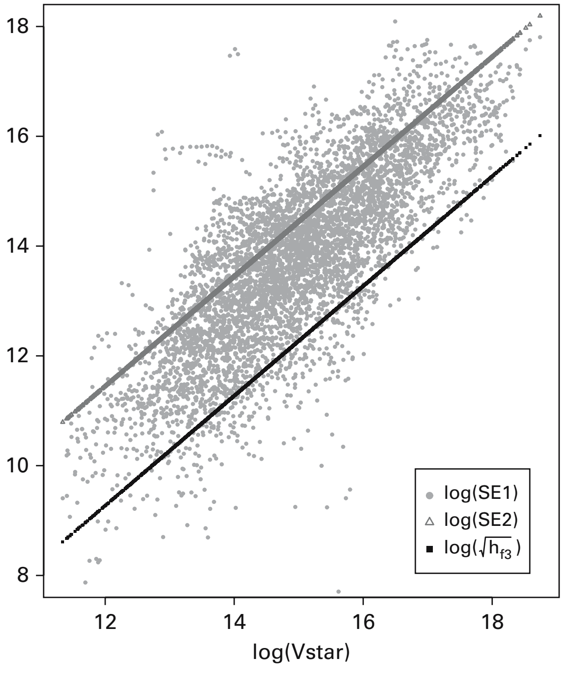

Fig. 4. Plot of ln (SE 1) (grey circles), ln (SE 2) (grey triangles) and ln ($\sqrt{h_{f3}}$) (black squares) against ln$\left(V^*_{pjt}\right)$ for the reduced form (25) of the regression model for the error terms r_{jt}. Here, SE 2, $\sqrt{h_{f3}}$ and V^*_{pjt} are all based on Model ($f3$)

First, we computed the fixed values

$$f_{1jt} = \gamma_0 + \gamma_1 \ln(x_{j0}) + \gamma_2 \ln(n_{jt}) + \gamma_3 \ln(t) \tag{30}$$

based on the numerical values of the coefficient vector γ for Model ($f1$) presented in Table 3 for all 5,160 combinations of domain j and month t considered in Section 5.

Second, we generated the normal $(0, \sigma^2_{q*})$ random variables $q^*_{jt(r)}$ for the 5,160 cases with σ^2_{q*} defined by Expression (C.6) using values of d_q specified in Table 8. We then computed

$$V_{pjt(r)} = \exp(f_{1jt} + q^*_{jt(r)}).$$

In addition, we generated $\hat{\theta}_{jt(r)}$ as independent normal (x_{j0}, V_{pjt}) random variables and generated $\epsilon^*_{jt(r)}$ as independent normal $(0, \sigma^2_{\epsilon*})$ random variables with $\sigma^2_{\epsilon*}$ defined by Expression (C.5) with $d_\epsilon = 6$. We then computed

$$\hat{V}_{pjt(r)} = V_{pjt(r)} \exp(\epsilon^*_{jt(r)}).$$

Based on the 5,160 vectors $\left(\hat{V}_{pjt(r)}, X_{jt}\right)$, where $X_{jt} = \left(1, \ln(x_{j0}), \ln(n_{jt}), \ln(t)\right)$, we carried out ordinary least squares regression of $\ln(\hat{V}_{pjt(r)})$ on X_{jt} to produce the coefficient vector estimate $\hat{\gamma}_{(r)}$; the term $\hat{\sigma}^2_{(r)}$ equal to the regression mean squared error; the term $\hat{\sigma}^2_{q*(r)}$ defined by Expression (C.6); and the predicted variances $V^{**}_{pjt(r)}$ defined by Expression (C.9). In addition, we computed the confidence intervals for θ_{jt},

$$\hat{\theta}_{jt(r)} \pm t_{d_\epsilon, 1-\alpha/2}\left(\hat{V}_{pjt(r)}\right)^{1/2} \tag{31}$$

