INTERVAL ANALYSIS OF NAVIGATIONAL PROBLEMS

ANALIZA PRZEDZIAŁOWA ZADAŃ NAWIGACYJNYCH

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Abstract: Algorithms of solutions to navigational problems usually comprise elements for numerical calculations. Apart from random errors, numerical errors of varying nature can be found in them. These errors are due to the level of precision of input data, the approximability of computing methods and errors generated by the computing very process itself in a computer. The latter category includes numerical precision (floating point of a numerical notation) and rounding off of numbers. These errors are analyzed as absolute and relative errors, rounding off errors and are regarded as random errors with a triangular distribution. Over the last decade or more the interval analysis of rounding off errors has been dynamically developing. Despite increased computing effort (costs), in some cases such analysis is necessary, particularly when we want to get a result of specific precision. It can be extended as an analysis of random errors with uniform distribution.

Keywords: absolute error, relative error, rounding – off, interval analysis

Streszczenie: Algorytmy rozwiązywania zagadnień nawigacyjnych zazwyczaj zawierają elementy o obliczeń numerycznych. Występują więc w nich, oprócz błędów losowych, również błędy numeryczne o różnym charakterze. Błędy te wywołane są poziomem precyzji danych wejściowych, przybliżonością metod obliczeniowych oraz błędami generowanymi samym procesem obliczeniowym w komputerze. Do tych ostatnich należą precyzja numeryczna (reprezentacja zmiennopozycyjna) oraz zaokrąglenia. Błędy te analizujemy jako błędy bezwzględne, błędy względne, błędy zaokrągleń oraz traktuje się je jako błędy losowe o rozkładzie trójkątnym. W ostatnich kilkunastu latach dynamicznie rozwija się analiza przedziałowa błędów zaokrągleń. Pomimo zwiększonego nakładu obliczeń (kosztów) jest ona w niektórych przypadkach niezbędna, szczególnie gdy chcemy otrzymać wynik o odpowiedniej precyzji. Można ją również rozszerzyć na analizę błędów losowych o rozkładzie jednostajnym.

Słowa kluczowe: błąd bezwzględny, błąd względny, zaokrąglenia liczb, analiza przedziałowa

1. Introduction

One of the basic problems in navigation is the speed of acquiring information on position coordinates, ship movement vector and its accuracy. An essential role in solving the problem belongs to navigational computing, that in fact is part of numerical analysis.

When attempting to solve navigational problems, we adopt a certain mathematical model of a given phenomenon. The choice of a specific mathematical model depends on the extent to which the considered process is known, on required accuracy and computing capacity. A correctly formulated problem and properly selected numerical method assure practically high accuracy of final results.

Mathematical models of navigational processes may be deterministic or probabilistic. In probabilistic models we can also separate a deterministic component, a trend of the navigational parameter involved. That is why navigational calculations can be restricted to considerations of deterministic models, while the estimation of navigational measurements and parameters can be regarded as a separate issue.

As mentioned before, we mainly use numerical methods in navigational computing for practical reasons, as these methods have its specifics suited to the computing method used. Further in the article we will present possible applications of interval analysis in numerical navigational problems. These, due to measurements used in them, are characterized by the fact that we know initial values with a finite accuracy (we can define an interval on which these values occur).

2. Analysis of navigational problems

Attempting to solve and algorithmize a navigational problem, we analyze the following partial problems:

- existence of a solution,
- sensitivity of the problem,
- convergence of solution,
- speed of solution,
- stability of solution,
- computing complexity,
- solution accuracy.

Bearing the above in mind, while developing an algorithm of navigational problem solution we should follow these fundamental criteria:

- simplicity,
- minimization of operations,

- maximization of results accuracy.

In general, we can state that computing needs in navigation refer to three main groups of problems, numerically related to each other. These are:

- linear algebra (voyage planning, estimation),
- non-linear equations and equation systems (calculation of coordinates, optimization),
- differential equations (various versions of dead reckoning, control problems).

Numerical solutions of navigational problems involve three types of error [2], [3], [7], [8]:

- error of mathematical model of a problem,
- error of input data (e.g. measurements),
- computing error.

The first mentioned error occurs when a mathematical model is only an approximation of the navigational process concerned, usually a physical phenomenon, which case occurs most of the time. The model error value depends on how well a given phenomenon is known and what accuracy of results is required in relation to the adopted mathematical model. Unfortunately, there are cases where the error cannot be estimated in advance, so it should be considered while analyzing a problem solution.

Another kind of error appears when data are physical quantities obtained from measurements or when we assume a finite representation of an infinite development of a real number.

