COEFFICIENT, DISTORTION AND GROWTH INEQUALITIES FOR CERTAIN p -VALENT CLOSE-TO-CONVEX FUNCTIONS

I.B. BAPANA AND SNEHA NANDWANA

Abstract

In the present paper, we introduce and investigate a new subclass $\chi_p^t(\gamma)$ of analytic and p -valent close-to-convex functions in the open unit disk U . For functions belonging to the class $\chi_p^t(\gamma)$, some interesting properties including the coefficient estimates, distortion theorems and subordination results are obtained.

Mathematics Subject Classification 2000: 30C45

Additional Key Words and Phrases: Analytic p -valent functions, Starlike functions, Close-to-convex functions, Hadamard product, Subordination.

1. Introduction

Let $\mathcal{A}(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\}), \quad (1.1)$$

which are analytic and p -valent in the unit disc $\mathbb{U} = \{z : z \in C, |z| < 1\}$. Let $\mathcal{S}^*$, $[\mathcal{S}_p^*(\alpha)]$ and $\mathcal{K}_p(\alpha)$ be the usual class of functions which are univalent starlike of order zero, p -valent starlike and p -valent convex of order α where $(0 \leq \alpha < p)$.

For two functions f and g , analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and we write $f(z) \prec g(z)$ ($z \in \mathbb{U}$), if there exists a Schwartz function $w(z)$, analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, such that $f(z) = g(w(z))$, ($z \in \mathbb{U}$). Indeed it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

More recently, Kowalczyk et. al [2] discussed $\mathcal{K}_s(\gamma)$ of analytic functions related to starlike function.

A function $f(z) \in \mathbb{U}$ is said to be in the class $\mathcal{K}_s(\gamma)$, $0 \leq \gamma < 1$, if it satisfies the inequality

$$\operatorname{Re} \left(\frac{-z^2 f'(z)}{g(z)g(-z)} \right) > \gamma$$

where $g(z) \in \mathcal{S}^*(\frac{1}{2})$.

Motivated by this class $\mathcal{K}_s(\gamma)$, we introduce a new class $\chi_p^t(\gamma)$ ($|t| \leq 1, t \neq 0, 0 \leq \gamma < p$).

Definition 1. Let f be a p -valent analytic function in $\mathbb{U}$ defined by (1.1) then f

is said to be in the class $\chi_p^t(\gamma)$ ($|t| \leq 1, t \neq 0, 0 \leq \gamma < p$) if there exist a function $g \in S^*(\frac{p}{2})$, such that

$$\operatorname{Re} \left(\frac{t^p z^{p+1} f'(z)}{g(z)g(tz)} \right) > \gamma \quad z \in \mathbb{U} \quad (1.2)$$

we see that $\chi_1^{-1}(\gamma) = K_s(\gamma)$ and $\chi_1^{-1}(0) = K_s$ and by simple calculation we see that

$$\left| \frac{t^p z^{p+1} f'(z)}{g(z)g(tz)} + p \right| < \left| \frac{t^p z^{p+1} f'(z)}{g(z)g(tz)} - p + 2\gamma \right| \quad z \in U \quad (1.3)$$

Cho and Zhou [2] and Knwalczyk and Le's-Bomba [3], have obtained properties for the function classes $\mathcal{K}_s$ and $\mathcal{K}_s(\gamma)$, respectively. Moreover, some other interesting results on the classes $\mathcal{K}_s$ and $\mathcal{K}_s(\gamma)$ were obtained in [4, 5]. In the present paper we investigate sufficient conditions, coefficient inequalities and distortion theorems for belonging to th the class $\chi_p^t(\gamma)$.

2.Main Results

Theorem 2.1 Let $g \in S^*(\frac{p}{2})$ and be given by

$$g(z) = z^p + \sum_{n=1}^{\infty} b_{n+p} z^{n+p}, \quad (z \in U) \quad (2.1)$$

If we put

$$F(z) = \frac{g(z)g(tz)}{t^p z^p} = z^p + \sum_{n=1}^{\infty} c_{p+n} z^{p+n} \quad (2.2)$$

where for $n = 1, 2, \dots$

$$c_{p+n} = b_{p+n} + b_2 b_{p+n-1} t + b_3 b_{p+n-2} t^2 + \dots + b_{p+n-1} b_2 t^{p+n-2} + b_{p+n} t^{p+n-1} \quad (2.3)$$