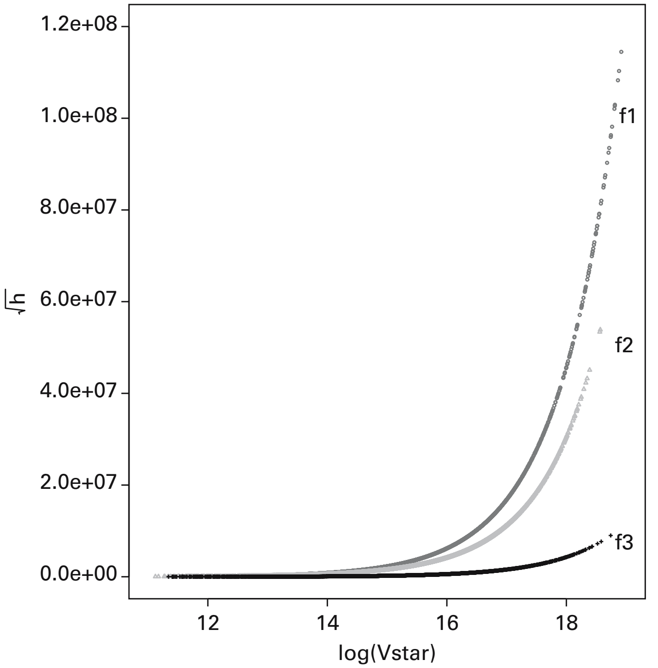

Fig. 5. Three overlaid plots of $\sqrt{h_f}$ against $ln\left(V^*_{pjt}\right)$. In the top curve (dark grey circles), both $\sqrt{h_{f1}}$ and $ln\left(V^*_{pjt}\right)$ are based on Model ($f1$) for $ln\left(V^*_{pjt}\right)$. In the middle curve (light grey triangles), both $\sqrt{h_{f2}}$ and $ln\left(V^*_{pjt}\right)$ are based on Model ($f2$). In the bottom curve (black crosses), both $\sqrt{h_{f3}}$ and $ln\left(V^*_{pjt}\right)$ are based on Model ($f3$). In all curves, $\sqrt{h_f}$ is based on the reduced Model (25) for r_{jt}.

based on the direct variance estimates $\hat{V}_{pjt(r)}$; and

$$\hat{\theta}_{jt(r)} \pm t_{d_q,1-\alpha/2}\left(V^{**}_{pjt(r)}\right)^{1/2} \tag{32}$$

based on the GVF predictors $V^{**}_{pjt(r)}$, where $t_{d,1-\alpha/2}$ is the upper $1-\alpha/2$ quantile of a t distribution on d degrees of freedom. Finally, taking averages over the R replicates, we computed estimates

$$R^{-1}\sum_{r=1}^{R}\left(\hat{\gamma}_{(r)}-\gamma\right) \tag{33}$$

of the biases of the coefficient estimates;

$$\left(n^{-1}R^{-1}\sum_{r=1}^{R}\sum_{t=1}^{12}\sum_{j=1}^{430}V_{pjt}\right)^{-1}\left(n^{-1}R^{-1}\sum_{r=1}^{R}\sum_{t=1}^{12}\sum_{j=1}^{430}\Delta_{pjt(r)}\right) \tag{34}$$

the aggregate relative bias of the predictors $V^{**}_{pjt(r)}$ where $\Delta_{pjt(r)}=V^{**}_{pjt(r)}-V_{pjt}$, and $n=J\times T=430\times 12=5,160$;

$$\left(n^{-1}R^{-1}\sum_{r=1}^{R}\sum_{t=1}^{12}\sum_{j=1}^{430}V^{-1}_{pjt}\Delta_{pjt(r)}\right) \tag{35}$$

Table 8. Simulation results for coefficient estimators and variance predictors under Model $(f1)$

		Coefficient Bias (Standard Deviation)				Var (predictors)		Confidence Interval Properties			
						rel bias aggregated $(*10^{-4})$	*rel bias domain*	*Cov. Rate*		*Mean Width* $(*10^{3})$	
d_q	σ^2_{q*}	$\hat{\gamma}_0$	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\gamma}_3$			$\hat{V}_{pjt}$	V^{**}_{pjt}	$\hat{V}_{pjt}$	V^{**}_{pjt}
4	0.645	0.0010	−0.0001	0.0009	−0.0010	4.068	0.906	0.96	0.98	9.17	10.74
		(0.247)	(0.026)	(0.034)	(0.061)						
6	0.395	0.0009	−0.0000	0.0006	−0.0008	3.906	0.484	0.96	0.98	8.89	8.89
		(0.215)	(0.022)	(0.030)	(0.054)						
8	0.284	0.0009	−0.0000	0.0006	−0.0007	3.883	0.328	0.96	0.97	8.77	8.15
		(0.200)	(0.021)	(0.028)	(0.050)						
16	0.133	0.0008	−0.0000	0.0004	−0.0005	4.002	0.143	0.96	0.96	8.60	7.21
		(0.175)	(0.018)	(0.025)	(0.044)						
30	0.069	0.0007	−0.0000	0.0002	0.0008	4.188	0.072	0.96	0.96	8.54	6.84
		(0.164)	(0.017)	(0.023)	(0.042)						
60	0.034	0.0007	−0.0000	0.0002	−0.0003	4.395	0.035	0.96	0.95	8.50	6.64
		(0.158)	(0.017)	(0.022)	(0.040)						
120	0.017	0.0007	−0.0000	0.0001	−0.0002	4.574	0.017	0.96	0.95	8.48	6.55
		(0.154)	(0.016)	(0.022)	(0.039)						
400	0.005	0.0006	0.0000	0.0000	−0.0001	4.799	0.006	0.96	0.95	8.47	6.48
		(0.152)	(0.016)	(0.021)	(0.038)						

the average domain-specific relative bias of V_{pjt}^{**}; and the coverage rates and mean widths for the confidence intervals (31) and (32).

