ON A NEW THEOREM INVOLVING GENERALIZED MELLIN-BARNES TYPE OF CONTOUR INTEGRAL AND SRIVASTAVA POLYNOMIALS

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Abstract

Motivated by recent work of Agarwal [1], the author is establish the new theorem associated with the $\bar{H}$ -function (Generalized Mellin-Barnes type of Contour Integral), which was introduced and study in a series of papers by Inayat -Hussain ([12], [13]). Theorem involves a product of the $\bar{H}$ -function, Generalized hypergeometric functions and Srivastava polynomials. The convergence and existence condition, basic properties of $\bar{H}$ -function were given by Buschman and Srivastava ([7]). Next, we obtain certain new integrals by the application of our theorem. These results, besides being of very general character have been put in a compact form avoiding the occurrence of infinite series and thus making them useful in applications. Our findings provide interesting unifications and extensions of a number of new results.

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Additional Key Words and Phrases: $\bar{H}$ -function, Srivastava Polynomial, expansion formulae.

1. INTRODUCTION

A lot of research work has recently come up on the study and development of a function that is more general than the Fox H-function, popularly known as $\bar{H}$ -function. It was introduced by Inayat-Hussain [12, 14] and now stands on a fairly firm footing through the research contributions of various authors [1-7, 10-14]. The $\bar{H}$ -function will be defined and represented as follows [7]:

$$\begin{aligned} \bar{H}_{P,Q}^{M,N}[z] &\equiv \bar{H}_{P,Q}^{M,N} \left[z \mid \begin{matrix} (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q} \end{matrix} \right] \\ &= \frac{1}{2\pi i} \int_L z^\xi \bar{\phi}(\xi) d\xi. \quad (z \neq 0) \end{aligned} \quad (1.1)$$

$$\bar{\phi}(\xi) = \frac{\prod_{j=1}^M \Gamma(b_j - \beta_j \xi) \prod_{j=1}^N \left\{ \Gamma(1 - a_j + \alpha_j \xi) \right\}^{A_j}}{\prod_{j=M+1}^Q \left\{ \Gamma(1 - b_j + \beta_j \xi) \right\}^{B_j} \prod_{j=N+1}^P \Gamma(a_j - \alpha_j \xi)} \quad (1.2)$$

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Buschman and Srivastava [7] has proved that the integral on the right hand side of (1.1) is absolutely convergent when

$$\Omega > 0 \quad \text{and} \quad \left| \arg z \right| < \frac{1}{2} \pi \Omega, \quad \text{where}$$

$$\Omega = \sum_{j=1}^M \beta_j + \sum_{j=1}^N a_j \alpha_j - \sum_{j=M+1}^Q b_j \beta_j - \sum_{j=N+1}^P \alpha_j > 0 \quad (1.3)$$

here, and throughout the paper $a_j (j=1, \dots, P)$ and $b_j (j=1, \dots, Q)$ are complex parameters, $\alpha_j \geq 0 (j=1, \dots, P)$, $\beta_j \geq 0 (j=1, \dots, Q)$ (not all zero simultaneously) and the exponents $A_j (j=1, \dots, N)$ and $B_j (j=M+1, \dots, Q)$ can take on non-negative values. For further details of $\overline{H}$ -function one can refer the original paper of Buschman and Srivastava [7].

The Srivastava polynomials defined and represented as follows [16, p.1, Eq. (1)]:

$$S_{n_1, \dots, n_s}^{m_1, \dots, m_s} [x_1, \dots, x_s] = \sum_{k_1}^{[n_1/m_1]} \dots \sum_{k_s}^{[n_s/m_s]} \frac{(-n_1)_{m_1 k_1}}{k_1!} \dots \frac{(-n_s)_{m_s k_s}}{k_s!} A[n_1 k_1, \dots, n_s k_s] x_1^{k_1} \dots x_s^{k_s}, \quad (1.4)$$

where $n_i = 0, 1, 2, \dots, \forall i = (1, \dots, s)$; $m_1, \dots, m_s$ are arbitrary positive integers and the coefficients $A[n_1 k_1, \dots, n_s k_s]$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A[n_1 k_1, \dots, n_s k_s]$, $S_{n_1, \dots, n_s}^{m_1, \dots, m_s} [x_1, \dots, x_s]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Lagurre polynomials, the Bessel's polynomials and several others [17, pp. 158-161].

