OUTPUT FEEDBACK CONTROL OF SWITCHED NONLINEAR SYSTEMS: A GAIN SCHEDULING APPROACH

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Switched controller design for nonlinear continuous and discrete-time systems under an arbitrary switching signal using the gain scheduling approach is addressed in this paper. The obtained controller design procedures for continuous and discrete-time systems are in the bilinear matrix inequality form. The proposed design procedure ensures multi parameter-dependent quadratic stability of the switched gain scheduled plant which is associative to a nonlinear plant model and optimal performance defined by quadratic gain scheduled parameters weighting cost function. Example demonstrates the effectiveness of the proposed approaches.

K e y w o r d s: switched controller, non-linear system, gain-scheduled control, controller design, structured controller, Lyapunov function

1 INTRODUCTION

Switched systems consist of a set of subsystems and a switching signal selecting a subsystem to be active during a time interval. Switched systems have many applications in many fields of real plant control. There are several approaches to model hybrid systems [16]. In [4] a model of a large class of hybrid systems is given that considers discrete event systems and continuous dynamics modeled by differential or difference equation. Such models are used to formulate a general stability analysis and controller synthesis framework for hybrid systems. Results for modeling and stability analysis of hybrid systems have been presented in [7, 16, 17].

In this paper we follow the class of hybrid systems known as switched systems [15]. The reader can find a survey of the present state of hybrid systems in the excellent paper [17] and book [16]. Stability and the controller design procedure are the most important issues in the study of switched systems, specially of nonlinear ones. Stability under arbitrary switching is guaranteed by the existence of a common Lyapunov function for all subsystems [9, 14, 24, 29]. For switched linear systems, finding the common Lyapunov function is relatively easy but for nonlinear systems it is difficult. A survey of switched nonlinear systems can be found in [2, 14, 20, 29]. Due to the switched controller design problems for continuous and discrete time nonlinear systems, in this note we pursue the idea to use – instead of a model of a nonlinear switched plant – a model of the switched gain scheduled plant and to design a switched gain scheduled controller guaranteeing closed loop stability and an optimal value of a given quadratic novel cost function for all operating points of the switched nonlinear system.

A number of papers deal with the design of a switched controller for discrete-time systems [4, 16, 17]. There are only some results in the field of switched (gain scheduled) controller design for continuous-time systems [10, 19, 23, 25] and stabilization of switched continuous time linear systems [3, 6], where in [19] the switching gain scheduled control technique is used to control the flexible ball-screw drive servo with a wide range of operating conditions. The framework of [25] is based on the linear matrix inequalities (LMI) formulation with parameter dependent Lyapunov functions. The results of the proposed synthesis is a gain scheduled controller that guarantees stability even for a nonlinear plant. In [23] a switched controller is proposed based on the inverse optimal theory. The obtained controller is then modified by gain scheduling to achieve the quality of the closed-loop system performance.

Two papers are representatives of the switched controller design for linear continuous time systems [3, 6]. In [6] the authors introduce the notion of the dwell-time T_d (minimal time interval between switching) into the switched controller design procedure. In the stability analysis condition of switched systems the dwell-time is included to the term $e^{A_c T_d}$ (A_c–closed-loop matrix (11)). In such a way the proposed design procedure for stability analysis and switching controller design for the real switching time interval $T > T_d$ becomes conservative. The "dwell-time term" for continuous-time systems very complicated the switched controller design procedure. In [3], sufficient conditions are given for the stability of linear systems with the dwell-time and with polytopic type parameter uncertainty. Lyapunov functions, in a quadratic form for each mode, which are non-increasing at the switching instants are assigned to each mode. During the dwell-time this function varies piecewise linearly

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in time after switching occurs. The proposed method was applied to stabilization via state feedback for both nominal and uncertain cases. Since within the dwell-time the Lyapunov function varies piecewise linearly and the real switching time interval $T > T_d$, the switching controller design procedure becomes rather conservative. The switched controller design procedure for continuous-time systems proposed in this paper does not use the approach of the "dwell-time", therefore there is no such drawback as mentioned in the above references.

In this paper a design procedure is proposed for an LMI switched gain scheduled controller for continuous and discrete-time systems the based on a novel switched plant model for a class of nonlinear systems. The proposed design procedure is based on a special type of Lyapunov function. The procedure can be easily extended to the case of the robust switched gain scheduled controller design. A great many references on the switched controller design for continuous time systems focuss on the case where switching can occur immediately (ideal case), for a large number of switched systems the realistic case is where the rate of change of the switching signal is finite (non-ideal case). The assumption of a non-ideal case of the switching variable will be used in this note which gives other opportunities for the designer. Some results about the stability of a class of uncertain linear varying systems (transform to switched systems) can be found in [22].