Then $F \in S^*$

Proof. Result (2.2) can be found easily. Also $|tz| < |z| < 1$, then from the definition of starlike function we have

$$\operatorname{Re} \left(\frac{zg'(z)}{g(z)} \right) > \frac{p}{2} \quad \text{and} \quad \operatorname{Re} \left(\frac{tzg'(tz)}{g(tz)} \right) > \frac{p}{2}$$

therefore

$$\begin{aligned} \operatorname{Re} \left(\frac{zF'(z)}{F(z)} \right) &= \operatorname{Re} \left(\frac{zg'(z)}{g(z)} \right) + \operatorname{Re} \left(\frac{tzg'(tz)}{g(tz)} \right) - p \\ \operatorname{Re} \left(\frac{zF'(z)}{F(z)} \right) &> 0 \end{aligned}$$

hence $F(z) \in S^*$, the theorem proved.

Theorem 2.2 A function $f(z) \in \chi_p^t(\gamma)$ ($0 \leq \gamma < p$) if and only if there exist $g \in S^*(\frac{p}{2})$ such that

$$\frac{t^p z^{p+1} f'(z)}{g(z)g(tz)} \prec \frac{1 + (p - 2\gamma)}{1 - z} \quad z \in \mathbb{U} \quad (2.4)$$

Proof Let $f(z) \in \chi_p^t(\gamma)$. then, there exist a function $g(z) \in S^*(\frac{p}{2})$ such that

$$\operatorname{Re} \left(\frac{t^p z^{p+1} f'(z)}{g(z)g(tz)} \right) > \gamma \quad z \in \mathbb{U}$$

in terms of subordination we can write that

$$\frac{t^p z^{p+1} f'(z)}{g(z)g(tz)} \prec \frac{1 + (p - 2\gamma)z}{1 - z} \quad (2.4)$$

because the function $h(z) = \frac{1+(p-2\gamma)z}{1-z}$ is analytic in $\mathbb{U}$ and $h(0) = H(0) = 1$, where $H(z) = \frac{t^p z^{p+1} f'(z)}{g(z)g(tz)}$. Conversely we assume that the subordination (2.4) holds, then there exist an analytic function in w in $\mathbb{U}$ such that $w(0) = 0$, $|w(z)| < 1$ and

$$\frac{t^p z^{p+1} f'(z)}{g(z)g(tz)} \prec \frac{1 + (p - 2\gamma)w(z)}{1 - w(z)}$$

therefore, by using the condition $|w(z)| < 1$, we obtain (1.3) which is equivalent to (1.2) so $f(z) \in \chi_p^t(\gamma)$.

Theorem 2.3 For $0 \leq \gamma < p$, if there function $f(z) \in \chi_p^t(\gamma)$, then

$$(p+n)|a_{p+n}| \leq |c_{p+n}| + 2(p-\gamma) \left(1 + \sum_{k=1}^{p+n-1} |c_k| \right) \quad (2.5)$$

Proof. Let

$$p(z) = \frac{1}{p-\gamma} \left(\frac{zf'(z)}{F(z)} - \gamma \right) \quad (2.6)$$

or

$$zf'(z) = [1 + (p-\gamma)(h(z)-1)]F(z) \quad (2.7)$$

also

$$h(z) = 1 + d_1 z + d_2 z^2 + \dots \quad (2.8)$$

where now $\operatorname{Re}(h(z)) > 0$ using (2.2), (2.8), in (2.7)

$$(p+n)a_{p+n} = (p-\gamma)(d_{n-1} + d_{n-2}c_{p+1} + \dots + d_1 c_{p+n})$$

since $\operatorname{Re}(p(z)) > 0$, then $|d_n| \leq 2, n \in \mathbb{N}$. Using this property and principle of mathematical induction we obtain

$$(p+n)|a_{p+n}| \leq |c_{p+n}| + 2(p-\gamma) \left(1 + \sum_{n=1}^{\infty} |c_{p+n}| \right)$$

this completes the proof of the theorem 2.2.

Corollary 2.1 Let $0 \leq \gamma < p$ if the function $f \in \chi_p^t(\gamma)$

$$|a_{p+n}| \leq 1 + (p+n-1)(p-\gamma) \quad (2.9)$$

Proof. From theorem (2.1) $F(z) \in S^*$, thus $|c_{p+n}| \leq p+n$ The assertion (2.9), can now easily derived from theorem(2.3).