2. MAIN RESULT

Theorem 2.1: Let $\alpha, \beta, \gamma, \lambda, u_i, \nu \in \mathbb{C}$ where $(i = 1, \dots, s)$ and if

$$(1-x)^{\alpha+\beta-\gamma-\frac{1}{2}} {}_2F_1[2\alpha, 2\beta; 2\gamma; x] = \sum_{r=0}^{\infty} \beta_r x^r \quad (2.1)$$

then there hold the formula

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$$\begin{aligned}
 & \int_0^1 x^\lambda {}_2F_1[\alpha, \beta; \gamma; x] {}_2F_1\left[\gamma - \alpha + \frac{1}{2}, \gamma - \beta + \frac{1}{2}; \gamma + 1; x\right] S_{n_1, \dots, n_s}^{m_1, \dots, m_s}[cx^{u_1}, \dots, cx^{u_s}] \overline{H}_{P, Q}^{M, N}[yx^v] dx \\
 &= \sum_{k_1}^{[n_1/m_1]} \dots \sum_{k_s}^{[n_s/m_s]} \sum_{r=0}^{\infty} \frac{(-n_1)_{m_1} k_1}{k_1!} \dots \frac{(-n_s)_{m_s} k_s}{k_s!} . A[n_1 k_1, \dots, n_s k_s] \frac{\left(\gamma + \frac{1}{2}\right)_r}{(\gamma + 1)_r} \beta_r(c^{k_1} \dots c^{k_s}) \\
 & \overline{H}_{P+1, Q+1}^{M, N+1} \left[y \left| \begin{matrix} \left(-\lambda - r - \sum_{i=1}^s u_i k_i, v; 1\right), (a_j, \alpha_j; A_j)_{1, N}, (a_j, \alpha_j)_{N+1, P} \\ (b_j, \beta_j)_{1, M}, (b_j, \beta_j; B_j)_{M+1, Q}, \left(-1 - \lambda - r - \sum_{i=1}^s u_i k_i, v; 1\right) \end{matrix} \right. \right]
 \end{aligned} \tag{2.2}$$

provided that:

$$\begin{aligned}
 & u_i > 0, v \geq 0, (i = 1, \dots, s) \quad ; -\frac{1}{2} < (\gamma - \alpha - \beta) < \frac{1}{2} \\
 & v_{\min_{1 \leq j \leq M} \left[\operatorname{Re} \left(\frac{b_j}{\beta_j} \right) \right]} + 1 > 0,
 \end{aligned}$$

and

$$\begin{aligned}
 & |\arg y| < \frac{1}{2} \pi \Omega, \text{ where} \\
 & \Omega = \sum_{j=1}^M \beta_j + \sum_{j=1}^N A_j \alpha_j - \sum_{j=M+1}^Q B_j \beta_j - \sum_{j=N+1}^P \alpha_j > 0
 \end{aligned}$$

Proof:

We start from the well known result due to Slater ([15], p.75):

$${}_2F_1[\alpha, \beta; \gamma; x] {}_2F_1\left[\gamma - \alpha + \frac{1}{2}, \gamma - \beta + \frac{1}{2}; \gamma + 1; x\right] = \sum_{r=0}^{\infty} \frac{\left(\gamma + \frac{1}{2}\right)_r}{(\gamma + 1)_r} \beta_r x^r \tag{2.3}$$

where β_r is given by (2.1).