Gain scheduling control is appealing to deal with systems subject to parametric variations, which include linear systems with time-varying parameters or nonlinear systems modeled as linear parameter varying systems [21]. In many applications, a gain scheduled controller must accommodate a plant with changing dynamics, where the dynamics is strongly dependent on the operating conditions. Reviews of the gain scheduled plant model can be found in [8, 13, 26]. Because of time varying gain scheduled parameters many researcher tackled the design problem of gain scheduled controllers for linear time-varying parameters system (LPV) using the linear matrix inequalities and the Lyapunov function approach [1, 18, 21, 28, 30]. In such cases, the designed gain-scheduled controller must be able to stabilize and guarantee a reasonable performance for all operating conditions.

The remainder of the manuscript is organized as follows. In Section 2 we present the development of the gain scheduled plant model for a class of nonlinear switched systems and switched gain scheduled control systems considered and some preliminaries. In Section 3 we address the output feedback PI switched gain scheduled controller design procedure for continuous and discrete-time systems. Finally, in Section 4 the proposed design procedure is demonstrated on a simple example.

Our notations are standard, $D \in \mathbb{R}^{m \times n}$ denotes the set of real $m \times n$ matrices. I_m is $m \times m$ identity matrix and 0_m denotes a zero matrix. If the size can be determined from the context, we will omit the subscript. $P > 0$ ($P \geq 0$) is a real symmetric, positive definite (semidefinite) matrix. $\sigma \in S$ indicates the arbitrary switching

algorithm and $\sigma + 1$ is the first next mode to mode σ for switching system. $*$ in matrices denotes the respective transposed term to make the matrix symmetric.

2 PROBLEM STATEMENT AND PRELIMINARIES

2.1 Gain scheduling model

Consider a family of nonlinear systems

$$
\begin{aligned}
\dot{z} &= f_\sigma(z, v, w) \\
\overline{y} &= h(z)
\end{aligned}
\tag{1}
$$

where $z \in \mathbb{R}^n$ is the state, the input $v \in \mathbb{R}^m$, the output $\overline{y} \in \mathbb{R}^l$, exogenous input $w \in \mathbb{R}^k$ which captures parametric dependence of the plant (1) on exogenous input (working point, outputs, and so on). The arbitrary switching algorithm $\sigma \in S = \{1, 2, \ldots, N\}$ is a piecewise constant, right continuous function which specifies at each time the index of the active system [20]. Assume that for every $\sigma \in S$, $f_\sigma(\ )$ is locally Lipschitz. A plant (1) equilibrium point z_e, v_e, w_e is such a point that $f_\sigma(z_e, v_e, w_e) = 0$. The following definition describes a set of equilibrium points which will be parameterized by the scheduling variable $\theta \in \mathbb{R}^p$.

DEFINITION 1 [26]. Functions z_e, v_e, w_e define an equilibrium family for the plant (1) if

$$
f_\sigma(z_e, v_e, w_e) = 0, \ \sigma \in S.
\tag{2}
$$

Associated with this equilibrium family is the measured output family

$$
\overline{y}_e = h(z_e).
\tag{3}
$$

Let us assume that the number of equilibrium points (working points) is equal to p, then corresponding to a specified p equilibrium plant family there is a linearized plant family model for each plant mode $\sigma \in S$ p in the following form

$$
\begin{aligned}
\dot{x} &= \overline{A}_{\sigma i} x + \overline{B}_{\sigma i} u, \\
y &= C x, \ i = 1, 2, \ldots, p, \ \sigma \in S = \{1, 2, \ldots, N\}
\end{aligned}
\tag{4}
$$

where $x = z - z_e$, $u = v - v_e$, $y = \overline{y} - \overline{y}_e$, $(z_e, v_e, \overline{y}_e)$ define the equilibrium family for plant (1). Assume, that for i-th equilibrium point one obtain the sets $x \in X_i$, $u \in U_i$, $y \in Y_i$, $i = 1, 2, \ldots p$. Summarizing above sets we get $x \in X = \bigcup_{i=1}^{p} X_i$, $u \in U = \bigcup_{i=1}^{p} U_i$, $y \in Y = \bigcup_{i=1}^{p} Y_i$.

$$
\overline{A}_{\sigma i} = \frac{\partial f_\sigma}{\partial z}(z_{ei}, v_{ei}, w_{ei}), \quad \overline{B}_{\sigma i} = \frac{\partial f_\sigma}{\partial v}(\xi_{ei}, v_{ei}, w_{ei})
$$

From (4) for each plant mode one obtains the gain scheduled plant model as a solution of the following linear equations

$$\overline{A}_{\sigma i} = A_{\sigma o} + \sum_{j=1}^{p} A_{\sigma j}\theta_j \,, \ \ i = 1, 2, \ldots p, \ \ \sigma = 1, 2, \ldots N \tag{5}$$

where $\theta^{\top} = [\theta_1 \ldots \theta_p] \in \Omega$ scheduling variable is treated as plant parameters which may be time varying and for which we assume that they are known, and lower and upper bounds are available for the parameters values and rate variations. Specifically:

- Each parameter θ_j ranges within the given interval

$$\theta_j \in \langle \underline{\theta}_j, \overline{\theta}_j \rangle \,, \ j = 1, 2, \ldots p, \ \theta \in \Omega \,. \tag{6}$$