We repeated these steps for the eight values of $d_q = 4, 6, 8, 16, 30, 60, 120$ and 400. Results are displayed in Table 8.

6.2. *Numerical Results*

The first two columns of Table 8 present the selected values of d_q and the corresponding values of σ_{q*}^2 based on Expression (C.6). Note that the value $d_q = 4$ corresponds approximately to the value of $d*$ for Model $(f1)$ in Table 7; and the value of $d_q = 400$ is slightly less than the value of $d*$ for Model $(f3)$ in Table 7.

The next four columns of Table 8 present the bias terms as given in Expression (33), with the corresponding simulated standard deviations placed in parentheses. Note that the bias terms are all small relative to the coefficient values in Table 3 and relative to the reported standard deviations.

The next two columns report the relative bias values given by Expressions (34) and (35), respectively. Note that the aggregate bias terms (34) are relatively small for all cases; while the relative bias terms (35) are fairly large for $d_q = 4$, and decline to values close to zero as d_q increases. The ninth through twelfth columns report coverage rates and mean widths for nominal 95% confidence intervals (31) and (32), respectively. Note that all coverage rates exceed the nominal value of 0.95.

For $d_q = 4$, the intervals (31) based on $\hat{V}_{pjt}$ have a mean width approximately 17% less than the intervals (32) based on V_{pjt}^{*}. This is not surprising, since in this case d_ϵ is greater than d_q. For $d_q = 6$, the intervals (31) and (32) have approximately the same mean width. As d_q increases in the remainder of Table 8, mean widths of the intervals (32) became progressively smaller relative to the widths of the interval (31). This reflects the increasing efficiency of V_{pjt}^{**} relative to $\hat{V}_{pjt}$ as d_q increases with d_ϵ held equal to 6. We observed similar patterns in comparisons of the quantiles of the widths of the confidence intervals (31) and (32); details are omitted here in the interest of space.

In addition, we produced month-specific forms of the final six columns of Table 8, and explored the numerical results for possible time effects. In results not detailed here, we did not identify any substantial time effects for the relative-bias results related to Expressions (34) and (35), nor for the coverage rates of confidence intervals for θ_{jt} based on Expressions (31) and (32), respectively. As one would expect from the positive coefficient γ_3 in Expression (30), the widths of the intervals (31) and (32) did increase over time, but for a given value of d_q, the relative widths of intervals (31) and (32) remained approximately the same.

7. Discussion

7.1. *Summary of Ideas and Methods*

This article has considered two related approaches to the evaluation of generalized variance functions for the analysis of complex survey data. First, an extension of standard estimating equation methods led to design-based variance estimators for the coefficient estimators of a GVF model. This in turn led to design-based inferences for these coefficients, as illustrated by the CES example in Tables 3 through 5. For many of the coefficients considered in Tables 3 through 5, the numerical values of the misspecification effect ratio (29) were substantially greater than one. Thus, in inference for the CES example, it was important to use the design-based variance estimator from (6) instead of the customary variance estimates obtained directly from standard OLS output. Second, additional conditions on the equation error terms q_{jt} led to approximations for the mean squared error of the GVF-based estimators V^*_{pjt}. A regression model for these MSE terms allowed the comparison of the predictive precision of the GVF V^*_{pjt} with the direct design-based variance estimators $\hat{V}_{pjt}$. Application of this second set of analyses in Tables 6 and 7 and in Figures 1 through 4 allowed the identification of some specific GVFs with smaller MSEs than $\hat{V}_{pjt}$ for our CES data.