The latter error has three main origins. Blunders make up one group. Another is simplification of the problem being solved, that is solving it in its approximated form. The direct reason is usually replacement of infinite operations (e.g. adding of series) or operations on infinitely small numbers (e.g. differentiation) by approximate finite operations. This type of errors is known as the *method error* (selected for an approximate solution) or *truncation error*, as it is often an effect of "truncation" of an infinite process to a finite form. Solving a specific problem, we should try to estimate the value of this error.

Finally, the third herein mentioned major source of errors is the fact that arithmetic operations almost never can be completed with utmost accuracy. It is well known that real numbers usually have an infinite decimal expansion and during computations they have to be rounded or truncated at a certain position of the expansion. If input data may have a finite decimal expansion, the result of their division may have an infinite expansion. The error we introduce by rounding of numbers is simply called a *rounding error*. Like measurement errors, the rounding error is to some degree random, which, alas, makes it difficult to estimate.

Analyzing the algorithm of navigational problem solution, we should take into account the following facts [1]:

- numerical calculations are made on a computer with limited precision, i.e. by using numbers with a finite digital representation;
- in each numerical operation excess digits are cut off, so the number value is burdened with a truncation error, that in an extreme case may amount to almost one least significant unit;
- result of a given operation – if it is not the final result – is an argument for further operations;
- when the number of operations increases (decreases), generally the result error also increases (decreases), as it is produced as a consequence of gradual transfer and cumulation of rounding and truncation errors on the way between input data, through indirect results, to final results. This happens due to the propagation and cumulation of those errors.

3. Interval analysis

Interval analysis was developed by Ramon E. Moore and published in 1966 [5]. It offered a solution to the problem of precise calculations where rounding errors occurred in the computing process, as well as the problem of finite digital representation of input data. Since Moore's first publication a lot of studies on interval analysis in engineering and scientific applications have come out ([4], [6] and many others). There is also some skepticism towards the method [3], connected with the fact that its use calls for increased computing effort; in the simplest case computing is twofold larger (operations on two numbers of the interval ends, instead of one). However, as it often happens in life, this theory has more applications than it was originally anticipated. In many engineering computations, also in navigation, input data are known not as accurate values, but are found in an interval. Operations on intervals have been so defined that the obtained final interval certainly contains a true result. Certain contraction mappings are used to enable getting minimum intervals containing the solution. The concept was adapted from functional analysis. These mappings enable getting intervals of minimum width comprising a true result.

Presented below are basic concepts of interval analysis, needed to understand the essence of its application in engineering computing. For a real variable x we assume that it belongs to a certain minimum interval, to our knowledge, that will be denoted as $X = [\underline{X}, \bar{X}]$, which means that $x \in [\underline{X}, \bar{X}]$, where $\underline{X}$ denotes the minimum value in the interval (its left-hand end), and $\bar{X}$ – maximum value (right-hand end). It should be noted that operations on intervals, that are specific sets, differ from operations on real numbers, even though they refer to real numbers.

We define adding of intervals as: $X + Y = [\underline{X} + \underline{Y}, \bar{X} + \bar{Y}]$.

Subtracting of intervals will be written as: $X - Y = [\underline{X} - \bar{Y}, \bar{X} - \underline{Y}]$.

Multiplying of intervals has this notation: $X \cdot Y = [\min S, \max S]$,

where $S = [\underline{X} \cdot \underline{Y}, \underline{X} \cdot \bar{Y}, \bar{X} \cdot \underline{Y}, \bar{X} \cdot \bar{Y}]$.

Dividing of intervals is defined as: $X/Y = X \cdot (1/Y)$, gdzie $1/Y = [1/\bar{Y}, 1/\underline{Y}]$.

Extraction of roots of intervals: $\sqrt{X} = [\sqrt{\underline{X}}, \sqrt{\bar{X}}]$ dla $\underline{X} \geq 0$.

A square function is written in this form:

$$f(X) = \begin{cases} [\underline{X}^2, \bar{X}^2]; & 0 \leq \underline{X} \leq \bar{X}, \\ [\bar{X}^2, \underline{X}^2]; & \underline{X} \leq \bar{X} \leq 0, \\ [0, \max\{\underline{X}^2, \bar{X}^2\}]; & \underline{X} \leq 0 \leq \bar{X}. \end{cases}$$

Besides, some new concepts were introduced: midpoint $m(X) = \frac{1}{2}(\underline{X} + \bar{X})$, interval width $w(X) = \bar{X} - \underline{X}$, absolute value $|X| = \max\{|\underline{X}|, |\bar{X}|\}$, function on the intervals and the theory was extended to include linear spaces. This enabled constructing coherent concepts for use in linear algebra and numerical computing (e.g. by Newton's method or least squares method).