Remark. Setting $p = 1, t = -1$ in theorem 2.2 we get the results due to Geo and

Zhou [2]

Theorem 2.4 Let $g \in s^*(\frac{p}{2})$ be a function given by (1.2), and $0 \leq \gamma \leq p$. If an analytic function f in $\mathbb{U}$ defined by (1.1) satisfies the inequality

$$2 \sum (n+p)|a_{n+p}| + (|p-2\gamma|+p) \sum_{n=1}^{\infty} |c_{n+p}| \leq 2(p-\gamma) \quad (2.10)$$

where $n = 1, 2, \dots$ and the coefficient c_{n+p} are given by (2.3), then $f \in \chi_p^t(\gamma)$.

Proof Let

$$\begin{aligned} \Lambda &= \left| z f'(z) - p \frac{g(z)g(tz)}{z^p t^p} \right| - \left| z f'(z) - \frac{(p-2\gamma)g(z)g(tz)}{z^p t^p} \right| \\ &= \left| \sum_{n=1}^{\infty} (n+p) a_{n+p} z^{n+p} - p \sum_{n=1}^{\infty} c_{n+p} z^{n+p} \right| \\ &\quad - \left| 2(p-\gamma)z^p + \sum_{n=1}^{\infty} (n+p) a_{n+p} z^{n+p} + (p-2\gamma) \sum_{n=1}^{\infty} c_{n+p} z^{n+p} \right| \quad (2.11) \\ &\leq \sum_{n=1}^{\infty} |(n+p) a_{n+p}| |z|^{n+p} + p \sum_{n=1}^{\infty} |c_{n+p}| |z|^{n+p} \\ &\quad - \left[2(p-\gamma) |z|^p - \sum_{n=1}^{\infty} (n+p) |a_{n+p}| |z|^{n+p} - \right. \\ &\quad \left. (|p-2\gamma|+p) \sum_{n=1}^{\infty} |c_{n+p}| |z|^{n+p} \right] \\ &\leq \left[-2(p-\gamma) + \sum_{n=1}^{\infty} (n+p) |a_{n+p}| + (|p-2\gamma|+p) \sum_{n=1}^{\infty} |c_{n+p}| \right] |z|^p \end{aligned}$$

therefore $\Lambda < 0$ and we have

$$\left| z f'(z) - p \frac{g(z)g(tz)}{z^p t^p} \right| < \left| z f'(z) - \frac{(p-2\gamma)g(z)g(tz)}{z^p t^p} \right|$$

which is equivalent to (1.3) and also to the inequality (1.2). Thus and it completes the proof.

Theorem 2.5 If $f \in \chi_p^t(\gamma)$ then we have

$$\frac{1 - (p-2\gamma)r}{(1+r)^3} \leq |f'(z)| \leq \frac{1 + (p-2\gamma)r}{(1+r)^3} \quad (|z| = r, 0 \leq r < 1) \quad (2.12)$$

and

$$\int_0^r \frac{1 - (p-2\gamma)\tau}{(1+\tau)^3} d\tau \leq |f(z)| \leq \int_0^r \frac{1 + (p-2\gamma)\tau}{(1+\tau)^3} d\tau \quad (|z| = r, 0 \leq r < 1) \quad (2.13)$$

These results are sharp.

Proof. Suppose that $f \in \chi_p^t(\gamma)$. From the definition of subordination between analytic functions, we deduce that

$$\begin{aligned} \frac{1 - (p - 2\gamma)r}{(1 + r)} &\leq \frac{1 - (p - 2\gamma)|w(z)|}{(1 + |w(z)|)} \leq \left| \frac{t^p z^{p+1} f'(z)}{g(z)g(tz)} \right| = \left| \frac{z f'(z)}{F(z)} \right| \leq \\ &\frac{1 + (p - 2\gamma)|w(z)|}{(1 + |w(z)|)} \leq \frac{1 + (p - 2\gamma)r}{(1 + r)} \end{aligned} \quad (2.14)$$

where w is Schwarz function with $w(0) = 0$ and $|w(z)| < 1, z \in U$. Since

$$\left[F(z) = \frac{g(z)g(tz)}{t^p z^p} \quad (|z| = r, 0 \leq r < 1) \right]$$

is an starlike function, it is well known that [1]

$$\frac{r}{(1 + r)^2} \leq |F(z)| \leq \frac{r}{(1 - r)^2} \quad (|z| = r < 1). \quad (2.15)$$

Now it follows from (2.14) and (2.15), that

$$\left[\frac{1 - (p - 2\gamma)r}{(1 + r)^3} \leq |f'(z)| \leq \frac{1 + (p - 2\gamma)r}{(1 + r)^3} \quad (|z| = r, 0 \leq r < 1) \right]$$

