The behavior of Jurík's nearest point approach algorithm on some 3-dimensional problems

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Abstract

In this note we highlight two issues with recently published Jurík's Nearest Point Approach algorithm for linear programming.

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1. INTRODUCTION

Tomáš Jurík in his doctoral thesis introduced three new algorithms for linear programming. Theoretical properties of these algorithms remain unknown, although initial computer experiments were encouraging. In this note we exhibit a couple of problems that one of these algorithms, the so called *nearest point approach* algorithm, has.

Let us briefly describe the workings of the algorithm. The algorithm performs gradient-like descent [Such2], although its iterative steps are formulated in geometric terms. Suppose P is a convex polytope in $\mathbf{R}^n$ and a n -dimensional vector c is given. Given an initial point x_0 lying in P we would like to find the point x^* that maximizes the linear functional $c^T x$ on P . One chooses a hyperplane H given by the equation $c^T x = z$ where $z > c^T x^*$. The linear programming problem then can be reformulated as trying to find a point x^* of polygon P that is closest to the hyperplane H under the standard Euclidean distance function $d(\cdot, \cdot)$.

It is easy to construct points x^0 and w^0 such that

- (S1) x^0 lies in P ,
- (S2) w^0 lies in H ,
- (S3) x^0 is the closest point to w^0 among all the points in P .

One then inductively proceeds to construct a series $\{L^k\}$ of lines lying in H and points $x^k \in P$, $w^k \in L^{k-1}$ such that

- (I1) the pair (x^k, w^k) is the closest pair of points from P and L^{k-1} ,
- (I2) L^k contains the projection of x^k on H .

One has

$$\begin{aligned}
 d(x^k, H) &\leq d(x^k, L^{k-1}) && \text{because } L^{k-1} \subset H \\
 d(x^k, L^{k-1}) &\leq d(x^{k-1}, L^{k-1}) && \text{because } x^k \text{ minimizes distance to } L^{k-1} \\
 d(x^k, L^{k-1}) &= d(x^{k-1}, H) && \text{because } L^{k-1} \text{ contains the projection of } x^{k-1}
 \end{aligned}$$

Jurík uses for definition of L^k the projection of the line passing through x^k and w^k , although it should be noted that only the condition (I2) is necessary for the sequence $\{c^T x^k\}$ to be non-decreasing. Similarly, L^k could be any proper linear subspace.

2. A THREE DIMENSIONAL EXAMPLE NEEDING MANY LINES TO CONVERGE

Numerical experiments in [Such1] indicated that there may exist cases when quite a few lines are needed to compute the optimum. Now we will show that this is indeed the case even for the case of the three-dimensional space.

Let H be a plane, O a point in H . Let p be the orthogonal projection of space onto H . Suppose that the convex set used is a half-line $P = \vec{OQ}$, originating in the point O and passing through a point $Q \neq O$. Let x^k and w^k be points on P and L^k minimizing distance between P and $L^k \subset H$. Let R be point of L^k closest to O and R' another point on L^k so that $L^k = RR'$. Denote by α the angle α between projection $Op(\vec{Q})$ and $\vec{OR}$. Note that if $\alpha \geq \pi/2$ then actually $x^k = O$.

Suppose $\alpha < \pi/2$. The direction vector of the line $x^k w^k$ is actually given as the cross product of vectors $\vec{OQ}$ and $R\vec{R}'$. But

$$\begin{aligned}
 \pm p(w^k x^k) &= p(\vec{OQ} \times R\vec{R}') \\
 &= p((Op(\vec{Q}) + p(\vec{Q})Q) \times R\vec{R}') \\
 &= p(Op(\vec{Q}) \times R\vec{R}') + p(p(\vec{Q})Q \times R\vec{R}')
 \end{aligned}$$

and since $Op(\vec{Q}) \times R\vec{R}'$ is orthogonal to H

$$\begin{aligned}
 &= p(p(\vec{Q})Q \times R\vec{R}') \\
 &= \pm \vec{OR}
 \end{aligned}$$

It follows that lines L^k and $L^{k+1} = p(w^k x^k)$ are orthogonal, and thus lines L^k and L^{k+2} are parallel. Since generation of points x^k , w^k and lines L^k is preserved under homotheties, the subsequence $\{|Ox^{2k}|\}$ is geometrically decreasing. Moving the set P away from H slightly so that H does not intersect P (and thus satisfies the assumptions of the nearest point approach algorithm), we easily deduce