- The rate of variation θ_j is well defined at all times and satisfies

$$\dot{\theta}_j \in \langle \underline{\dot{\theta}}_j, \overline{\dot{\theta}}_j \rangle, \ \dot{\theta} \in \Omega_t \,. \tag{7}$$

To obtain the gain scheduled plant model, first $A_{\sigma 0}$ $(B_{\sigma 0})$ is calculated as $A_{\sigma o} = \frac{1}{p} \sum_{i=1}^{p} \overline{A}_{\sigma i}$ and then from (5) one can obtain $A_{\sigma j}$ substituting the lower and upper bounds for θ_j. Equations (4), (5) established a relationship between the nonlinear model (1) and corresponding family of linear ones (5), [13] and [26]. Note that due to (5) for each mode of the plant the obtained set of linear models (4) is retrieved by one gain scheduled plant model in the form

$$\dot{x} = \Big(A_{\sigma 0} + \sum_{j=1}^{p} A_{\sigma j}\theta_j\Big)x + \Big(B_{\sigma 0} + \sum_{j=1}^{p} B_{\sigma j}\theta_j\Big)u \,, \tag{8}$$
$$y = Cx, \ \ \sigma \in S, \ \ \theta \in \Omega \,.$$

For more details how to obtain the gain scheduled plant model, see [8, 13, 26].

2.2 Problem formulation and preliminaries

For each plant mode of the switched gain scheduled LPV plant model the following system is considered

$$\delta x(t) = \Big(A_{\sigma 0} + \sum_{j=1}^{p} A_{\sigma j}\theta_j\Big)x(t)$$
$$+ \Big(B_{\sigma 0} + \sum_{j=1}^{p} B_{\sigma j}\theta_j\Big)u(t) \tag{9}$$
$$y(t) = Cx(t), \ \ \sigma \in S$$

where
1) $\delta x(t) = \dot{x}(t)$ for continuous-time system and
2) $\delta x(t) = x(t+1)$ for discrete-time system.

For each switched plant mode the output feedback gain scheduled control law is considered in the form

$$u(t) = \Big(F_{\sigma 0} + \sum_{j=1}^{p} F_{\sigma j}\theta_j\Big)y(t) \,, \ \sigma \in S, \ \theta \in \Omega \tag{10}$$

where $F_{\sigma k} \in \mathbb{R}^{m \times l}, \ k = 0, 1, 2, \ldots p$ are constant output feedback gain matrices. For each plant mode the output feedback gain matrices can be structured in the decentralized, centralized and so on form. The closed-loop for the switched gain scheduled system is

$$\delta x(t) = A_c(\theta, \alpha)x(t) \tag{11}$$

where

$$A_c(\theta, \alpha) = \sum_{\sigma=1}^{N} A_{\sigma}(\theta)\alpha_{\sigma} \,, \tag{12}$$

$$A_{\sigma}(\theta) = A_{\sigma 0} + \sum_{j=1}^{p} A_{\sigma j}\theta_j$$
$$+ \Big(B_{\sigma 0} + \sum_{j=1}^{p} B_{\sigma j}\theta_j\Big)\Big(F_{\sigma 0} + \sum_{j=1}^{p} F_{\sigma j}\theta_j\Big)C \,,$$

$$\alpha^{\top} = [\alpha_1, \ldots, \alpha_N] \,, \quad \sum_{\sigma=1}^{N} \alpha_{\sigma} = 1 \,, \quad \sum_{\sigma=1}^{N} \dot{\alpha}_{\sigma} = 0 \,,$$
$$\dot{\alpha}_{\sigma} \in \langle \underline{\dot{\alpha}}_{\sigma}, \overline{\dot{\alpha}}_{\sigma} \rangle \in \Omega_d \,, \quad \alpha_{\sigma} \in \Omega_{\alpha}, \quad \alpha_{\sigma} \geq 0$$

and $\alpha_j = 1$ when σ_j is an active plant mode, else $\alpha_j = 0$. Assume $\alpha \in \Omega_{\alpha}, \ \dot{\alpha} \in \Omega_d$.

To assess the system performance, we consider an original weighted scheduled quadratic cost function which allows to obtain the desired performance in the different plant equilibrium points for

- continuous-time system

$$J = \int_{t=0}^{\infty} J(t)\mathrm{d}t \,, \tag{13}$$
$$J(t) = x(t)^{\top}Q(\theta)x(t) + u^{\top}(t)Ru(t) \,,$$

- discrete-time case

$$J_d = \sum_{t=0}^{\infty} J_d(t) \,, \tag{14}$$
$$J_d(t) = x(t)^{\top}Q(\theta)x(t) + u^{\top}(t)Ru(t)$$

where $Q(\theta) = Q_0 + \sum_{j=1}^{p} Q_j\theta_j \geq 0, \ R > 0$.