4. Examples

Here are two examples illustrating the use of interval analysis in navigation. In the first example we are estimating an interval comprising the value of a result dependent on two other quantities, of which only approximate values are known. The other example shows an interval analysis application for solving linear equations.

a) Calculation of a distance

Let us examine a case of distance computed between two points, whose coordinates are approximately determined. Let us assume the following data: point P_1 with the coordinates $x_1 \in [0, 10]$, $y_1 \in [0, 10]$ and point P_2 with the coordinates $x_2 \in [1000, 1010]$, $y_2 \in [1000, 1010]$. The distance between the two points on a plane is expressed by this formula:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

The formula includes these operations on intervals: adding, subtracting, extraction of a root and involution. Thus we obtain

$$\begin{aligned} X_2 - X_1 &= [1000, 1010] - [0, 10] = [990, 1010], \\ Y_2 - Y_1 &= [1000, 1010] - [0, 10] = [990, 1010], \end{aligned}$$

$$\begin{aligned} (X_2 - X_1)^2 &= [990^2, 1010^2], \\ (Y_2 - Y_1)^2 &= [990^2, 1010^2], \end{aligned}$$

$$\begin{aligned} D &= \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2} = \sqrt{[990^2, 1010^2] + [990^2, 1010^2]} = \\ &= \sqrt{2 \cdot [990^2, 1010^2]} = [990\sqrt{2}, 1010\sqrt{2}] = [1400,07, 1428,36], \end{aligned}$$

that is $d \in [1400,07, 1428,36]$ (for midpoint coordinates we would get $d = 1421,28$).

b) Newton's method

Newton's method of solving non-linear equations will be illustrated by an example of computing position coordinates from a bearing and radar range. Solving the problem on a plane, let us assume the origin of the coordinate system is at the position of a navigational mark, and position lines have these forms:

- position line from a measurement of radar range as a circle described by the equation: $x^2 + y^2 = 4$.
- position line from a radar bearing as a straight line: $y = 2x + 1$.

Consequently, we obtain a non-linear equation:

$$f(x) = 5x^2 + 4x - 3,$$

that will be solved in the interval $X^{(0)} = [0, 1]$.

The interval extension of the function $f'(x) = 10x + 4$ is determined by the relationship $F'(X) = 10X + 4$.

Newton's method in the interval version has this form:

$$N(X) = m(X) - \frac{f(m(X))}{2X}$$

and

$$X^{(k+1)} = X^{(k)} \cap N(X) = X^{(k)} \cap \left\{ m(X^{(k)}) - \frac{[m(X^{(k)})]^2 - 5}{2X^{(k)}} \right\}.$$

Taking into account that $X^{(0)} = [0, 1]$, we will have

$$X^{(1)} = [0.4375000000, 0.4821428571], \quad w(X) = 0.0446428571,$$

$$X^{(2)} = [0.4712915017, 0.4718406593], \quad w(X) = 0.0005491577,$$

$$\begin{aligned} X^{(3)} &= [0.4717797525, 0.4717797908], & w(X) &= 0.0000000384, \\ X^{(4)} &= [0.471779788708135, 0.471779788708135], & w(X) &= 0. \end{aligned}$$

The fourth step has led to the solution with an accuracy higher than 10^{-16} . Putting the computing result into the straight line equation describing the measured bearing, we will get the other coordinate: $y = 01.943559577$.

5. Summary

Interval analysis is one few theoretical approaches that constructively gives a solution to a problem of limited precision of numerical calculations. One might think that troubles connected with this issue refer only to areas of theoretical physics. However, if we look at the transformation of geographical data and position computing in this system, we will clearly see that in navigation and geodesy alike there occur problems resulting from cumulation of rounding errors. Interval analysis, therefore, offers a simple-concept method for an analysis of computations burdened with interval uncertainty, absolute errors or a global error of rounding. In numerical computing the stopping criterion is connected with testing the difference between subsequent results. In spite of its rather common use, there is no theoretical reason for this stopping criterion. In case of alternate series of results stopping criterion is correct, but in slowly convergent results with the same sign it may lead to incorrect estimation. On the other hand, in interval analysis the stopping criterion naturally results from the width of the last interval, because the method is always convergent, and the error is not larger than half the width of the interval.

The existing extension of interval analysis, including vector space, makes it a perfect tool for an analysis of numerical errors and measurement uncertainties, other than random ones, in navigational problems. Besides, a computer tool called INTLAB [6] was developed in the MATLAB environment.

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