Let $z = re^{i\theta}$ ($0 \leq r < 1$). If L denotes that closed line segment in the complex ζ plane from $\zeta = 0$ to $\zeta = z$, then we have

$$f(z) = \int_L f'(\zeta) d\zeta = \int_0^r f'(\tau e^{i\theta}) d\tau \quad (|z| = r, 0 \leq r < 1)$$

Thus by using upper estimate in (2.12), we have

$$|f(z)| = \left| \int_0^z f'(\zeta) d\zeta \right| \leq \int_0^r f'(\tau e^{i\theta}) d\tau \leq \frac{1 + (p - 2\gamma)\tau}{(1 + \tau)^3} d\tau \quad (|z| = r, 0 \leq r < 1)$$

which yields the right hand side of the inequality in (2.13). In order to prove the lower bound in (2.13), it is sufficient to show that it holds true for z_0 nearest to zero, where $|z_0| = r$ ($0 < r < 1$). Moreover, we have

$$|f(z)| \geq |f(z_0)| \quad (|z| = r, 0 \leq r < 1)$$

Since $f(z)$ is close-to-convex function in the open unit disk U , it is univalent in U . We deduce that the original image of the closed line segment L_0 in the complex ζ plane from $\zeta = 0$ to $\zeta = z_0$ is a piece of arc Γ in the disk U_r , given by

$$U_r = \{z : z \in C \text{ and } |z| = r, 0 \leq r < 1\}$$

Since, in accordance with (2.12), we have

$$|f(z)| = \int_{f(\Gamma)} |d(w)| = \int_{\Gamma} |f'(z)| |dz| \geq \int_0^r \frac{1 + (p - 2\gamma)\tau}{(1 + \tau)^3} d\tau \quad (|z| = r, 0 \leq r < 1)$$

This completes the proof of Theorem 2.5.

In order to give the inclusion relationship for the class $\chi_p^t(\gamma)$, we shall require the following lemma.

lemma 1. let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$. Then

$$\frac{1 + A_1(z)}{1 + B_1(z)} \prec \frac{1 + A_2(z)}{1 + B_2(z)}$$

Theorem 2.6 let $0 \leq \gamma_1 \leq \gamma_2 < p$.

then we have $\chi_p^t(\gamma_1) \subset \chi_p^t(\gamma_2)$

Proof Suppose that $f \in \chi_p^t(\gamma_1)$, by theorem 1 we have

$$\frac{zf'(z)}{g(z)} \prec \frac{1-(p-2\gamma_1)z}{(1-z)}$$

since $0 \leq \gamma_1 \leq \gamma_2$, then we have

$$-1 \leq p - 2\gamma_2 \leq p - 2\gamma_1 < 1$$

thus by lemma 1 , we have

$$\frac{zf'(z)}{g(z)} \prec \frac{1-(p-2\gamma_2)z}{(1-z)}$$

that is $f \in \chi_p^t(\gamma_2)$, This means $\chi_p^t(\gamma_1) \subset \chi_p^t(\gamma_2)$. Hence the proof is complete.

REFERENCES

- I. Graham and G. Kohr, Geometric Function Theory in One and Higher Dimensions, Marcel Dekker, Inc. (2003).
- C. Geo, S. Zhou, On a class of analytic functions related to the class to the starlike functions, Kyungpook Math. J. 45(2005), 123-130.
- J. Kowalczyk and E. Le ´s-Bomba, On a subclass of close-to-convex functions, Appl. Math. Letters 23(2010), 1147-1151.
- Qing-Hua Xu, H. M. Srivastava and Zhou Li, A certain subclass of analytic and close-to-convex functions, Appl. Math. Letters 24(2011), 396-401.
- Zhi-Gang Wang and Da-Zhao Chen, On a certain subclass of close-to-convex functions, Hacet. J. Math. Stat. 38(2)(2009), 95-101.

I.B. Bapana,

Department of Mathematics, Slovak Technical University,

Department of Mathematics, Govt. Postgraduate College,

Bhilwara-311001, Rajasthan,

India

E-mail: bapnain@yahoo.com

Sneha Handwana,

Department of Mathematics, Slovak Technical University,

Department of Mathematics, Govt. Postgraduate College,

Bhilwara-311001, Rajasthan,

India

E-mail: bapnain@yahoo.com

Received September 2013