7.2. Possible Extensions

In closing, we note several possible extensions of the current work. First, we have focused on modeling of the variance of sampling error alone. In some work with small domain estimation, there is also interest in modeling of the variances of prediction errors, which may include components of both sampling error and model error. Second, one may develop additional diagnostics that are specifically focused on evaluation of the effect of GVF lack of fit on specific statistics, that is, confidence intervals for finite population means or variance-based weights in construction of weighted least squares estimators. Third, in keeping with the comments at the end of Subsection 4.4, one could consider estimators of $E(q_{jt}^2|X_{jt})$ based on restricted maximum likelihood methods from the variance component literature. Fourth, Valliant (1987) explored questions regarding use of ordinary least squares or weighted least squares methods in estimation of the coefficients of a GVF model. It would be useful to extend his approach to the context defined in the current article, especially for estimation of the coefficients of the h_f models like (24) and (25). Fifth, the numerical work in this article used the assumption that the equation errors q_{jt} and estimation errors ϵ_{jt} followed lognormal distributions. One could consider extensions of this work to cases in which q_{jt} and ϵ_{jt} follow chi-square distributions or other distributions in the gamma family. Finally, the simulation-based evaluations in Section 6 used values $q^*_{jt(r)}, \epsilon^*_{jt(r)}$ and $\hat{\theta}_{jt(r)}$ generated from independent normal distributions. As suggested by a referee, one could carry out related simulation work by expanding the available CES data into a fixed finite population, and then drawing multiple stratified samples from that population.

Appendix A

Development of the Variance Estimator $\hat{V}(\hat{\omega})$

Subsection 3.2 developed variance estimators $\hat{V}(\hat{\gamma})$ for the GVF coefficient estimators $\hat{\gamma}$. To develop a similar estimator for the variance of the approximate distribution of $\hat{\omega}$, define r, $\mathbf{Z}$, J and C as in Subsection 4.5. Under regularity conditions, $\hat{\omega}$ follows approximately a multivariate normal distribution with mean ω and variance-covariance matrix $\mathbf{V}(\hat{\omega})$. An estimator of $\mathbf{V}(\hat{\omega})$ is

$$\hat{V}(\hat{\omega}) = \left\{\hat{\mathbf{w}}^{(1)}(\omega*)\right\}^{-1}\hat{V}\{\hat{\mathbf{w}}(\hat{\omega})\}\left[\left\{\hat{\mathbf{w}}^{(1)}(\omega*)\right\}'\right]^{-1},$$

where $\hat{\mathbf{w}}^{(1)}(\omega*) = \frac{\partial \hat{\mathbf{w}}(\omega)}{\partial \omega}\Big|_{\omega=\omega*} = \mathbf{Z}'\mathbf{Z}$. For Model (20),

$$\hat{\omega} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{Y}$$
$$\hat{\mathbf{w}}(\hat{\omega}) = \mathbf{Z}'\mathbf{Y} - \mathbf{Z}'\mathbf{Z}\hat{\omega}$$
$$\hat{V}\{\hat{\mathbf{w}}(\hat{\omega})\} = (J-1)^{-1}J\sum_{j\in\mathcal{D}}\left\{\hat{\mathbf{w}}_{jt}(\hat{\omega}) - \bar{\hat{\mathbf{w}}}(\hat{\omega})\right\}\left\{\hat{\mathbf{w}}_{jt}(\hat{\omega}) - \bar{\hat{\mathbf{w}}}(\hat{\omega})\right\}'$$
$$\text{and} \quad \bar{\hat{\mathbf{w}}}(\hat{\omega}) = J^{-1}\sum_{j\in\mathcal{D}}\hat{\mathbf{w}}_{jt}(\hat{\omega}).$$

Under additional regularity conditions, $d\hat{V}(\hat{\omega})$ follows approximately a Wishart $\left(d, V(\hat{\omega})\right)$ distribution. Standard arguments (e.g., Korn and Graubard 1990) indicate that for a fixed $p\times C$ dimensional matrix $\mathbf{A}$, if we define the quadratic form

$$W = (\mathbf{A}\hat{\omega})'\left[\mathbf{A}\hat{V}(\hat{\omega})\mathbf{A}'\right]^{-1}(\mathbf{A}\hat{\omega}),$$

then $(W/d)\{(d-p+1)/p\}$ has approximately a noncentral F distribution with p and $(d-p+1)$ degrees of freedom and noncentrality parameter $W_0 = (\mathbf{A}\omega)'\left[\mathbf{A}V(\hat{\omega})\mathbf{A}'\right]^{-1}(\mathbf{A}\omega)$.

Appendix B

Ad Hoc "Degrees of Freedom" Measures for Estimation and Prediction Errors Under Variance Function Models

Numerical work in this article uses the assumption that the errors q_{jt} and ϵ_{jt} follow lognormal distributions. However, direct statements about the moments of q_{jt} and ϵ_{jt} may be somewhat difficult to interpret. Consequently, it is useful to provide the following ad hoc "degrees of freedom" measures related to the moments of q_{jt} and ϵ_{jt}.