Multiplying both sides of (2.3) by $S_{n_1, \dots, n_s}^{m_1, \dots, m_s}[\dots] \overline{H}_{P, Q}^{M, N}[yx^v]$ and integrating with respect to x between the limits 0 to 1. Next, expressing the general class of polynomials in the series form with the help of (1.4) and the $\overline{H}$ -function in terms of Mellin-Barnes contour integral by (1.1), interchanging the order of integration and summation in the right hand side, we get the following result after a little simplification say(I):

$$I = \sum_{r=0}^{\infty} \frac{\left(\gamma + \frac{1}{2}\right)_r}{(\gamma+1)_r} \beta_r \int_0^1 x^{\lambda+r} S_{n_1, \dots, n_s}^{m_1, \dots, m_s} [cx^{u_1}, \dots, cx^{u_s}] \overline{H}_{P,Q}^{M,N} [yx^v] dx \quad (2.4)$$

Finally, evaluating the integral on the right hand side with the help of [8] and [9], we arrive at the required result after a little simplification.

3. APPLICATIONS AND SPECIAL CASES

On account of the most general nature of $\overline{H}$ -function and $S_{n_1, \dots, n_s}^{m_1, \dots, m_s} [cx^{u_1}, \dots, cx^{u_s}]$ occurring in our main theorem given by (2.2), a large number of integrals involving simpler functions of one variable can be easily obtained as their special cases. We however gave here only some special cases by way of illustration:

- If we put $\gamma = \alpha$ in the main theorem, the value of β_r in (2.1) comes out to be $\frac{\left(\beta + \frac{1}{2}\right)_r}{r!}$

equal to $r!$ and the result (2.2) yields the following interesting integral:

$$\begin{aligned} & \int_0^1 x^{\lambda} {}_2F_1 \left[\alpha + \frac{1}{2}, \beta + \frac{1}{2}; \alpha + 1; x \right] S_{n_1, \dots, n_s}^{m_1, \dots, m_s} [cx^{u_1}, \dots, cx^{u_s}] \overline{H}_{P,Q}^{M,N} [yx^v] dx \\ &= \sum_{k_1}^{[n_1/m_1]} \dots \sum_{k_s}^{[n_s/m_s]} \sum_{r=0}^{\infty} \frac{(-n_1)_{m_1 k_1}}{k_1!} \dots \frac{(-n_s)_{m_s k_s}}{k_s!} A[n_1 k_1, \dots, n_s k_s] \frac{\left(\alpha + \frac{1}{2}\right)_r}{(\alpha+1)_r} \frac{\left(\beta + \frac{1}{2}\right)_r}{r!} (c^{k_1} \dots c^{k_s}) \\ & \overline{H}_{P+1, Q+1}^{M, N+1} \left[y \left| \begin{matrix} \left(-\lambda - r - \sum_{i=1}^s u_i k_i, v; 1 \right), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q}, \left(-1 - \lambda - r - \sum_{i=1}^s u_i k_i, v; 1 \right) \end{matrix} \right. \right] \end{aligned} \quad (3.1)$$

where the conditions of validity of (3.1) easily follows from those given in (2.1).

- Take $\beta = \alpha + \frac{1}{2}$ then $\alpha + \frac{1}{2} = -f$ (f is non-negative integer) in (3.1), we have:

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$$\begin{aligned}
& \int_0^1 x^\lambda (1-x)^f S_{n_1, \dots, n_s}^{m_1, \dots, m_s} [cx^{u_1}, \dots, cx^{u_s}] \overline{H}_{P,Q}^{M,N} [yx^v] dx \\
&= \sum_{k_1}^{[n_1/m_1]} \dots \sum_{k_s}^{[n_s/m_s]} \sum_{r=0}^f \frac{(-n_1)_{m_1 k_1}}{k_1!} \dots \frac{(-n_s)_{m_s k_s}}{k_s!} .A[n_1 k_1, \dots, n_s k_s] \frac{(-f)_r}{r!} (c^{k_1} \dots c^{k_s}) \\
& \overline{H}_{P+1,Q+1}^{M,N+1} \left[y \left| \begin{matrix} \left(-\lambda - r - \sum_{i=1}^s u_i k_i, v; 1 \right), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q}, \left(-1 - \lambda - r - \sum_{i=1}^s u_i k_i, v; 1 \right) \end{matrix} \right. \right]
\end{aligned} \tag{3.2}$$

where conditions of validity of (3.2) easily follow from those given in (2.1).