DEFINITION 2. Consider a stable closed-loop switched system (11) with N modes. If there is a control law $u(t)$ (10) and a positive scalar J^* such that the closed-loop cost function (13) or (14) satisfies $J \leq J^*$ $(J_d \leq J^*)$ for all $\theta \in \Omega, \ \alpha \in \Omega_{\alpha}$ then J^* is said to be a guaranteed cost and $u(t)$ is said to be a guaranteed cost control law for arbitrary switching system (11).

LEMMA 1 [12]. *Control algorithm (10) is the guaranteed cost control law for the switched closed-loop system (11) if and only if there are a Lyapunov function*
$V(x(t),\, \theta(t),\, \alpha(t),\, t)$, *matrices* $Q(\theta)$, R *and gain matrices* $F_{\sigma k}$, $k = 0, 1, 2, \ldots p$, $\sigma \in S$ *such that the following inequality holds*

$$B_e(t) = \delta V\big(x(t),\, \theta(t),\, \alpha(t),\, t\big) + J(t) < 0 \qquad (15)$$

where

- $\delta V\big(x(t), \theta(t), \alpha(t), t\big) = \dfrac{dV(x(t), \theta(t), \alpha(t), t)}{dt}$
  for continuous-time system,
- $\delta V\big(x(t), \theta(t), \alpha(t), t\big) = V\big(x(t+1), \theta(t+1), \alpha(t+1), t+1\big) - V(x(t), \theta(t), \alpha(t), t)$ *for discrete-time system.*

We proceed with the notion of multi-convexity of a scalar quadratic function [5].

LEMMA 2. *Consider a scalar quadratic function of* $\theta \in R^p$

$$f(\theta) = a_0 + \sum_{j=1}^{p} a_j \theta_j + \sum_{j=1}^{p} \sum_{i>j}^{p} a_{ji} \theta_i \theta_j + \sum_{j=1}^{p} a_{jj} \theta_j^2 \quad (16)$$

and assume that if $f(\theta)$ *is multi-convex, that is*

$$\frac{\partial^2 f(\cdot)}{\partial \theta_j^2} = 2a_{jj} \geq 0\,, \quad j = 1, 2, \ldots p\,,$$

then $f(\theta)$ *is negative in the hyper rectangle (6) if and only if it takes negative values at the vertices of (6), that is if and only if* $f(\theta) < 0$ *for all vertices of the set given by (6).*

3 MAIN RESULTS

This section formulates the theoretical approach to the switched gain scheduled controller design with control law (10) which ensures closed-loop system multi parameter-dependent quadratic stability and guaranteed cost for arbitrary switched algorithm $\sigma \in S$ for all gain scheduled parameters $\theta \in \Omega$, $\alpha \in \Omega_\alpha$ and $\dot{\theta} \in \Omega_t$ ($\Delta \theta \in \Omega_{td}$), $\dot{\alpha} \in \Omega_d$.

3.1 Design of switched gain scheduled controller: Continuous-time case

Assume that in Lemma 1 the Lyapunov function is in the form

$$V(x(t), \theta(t), \alpha(t), t) = x(t)^\top P(\theta, \alpha) x(t) \qquad (17)$$

where the novel Lyapunov matrix is

$$P(\theta, \alpha) = P_0 + \sum_{\sigma=1}^{N} \Big(P_{\sigma 0} + \sum_{j=1}^{p} P_{\sigma j} \theta_j \Big) \alpha_\sigma\,.$$

For time derivative of (17) one obtains

$$\dot{V}(\cdot) = \begin{bmatrix} \dot{x}(t)^\top & x(t)^\top \end{bmatrix} \begin{bmatrix} 0 & P(\theta, \alpha) \\ P(\theta, \alpha) & \dot{P}(\theta, \alpha) \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ x(t) \end{bmatrix} \quad (18)$$

where

$$\dot{P}(\theta, \alpha) = \sum_{\sigma=1}^{N} \Big(\sum_{\sigma=1}^{N} P_{\sigma 0} \dot{\alpha}_\sigma + \sum_{j=1}^{p} P_{\sigma j} \dot{\theta}_j + \sum_{j=1}^{p} \sum_{\sigma=1}^{N} P_{\sigma j} \dot{\alpha}_\sigma \theta_j \Big) \alpha_\sigma.$$

Using the equality

$$\big(2G_1 \dot{x} + 2G_2 x\big)^\top \big(\dot{x} - A_c(\theta, \alpha)\big) = 0 \qquad (19)$$

where $G_1, G_2 \in R^{n \times n}$ are auxiliary matrices and summing (18) with (19) (after some manipulations) one obtains

$$\frac{dV(\cdot)}{dt} = \sum_{\sigma=1}^{N} \begin{bmatrix} \dot{x}(t)^\top & x(t)^\top \end{bmatrix} L_\sigma \begin{bmatrix} \dot{x}(t)^\top & x(t)^\top \end{bmatrix}^\top \alpha_\sigma \quad (20)$$

where

$$L_\sigma = \begin{bmatrix} L_{\sigma 11} & L_{\sigma 12} \\ * & L_{\sigma 22} \end{bmatrix}, \qquad (21)$$