THEOREM 2.1. (*Conjecture B from [Such1]*) *There is no function of the number of constraints and the dimension of the linear programming problem that bounds the number of iterations of the nearest point approach algorithm.*

EXAMPLE 2.2. *Let $O=(0,0,h)$, $Q=(3,0,1)$, and take for L^1 the line $(x,y,z) = (4-t, t, 0)$ in the hyperplane $z=0$. The number of steps to reach the optimum O increases linearly with the $-\log(h)$ as can be seen in the following table.*

h	0.1	0.01	0.001	0.0001	0.00001
steps	16	25	36	48	59

3. A THREE DIMENSIONAL COUNTEREXAMPLE TO JURÍK'S ITERATION FORMULAS

Reasoning in deduction of iteration formulas in [Jurík] contains a mistake, namely it is not the case that $a_i^T \Delta x = 0$ implies $(R_{k,l} \Delta x)_i = 0$. This implication is however needed in the argument preceding equation (18) in [Jurík]. We will now show an explicit example which demonstrates the problem.

Consider the optimization problem, which is a special case of Klee-Minty's cube [Klee- Minty]

$$\max_{x \in \mathbf{R}^3} \{c^T x : Ax \leq b\}$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 2/5 & 1 & 0 \\ 2/5 & -1 & 0 \\ 0 & 2/5 & 1 \\ 0 & 2/5 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 1/3 \\ 7/10 \\ -7/5 \end{pmatrix}$$

Jurík's algorithm (see [Jurík] for notation and formulas) requires an initial point x^0 lying inside the cube and a choice of a hyperplane not passing through the cube. We choose $x^0 = (0, 0, 0)'$ and hyperplane $c^T w = 3$. In Step 0 of the algorithm a point $w^{0,0}$ lying on the hyperplane closest to $x^{0,0} = x^0$ is constructed. In our example

$$w^{0,0} = x^0 + \frac{25}{14} \begin{pmatrix} 0 \\ 2/5 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 5/7 \\ -25/14 \end{pmatrix}$$

In Step 1, point $x^{0,0}$ is tested for optimality. In fact, the optimum lies at point $(1, 3/5, 6/25)^T$. Therefore we proceed to Step 2, where direction s^1 is computed

$$s^1 = \begin{pmatrix} 180/461 \\ 341/3227 \\ 941/6454 \end{pmatrix}, \quad R_{1,0} = \begin{pmatrix} 1 & -2/5 & -4/25 \\ 0 & -1 & -2/5 \\ 0 & 0 & -1 \end{pmatrix}$$

Therefore

$$R_{1,0}s^1 = \begin{pmatrix} -5256/11525 \\ -378/2305 \\ -941/6454 \end{pmatrix} \text{ and } R_{1,0}(w^{0,0} - x^{0,0}) = \begin{pmatrix} 0 \\ 0 \\ 25/14 \end{pmatrix}$$

It follows that $J_{1,0} = \{6\}$ and thus $\Delta x^{1,0} = (180/461, 1890/13369, 756/13369)^T$ but

then

$$A\Delta x^{1,0} = \begin{pmatrix} 180/461 \\ -180/461 \\ 3978/13369 \\ 198/13369 \\ 1512/13369 \\ 0 \end{pmatrix}$$

Since the fourth coordinate of $A\Delta x^{1,0}$ is positive we obtain $t_2 = 0$, and thus the algorithm cannot move from the initial point. This shows that the iteration formulas presented in [Jurík] are not correct.

4. CONCLUSION

We have exhibited two problems with Jurík's algorithm. It is, however, not clear that the problems are impossible to surmount.

The first problem can be attacked in several ways. The condition when the active set repeats itself is easy to detect and long iterations can be circumvented by a move along the hyperedge. An alternative approach is to choose the initial hyperplane H sufficiently far from the convex polytope, which should prevent this degenerate situation from occurring, or to apply an affine rescaling transformation, as is typically done in subgradient methods [Shor].

The second problem appears harder to fix. A partial result, a fast algorithm to project on isotone cones, is exhibited in recent work [Neme]. It well may be that there is no simple formula for the local direction of nearest approach, but there are of course plenty of algorithms that solve the underlying quadratic programming problem. The algorithm could be also modified to work with higher dimensional subset L^k instead of lines.

Ultimately however, the usefulness of Jurík's idea will be decided by practical considerations – whether its implementations will perform better than currently used algorithms.

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