- Now evaluating the integral on the left of (3.2) with the help of a know result ([8]), we establish the following interesting expansion formula :

$$\begin{aligned}
& \sum_{k_1}^{[n_1/m_1]} \dots \sum_{k_s}^{[n_s/m_s]} \sum_{r=0}^f \frac{(-n_1)_{m_1 k_1}}{k_1!} \dots \frac{(-n_s)_{m_s k_s}}{k_s!} .A[n_1 k_1, \dots, n_s k_s] \frac{(-f)_r}{r!} (c^{k_1} \dots c^{k_s}) \\
& \overline{H}_{P+1,Q+1}^{M,N+1} \left[y \left| \begin{matrix} \left(-\lambda - r - \sum_{i=1}^s u_i k_i, v; 1 \right), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q}, \left(-1 - \lambda - r - \sum_{i=1}^s u_i k_i, v; 1 \right) \end{matrix} \right. \right] \\
&= \Gamma \left(f + \sum_{i=1}^s u_i k_i + 1 \right) \sum_{k_1}^{[n_1/m_1]} \dots \sum_{k_s}^{[n_s/m_s]} \frac{(-n_1)_{m_1 k_1}}{k_1!} \dots \frac{(-n_s)_{m_s k_s}}{k_s!} .A[n_1 k_1, \dots, n_s k_s] (c^{k_1} \dots c^{k_s}) \\
& \overline{H}_{P+1,Q+1}^{M,N+1} \left[y \left| \begin{matrix} (0, v; 1), (a_j, \alpha_j; A_j)_{1,N}, (a_j, \alpha_j)_{N+1,P} \\ (b_j, \beta_j)_{1,M}, (b_j, \beta_j; B_j)_{M+1,Q}, (-1 - f - uk, v; 1) \end{matrix} \right. \right]
\end{aligned} \tag{3.3}$$

provided both sides exist.

- If we take $M=1, N=P$ and $Q=Q+1$ then the $\overline{H}$ -function occurring therein breaks up into the $\overline{\psi}$ given by [10] and the integral relation (2.2) takes the following form after a little simplification which is also believed to be new:

$$\begin{aligned}
 & \int_0^1 x^\lambda {}_2F_1[\alpha, \beta; \gamma; x] {}_2F_1\left[\gamma - \alpha + \frac{1}{2}, \gamma - \beta + \frac{1}{2}; \gamma + 1; x\right] S_{n_1, \dots, n_s}^{m_1, \dots, m_s}[cx^{u_1}, \dots, cx^{u_s}]_p \bar{\psi}_Q[-yx^v] dx \\
 &= \sum_{k_1}^{[n_1/m_1]} \dots \sum_{k_s}^{[n_s/m_s]} \sum_{r=0}^{\infty} \frac{(-n_1)_{m_1 k_1}}{k_1!} \dots \frac{(-n_s)_{m_s k_s}}{k_s!} A[n_1 k_1, \dots, n_s k_s] \frac{\left(\gamma + \frac{1}{2}\right)_r}{(\gamma + 1)_r} \beta_r(c^{k_1} \dots c^{k_s}) \\
 & \quad {}_{p+1}\bar{\psi}_{Q+1} \left[y \left| \begin{array}{l} \left(1 + r + \lambda + \sum_{i=1}^s u_i k_i, v; 1\right) (1 - a_j, \alpha_j; A_j)_{1,p} \\ (1 - b_j, \beta_j; B_j)_{1,q}, \left(2 + r + \lambda + \sum_{i=1}^s u_i k_i, v; 1\right) \end{array} \right. \right]
 \end{aligned} \tag{3.4}$$

where conditions of validity of (3.4) easily follow from those given in (2.1).

- If we take $\lambda = 0, s = 1$ in our main result than we have a known result [1].

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