$$L_{\sigma 11} = G_1^\top + G_1\,,$$

$$L_{\sigma 12} = -G_1^\top A_\sigma + G_2 + P_0 + P_{\sigma 0} + \sum_{j=1}^{p} P_{\sigma j} \theta_j\,,$$

$$L_{\sigma 22} = -G_2^\top A_\sigma - A_\sigma^\top G_2 + \sum_{\sigma=1}^{N} P_{\sigma 0} \dot{\alpha}_\sigma$$

$$+ \sum_{j=1}^{p} P_{\sigma j} \dot{\theta}_j + \sum_{j=1}^{p} \sum_{\sigma=1}^{N} P_{\sigma j} \dot{\alpha}_\sigma \theta_j$$

Note that if $\dot{V}(\cdot) < 0$ or $L_\sigma < 0$ (20) for all $\sigma \in S$, $\theta \in \Omega$, $\dot{\theta} \in \Omega_t$, $\dot{\alpha} \in \Omega_d$, the closed loop gain scheduled system is parameter-dependent quadratically stable for an arbitrary switching algorithm.

Substituting (10) to (13), after some manipulation one obtains

$$J(t) = x(t)^\top N(t, \theta) x(t) \qquad (22)$$

where

$$N(t, \theta) = N_0 + \sum_{j=1}^{p} N_j \theta_j + \sum_{j=1}^{p} \sum_{k>j}^{p} N_{jk} \theta_j \theta_k + \sum_{k=1}^{p} N_{kk} \theta_k^2,$$

$$N_0 = Q_0 + C^\top F_{\sigma 0}^\top R F_{\sigma 0} C\,,$$

$$N_j = Q_j + C^\top \big(F_{\sigma j}^\top R F_{\sigma 0} + F_{\sigma 0}^\top R F_{\sigma j} \big) C\,,$$

$$N_{jk} = C^\top \big(F_{\sigma j}^\top R F_{\sigma k} + F_{\sigma k}^\top R F_{\sigma j} \big) C\,,$$

$$N_{kk} = C^\top F_{\sigma k}^\top R F_{\sigma k} C\,, \quad \sigma \in S\,.$$

The model of σ plant modes (12) can be rewritten to the form

$$A_\sigma(\theta) = M_0 + \sum_{j=1}^p M_j \theta_j + \sum_{j=1}^p \sum_{k>j}^p M_{jk} \theta_j \theta_k + \sum_{k=1}^p M_{kk} \theta_k^2 \quad (23)$$

where

$$M_0 = A_{\sigma 0} + B_{\sigma 0} F_{\sigma 0} C ,$$
$$M_j = A_{\sigma j} + (B_{\sigma 0} F_{\sigma j} + B_{\sigma j} F_{\sigma 0}) C ,$$
$$M_{jk} = (B_{\sigma j} F_{\sigma k} + B_{\sigma k} F_{\sigma j}) C ,$$
$$M_{kk} = B_{\sigma k} F_{\sigma k} C , \quad \sigma \in S .$$

Substituting (20), (22) and (23) to Lemma 1 and due to Lemma 2 the closed-loop switched gain scheduled system with an arbitrarily switching algorithm is multi parameter-dependent quadratically stable if the following two inequalities hold

$$W_0 + \sum_{j=1}^p W_j \theta_j + \sum_{j=1}^p \sum_{k>j}^p W_{jk} \theta_j \theta_k + \sum_{k=1}^p W_{kk} \theta_k^2 \leq 0 ,$$
$$(24)$$
$$W_{kk} \geq 0 , \quad \sigma \in S ,$$

where

$$W_0 = \begin{bmatrix} W_{011} & W_{012} \\ * & W_{011} \end{bmatrix} ,$$

$$W_j = \begin{bmatrix} 0 & W_{j12} \\ * & W_{j22} \end{bmatrix} ,$$

$$W_{jk} = \begin{bmatrix} 0 & -G_1^\top M_{jk} \\ * & -G_2^\top M_{jk} - M_{jk}^\top G_2 + N_{jk} \end{bmatrix} ,$$

$$W_{kk} = \begin{bmatrix} 0 & -G_1^\top M_{kk} \\ * & -G_2^\top M_{kk} - M_{kk}^\top G_2 + N_{kk} \end{bmatrix} ,$$

$$W_{011} = G_1^\top + G_1 ,$$

$$W_{012} = -G_1^\top M_0 + P_0 + P_{\sigma 0} + G_2 ,$$

$$W_{022} = -G_2^\top M_0 - M_0^\top G_2$$
$$+ \sum_{\sigma=1}^N P_{\sigma 0} \dot\alpha_\sigma + \sum_{j=1}^p P_{\sigma j} \dot\theta_j + N_0 ,$$

$$W_{j12} = -G_1^\top M_j + P_{\sigma j} ,$$

$$W_{j22} = -G_2^\top M_j - M_j^\top G_2 + \sum_{\sigma=1}^N P_{\sigma j} \dot\alpha_\sigma + N_j .$$

The main results of the switched gain scheduled controller design for a continuous-time system are given in the next theorem.