Let A be a positive random variable with finite positive mean and variance. Then under a standard approach (e.g., Satterthwaite 1941 and Kendall and Stuart 1968, p. 83), the random variable $\{E(A)\}^{-1}dA$ has the same first and second moments as those of a χ^2_d random variable, where we define the "degrees of freedom" term

$$d = \{V(A)\}^{-1}2\{E(A)\}^2. \tag{B.1}$$

Specifically, for the random variables V_{pjt} and $\hat{V}_{pjt}$ defined in Expressions (1) and (2), $\{f(X_{jt},\gamma)\}^{-1}d_{q_{jt}}V_{pjt}$ has the same first and second moments as a $\chi^2_{d_{qjt}}$ random variable, where

$$d_{q_{jt}} = \{V(q_{jt})\}^{-1}\,2\{f(X_{jt},\gamma)\}^2. \tag{B.2}$$

Similarly, conditional on V_{pjt}, $\left(V_{pjt}\right)^{-1}d_{\epsilon_{jt}}\hat{V}_{pjt}$ has the same first and second moments as a $\chi^2_{d_{\epsilon jt}}$ random variable, where

$$d_{\epsilon_{jt}} = \left\{V(\epsilon_{jt}|X_{jt})\right\}^{-1}2\left(V_{pjt}\right)^2. \tag{B.3}$$

Appendix C

Predictors V^*_{jt} of the Design Variance V_{jt} Under Lognormal Models for Equation Error and Estimation Error

Under the model defined by Expressions (2) and (3), define $\epsilon^*_{jt} = \ln(\hat{V}_{jt}) - \ln(V_{jt})$ and assume that

$$q^*_{jt} \sim N(0, \sigma^2_{q*}) \tag{C.1}$$

and

$$\epsilon^*_{jt} \sim N\left(0, \sigma^2_{\epsilon*}\right). \tag{C.2}$$

Then routine calculations show that

$$E(V_{jt}|X_{jt}) = \exp\left(X_{jt}\gamma + 2^{-1}\sigma^2_{q*}\right). \tag{C.3}$$

Let $\hat{\sigma}^2_e$ be the customary mean squared error term from the regression of $\ln(\hat{V}_{pjt})$ on X_{jt} under the model defined by Expressions (2) and (3). Under additional regularity conditions, $\hat{\sigma}^2_e$ is a consistent estimator for the sum $\sigma^2_{q*} + \sigma^2_{\epsilon*}$.
If one does not have satisfactory information about the estimation-error variance term $\sigma^2_{\epsilon*}$, then one may consider use of the predictor

$$V^*_{pjt} = \exp\left(X_{jt}\hat{\gamma} + 2^{-1}\hat{\sigma}^2_e\right). \tag{C.4}$$

Expression (C.4) provides a predictor of the true variance V_{pjt} that is conservative in the sense that $E(V^*_{pjt})$ will tend to be larger than $E(V_{pjt})$. To develop a less conservative predictor of V_{pjt}, suppose that under Expression (B.3), the term $d_{\epsilon_{jt}}$ is known (up to a reasonable level of approximation) and equals the constant d_ϵ for all j and t. Additional calculations for the moments of the lognormal distribution then show that

$$\sigma^2_{\epsilon*} = \Psi\left(1, 2^{-1}d_\epsilon\right) \tag{C.5}$$

where $\Psi(a, b)$ is the Ψ function with arguments a and b (Abramowitz and Stegun 1972, p 258). Similarly, under the lognormal model (C.1), define $d_q = \{V(q_{jt})\}^{-1}2\{E(V_{jt}\}^2$, then

$$\sigma^2_{q*} = \Psi\left(1, 2^{-1}d_q\right) \tag{C.6}$$

In addition, define the function $c(d) = \Psi\left(1, 2^{-1}d\right)$. Expression (C.5) then leads to the estimators

$$\hat{\sigma}^2_{q*} = \hat{\sigma}^2_e - \sigma^2_{\epsilon*} \tag{C.7}$$

and

$$\hat{d}_q = c^{-1}\left(\hat{\sigma}^2_{q*}\right) \tag{C.8}$$

Finally, based on substitution of $\hat{\gamma}$ for γ and $\hat{\sigma}^2_{q*}$ for σ^2_{q*} in Expression (C.3), define the predictor

$$V^{**}_{pjt} = \exp\left(X_{jt}\hat{\gamma} + 2^{-1}\hat{\sigma}^2_{q*}\right). \tag{C.9}$$

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