THEOREM 1. *The closed-loop continuous time system* (11) *with a switched gain scheduled controller* (10) *and an arbitrarily switching control algorithm is multi parameter-dependent quadratically stable with guaranteed cost for all* $\sigma \in S$, $\theta \in \Omega$, $\dot\theta \in \Omega_t$, $\alpha \in \Omega_\alpha$ *and* $\dot\alpha \in \Omega_d$ *if the inequalities* (24) *hold.*

P r o o f . The proof of theorem sufficient condition is clear from previous considerations. Here, the proof is repeated only in basic steps. Substituting (20), (22) and (23) to Lemma 1 one obtains inequalities (24), which proves the sufficient conditions of Theorem 1.

Note that the condition "if and only if" for a concrete structure of the Lyapunov matrix in Lemma 1 reduces to "if".

3.2 Design of switched gain scheduled controller: Discrete-time case

In this part of the note we assume that for discrete-time systems the switching can occur immediately (ideal case), therefore with respect to switching variable σ the quadratic stability approach will be used. Because of quadratic stability with respect to σ let us assume that for discrete-time cases the Lyapunov function is in the form

$$V\big(x(t), t, \theta(t), \sigma(t)\big) = x(t)^\top P\big(\theta(t), \sigma(t)\big) x(t) \quad (25)$$

The first difference of (25) is

$$\delta V\big(x(t), t, \theta(t), \sigma(t)\big) = V\big(t+1, \theta(t+1), \sigma+1\big)$$
$$- V\big(t, \theta(t), \sigma\big) = x(t+1)^\top P\big(\theta(t+1), \sigma+1\big) x(t+1)$$
$$- x(t)^\top P\big(\theta(t), \sigma\big) x(t) \quad (26)$$

where

$$P\big(\theta(t), \sigma\big) = P_{\sigma 0} + \sum_{j=1}^p P_{\sigma j} \theta_j$$

and $\sigma + 1 \in S$ is the first next mode to mode $\sigma \in S$ for an arbitrary switching algorithm;

$$P\big(\theta(t+1), \sigma+1\big) = P\big(\theta(t), \sigma+1\big) + \sum_{j=1}^p P_{\sigma+1 j} \Delta\theta_j(t)$$

where $\Delta\theta(t) = \theta(t+1) - \theta(t) \in \langle \Delta\underline{\theta}, \Delta\overline{\theta} \rangle \in \Omega_{td}$. After some manipulation for the first difference of the Lyapunov function (25) one obtains

$$\delta V(\cdot) = \begin{bmatrix} x(t+1)^\top & x(t)^\top \end{bmatrix} L_{\sigma d} \begin{bmatrix} x(t+1)^\top & x(t)^\top \end{bmatrix}^\top \quad (27)$$

where

$$L_{\sigma d} = \begin{bmatrix} L_{\sigma d 11} & 0 \\ 0 & L_{\sigma d 22} \end{bmatrix} ,$$

$$L_{\sigma d 11} = P_{\sigma+1 0} + \sum_{j=1}^p P_{\sigma+1 j} \big(\theta_j(t) + \Delta\theta_j(t)\big),$$

$$L_{\sigma d 22} = P_{\sigma 0} + \sum_{j=1}^p P_{\sigma j} \theta_j(t) .$$

Using the procedure given by equations. (19)–(23) for continuous-time systems the following parameter-dependent quadratic stability conditions with guaranteed cost and arbitrarily switching control algorithm for the design of a discrete-time switched gain scheduled controller are obtained

$$V_0 + \sum_{j=1}^{p} V_j \theta_j + \sum_{j=1}^{p} \sum_{k>j}^{p} V_{jk} \theta_j \theta_k + \sum_{k=1}^{p} V_{kk} \theta_k^2 < 0, \quad (28)$$
$$V_{kk} \geq 0$$

for all $\sigma \in S$ where

$$V_0 = \begin{bmatrix} V_{011} & V_{012} \\ * & V_{022} \end{bmatrix},$$

$$V_j = \begin{bmatrix} V_{j11} & V_{j12} \\ * & V_{j22} \end{bmatrix},$$

$$V_{jk} = \begin{bmatrix} 0 & -G_1^\top M_{jk} \\ * & -G_2^\top M_{jk} - M_{jk}^\top G_2 + N_{jk} \end{bmatrix},$$

$$V_{kk} = \begin{bmatrix} 0 & -G_1^\top M_{kk} \\ * & -G_2^\top M_{kk} - M_{kk}^\top N_G + N_{kk} \end{bmatrix},$$

$$V_{011} = G_1^\top + G_1 + P_{\sigma+1_0} + \sum_{j=1}^{p} P_{\sigma+1_j} \Delta\theta_j(t),$$

$$V_{012} = -G_1^\top M_0 + G_2,$$

$$V_{022} = -G_2^\top M_0 - M_0^\top G_2 + N_0 + P_{\sigma 0},$$

$$V_{j11} = P_{\sigma+1_j},$$

$$V_{j12} = -G_1^\top M_j,$$

$$V_{j22} = -G_2^\top M_j - M_j^\top G_2 + N_j + P_{\sigma j},$$

$$V_{jk12} = -G_1^\top M_{jk},$$

$$V_{jk22} = -G_2^\top M_{jk} - M_{jk}^\top G_2 + N_{jk},$$

$$V_{kk12} = -G_1^\top M_{kk},$$

$$V_{kk22} = -G_2^\top M_{kk} - M_{kk}^\top G_2 + N_{kk}.$$

THEOREM 2. *The closed-loop discrete-time system* (11) *with switched gain scheduled controller* (10) *and an arbitrarily switching control algorithm is parameter-dependent quadratically stable with guaranteed cost for all* $\sigma \in S$, $\theta \in \Omega$, $\Delta\theta \in \Omega_{td}$ *if inequalities* (28) *hold for all vertices defined by* (6) *and* (28).

P r o o f . Proof of the theorem sufficient condition is clear from the previous consideration. Here, the proof is repeated only in basic steps. Substituting (22), (23) and (27) to Lemma 1 one obtains inequalities (28), which proves the sufficient conditions of Theorem 2.

4 NUMERICAL EXAMPLE

In this section the proposed design procedure is applied to a continuous-time system to design a switched gain scheduled controller. The problem is to design a PI

switched gain scheduled controller for a second order nonlinear switched continuous-time system with two modes and an arbitrary switching algorithm to guarantee the multi parameter-dependent quadratic stability and guaranteed value of a given structured cost function.

Consider the switched gain scheduled system (9) with two modes and two gain scheduled parameters. For the first and second plant modes the gain scheduled extended plant model [31] due to PI controller is as

First mode

$$A_{10} = \begin{bmatrix} -.5 & 0.2 & 0 \\ 0.35 & -0.1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} -0.01 & -0.02 & 0 \\ 0.035 & 0.02 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} -0.05 & 0.02 & 0 \\ 0.02 & 0.04 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} B_{10}^\top &= \begin{bmatrix} 2 & 1 & 0 \end{bmatrix}, \\ B_{11}^\top &= \begin{bmatrix} .2 & 0.5 & 0 \end{bmatrix}, \\ B_{12}^\top &= \begin{bmatrix} 0.01 & 0.03 & 0 \end{bmatrix}, \end{aligned}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Second mode

$$A_{10} = \begin{bmatrix} -.7 & 0.5 & 0 \\ 0.6 & -0.3 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 0.01 & -0.005 & 0 \\ 0.02 & -0.05 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$A_{12} = \begin{bmatrix} 0.001 & -0.03 & 0 \\ 0.02 & -0.04 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} B_{10}^\top &= \begin{bmatrix} 3 & 1 & 0 \end{bmatrix}, \\ B_{11}^\top &= \begin{bmatrix} .02 & 0.02 & 0 \end{bmatrix}, \\ B_{12}^\top &= \begin{bmatrix} 0.02 & 0.05 & 0 \end{bmatrix}, \end{aligned}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Parameters of the cost function (13) are $R = rI_r$, $r = 1$, $Q(\theta) = Q_0 + Q_1\theta_1 + Q_2\theta_2$ (in the term $Q(\theta)$ the gain scheduled parameter θ serves to ensure the desired performance in different plant working points) where $Q_i = q_iI$, $i = 0, 1, 2$, $q_0 = 0.1$, $q_1 = 0.02$, $q_2 = 0.02$, for the Lyapunov matrix it holds $0 < P(\theta, \sigma) < roI$, $ro = 1000$, $\theta_i \in \langle -1, 1 \rangle$, $\dot{\theta}_i \in \langle -4, 4 \rangle$, $i = 1, 2$ and the rate of mode changes (18) $\dot{\alpha}_i \in \langle -200, 200 \rangle$. For the above parameters as the solution of (25) the following gain scheduled controllers are obtained:
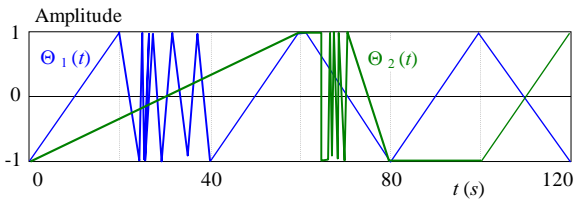First plant mode gain scheduled controller (10)

$$F_{10} = -1.2426 - \frac{0.4353}{s}; \quad F_{11} = 0.657 + \frac{0.1033}{s};$$

$$F_{12} = \left(0.6195 + \frac{0.5741}{s}\right)10^{-6}.$$

Second plant mode gain scheduled controller (10)

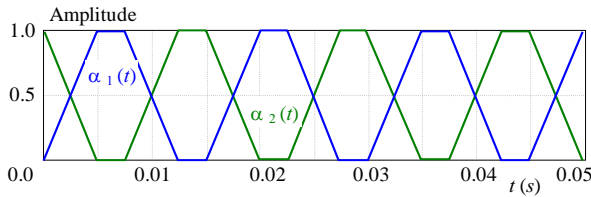$$F_{20} = -0.8843 - \frac{0.2914}{s}; \quad F_{21} = \left(0.4355 - \frac{0.3481}{s}\right)10^{-7};$$

$$F_{22} = \left(0.1460 + \frac{0.2294}{s}\right)10^{-3}. \quad (29)$$
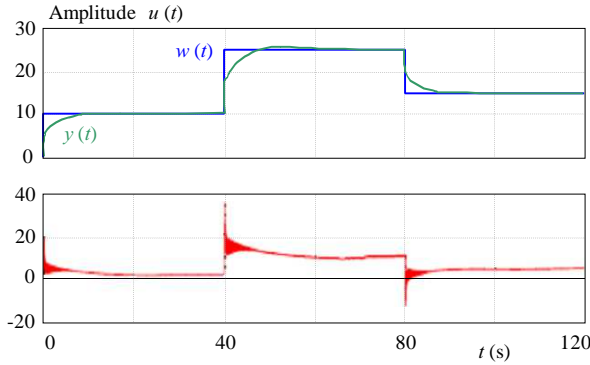
Fig. 1. Simulation results of output and input when $\dot{\theta}(t) \in \langle -4, 4 \rangle$, $\dot{\alpha} \in \langle -200, 200 \rangle$

Fig. 2. Development of the gain scheduled parameters (working point changes) $\dot{\theta}(t) \in \langle -4, 4 \rangle$ during simulation

Fig. 3. Detail of two non-ideal switching parameters development when $\dot{\alpha} \in \langle -200, 200 \rangle$ during simulation

Note that the gain scheduled controller is in the form

$$\text{Controller}_i(s) = F_{i0} + F_{i1}\theta_1 + F_{i2}\theta_2, \quad i = 1, 2.$$

The maximal eigenvalue of the closed-loop switched system for the case of $\theta_1 = \theta_2 = 0$ is -0.2471. The maximum rate of change of value (18) is about $\dot{\alpha}_{max} \in \langle -250, 250 \rangle$. For a larger value of $\dot{\alpha}$ the gain scheduled controller design procedure is not feasible. Dynamic behaviors of the closed-loop switched gain scheduled system are given in Figs. 1, 2 and 3.

R e m a r k 1. For decreasing the calculation loads of the design procedure for (17) one could choose the following inequalities

$$0 < P_c < ro\, I, \quad 0 < P_{\sigma j} < ro_{\sigma j} I$$

then in (24) substitute to W_0, W_j, V_0 the following inequalities

$$P_{\sigma j}\dot{\alpha}_\sigma \leq P_{\sigma j}\overline{\dot{\alpha}}_\sigma, \quad \dot{\theta}_j \leq \overline{\dot{\theta}}_j, \quad \Delta\theta(t) \leq \Delta\overline{\theta}_j.$$

R e m a r k 2. To assess the system performance we consider an original weighted scheduled quadratic cost function which allows to obtain the desired (different or the same) performance in the different plant equilibrium points.

R e m a r k 3. In this example due to (12) for non-ideal switching variables it holds $\alpha_1 + \alpha_2 = 1$, $\dot{\alpha}_1 + \dot{\alpha}_2 = 0$. Time developments of the switching variables are given in Fig. 3.

5 CONCLUSION

A new method for the stability analysis and design of a switched controller for non-linear system using the gain scheduling approach for continuous and discrete-time systems was introduced. In this paper the novel switched plant model for a class of nonlinear system was proposed.

The proposed switched controller design procedure is based on a novel plant model, a special type of the Lyapunov function and the Lyapunov function stability theory. For a continuous-time switched system the gain scheduled controller design procedure ensures multi parameter-dependent quadratic stability and guaranteed cost for an arbitrary switching law and a finite (infinite) rate of switching signal change. The switched controller design procedure for continuous-time systems proposed in this paper does not use the approach of the "dwell-time" which very complicated the switched controller design procedure. In this note, we introduce the assumption of a non-ideal case of the switching variable (rate of switching variable is finite) which gives other opportunities for the designer. The working plant parameters may change very rapidly but the rate of parameters change are bounded ($\overline{\dot{\theta}}_i$ is bounded). For discrete-time switched systems due to the novel plant a new design procedure is obtained which ensures parameter-dependent quadratic stability and guaranteed cost for an arbitrary switching law and for a given finite rate of change of the plant parameters.

To guarantee the desired system performance, in this paper we consider an original weighted scheduled quadratic cost function which allows to obtain the desired performance in different equilibrium points of the plant. The designed design procedure can easily be extended to the case of a robust switched gain scheduled controller using the idea of paper [27]. The numerical example illustrates the effectiveness of the proposed approach.

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