A NEW INFORMATION INEQUALITY AND ITS APPLICATION IN ESTABLISHING RELATION AMONG VARIOUS f-DIVERGENCE MEASURES

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Abstract

An Information inequality by using convexity arguments and Jensen inequality is established in terms of Csiszar f-divergence measures. This inequality is applied in comparing particular divergences which play a fundamental role in Information theory, such as Kullback-Leibler distance, Hellinger discrimination, Chi-square distance, J-divergences and others.

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1. INTRODUCTION

Let

$$\Gamma_n = \left\{ P = (p_1, p_2, \dots, p_n) \mid p_i \geq 0, \sum_{i=1}^n p_i = 1 \right\}, n \geq 2$$

be the set of all complete finite discrete probability distributions. There are many information and divergence measures exists in the literature on information theory and statistics. Csiszar [4] & [5] introduced a generalized measure of information using f-divergence measure

$$I_f(P, Q) = \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right) \quad (1.1)$$

where $f: \mathbf{R}_+ \rightarrow \mathbf{R}_+$ is a convex function and $P, Q \in \Gamma_n$.

As in Csiszar [5], we have interpret undefined expressions by $f(0) = \lim_{t \rightarrow 0^+} f(t)$, $0 f\left(\frac{0}{0}\right) = 0$, $0 f\left(\frac{a}{0}\right) = \lim_{\varepsilon \rightarrow 0^+} \varepsilon f\left(\frac{a}{\varepsilon}\right) = a \lim_{t \rightarrow \infty} \frac{f(t)}{t}$, $a > 0$

Here we shall give some example of divergence measures in the category of Csiszar's f-divergence measure.

• **Kullback-Leibler distance** [10]:-

$$D(P, Q) = \sum_{i=1}^n p_i \log \frac{p_i}{q_i} \quad (1.2)$$

• **Relative information of type s** [14]:-

$$D_s(P, Q) = [s(s-1)]^{-1} \left[\sum_{i=1}^n p_i^s q_i^{1-s} - 1 \right], s \neq 0, 1 \quad (1.3)$$

which is one parametric generalization of measure (1.2) for $s = 0$ and $s = 1$ it reduces to measure $D(P, Q)$ and $D(Q, P)$ respectively given by (1.2)

• **Chi-square Divergence** [11]:-

$$\chi^2(P, Q) = \sum_{i=1}^n \frac{(p_i - q_i)^2}{q_i} = \sum_{i=1}^n \frac{p_i^2}{q_i} - 1 \quad (1.4)$$

• **Relative Jensen-Shannon divergence** [15]:-

$$F(P, Q) = \sum_{i=1}^n p_i \log \left(\frac{2p_i}{p_i + q_i} \right) \quad (1.5)$$

• **Relative arithmetic-geometric divergence** [3]:-

$$G(P, Q) = \sum_{i=1}^n \left(\frac{p_i + q_i}{2} \right) \log \left(\frac{p_i + q_i}{2p_i} \right) \quad (1.6)$$

• **Unified relative Jensen-Shannon and Arithmetic-Geometric Divergence of type s**

[14] :- $FG_s(P, Q) = [s(s-1)]^{-1} \left[\sum_{i=1}^n p_i \left(\frac{p_i + q_i}{2q_i} \right)^s - 1 \right], s \neq 0, 1$

(1.7)

which is one parametric generalization for the measures (1.5) & (1.6) for $s=0$ and $s=1$ it reduces to (1.5) & (1.6) respectively.

• **Hellinger Discrimination [1]:-**

$$h(P, Q) = 1 - B(P, Q) = \frac{1}{2} \sum_{i=1}^n \left(\sqrt{p_i} - \sqrt{q_i} \right)^2 \quad (1.8)$$

where $B(P, Q) = \sum_{i=1}^n \sqrt{p_i q_i}$ is known as Bhattacharya divergence measure [2]

• **Triangular Discrimination [6]:-**

$$\Delta(P, Q) = 2[1 - W(P, Q)] = \sum_{i=1}^n \frac{(p_i - q_i)^2}{p_i + q_i} \quad (1.9)$$

where $W(P, Q) = \sum_{i=1}^n \frac{2 p_i q_i}{p_i + q_i}$ is known as harmonic mean divergence measure

• **Chi-square Divergence [8]:-**

$$\psi(P, Q) = \chi^2(P, Q) + \chi^2(Q, P) = \sum_{i=1}^n \frac{(p_i - q_i)^2 (p_i + q_i)}{p_i q_i} \quad (1.10)$$

• **J-Divergence measure [5]:-**

$$J(P, Q) = \sum_{i=1}^n (p_i - q_i) \log \frac{p_i}{q_i} \quad (1.11)$$

$$= D(P, Q) + D(P, Q)$$

• **Relative J-Divergence measure [12]:-**

$$J_R(P, Q) = \sum_{i=1}^n (p_i - q_i) \log \frac{p_i + q_i}{2q_i} \quad (1.12)$$

• **Arithmetic-geometric divergence [13]: –**

$$T(P, Q) = \frac{1}{2} [G(P, Q) + G(Q, P)] = \sum_{i=1}^n \left(\frac{p_i + q_i}{2} \right) \log \left(\frac{p_i + q_i}{2\sqrt{p_i q_i}} \right) \quad (1.13)$$

• **Symmetric Chi-square, Arithmetic and Geometric mean divergence measure [9]:-**

$$E_1^*(P, Q) = \sum_{i=1}^n \frac{(p_i - q_i)^2}{\sqrt{p_i q_i}} \quad (1.14)$$

• **Jensen-Shannon divergence measure [3, 16]:-**

$$I(P, Q) = \frac{1}{2} [F(P, Q) + F(Q, P)] = \frac{1}{2} \left[\sum_{i=1}^n p_i \log \left(\frac{2p_i}{p_i + q_i} \right) + \sum_{i=1}^n q_i \log \left(\frac{2q_i}{p_i + q_i} \right) \right] \quad (1.15)$$

2. New information inequality: -We prove the following theorem regarding information inequality using Jensen inequality.

Theorem 2.1:-Let $f : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ be the differentiable convex function and normalized i.e. $f(1) = 0$. Then for all $P, Q \in \Gamma_n$ we have the following inequality

$$\sum_{i=1}^n q_i f \left(\frac{p_i + q_i}{2q_i} \right) \leq \frac{1}{2} \sum_{i=1}^n q_i f \left(\frac{p_i}{q_i} \right)$$

$$S_f(P, Q) \leq \frac{1}{2} I_f(P, Q)$$

(2.1)

equality holds in (2.1) iff $\frac{p_1}{q_1} = \frac{p_2}{q_2} = \dots = \frac{p_n}{q_n}$

where $S_f(P, Q) = \sum_{i=1}^n q_i f\left(\frac{p_i + q_i}{2q_i}\right)$ we call a new f-divergence measure.

and $I_f(P, Q) = \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right)$ same as (1.1)

Proof:- Let $f: I \subset \mathbf{R} \rightarrow \mathbf{R}$ be a differentiable convex function on the interval I ,
 $t_i \in I$ (I is interior of I).

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Gamma_n$, then the Jensen inequality is

$$f\left(\sum_{i=1}^n \lambda_i t_i\right) \leq \sum_{i=1}^n \lambda_i f(t_i)$$

If $\lambda_1 = \lambda_2 = \frac{1}{2}$, $\lambda_3 = \lambda_4 = \dots = \lambda_n = 0$, then equation (2.2) reduced to

$$f\left(\frac{t_1 + t_2}{2}\right) \leq \frac{1}{2}[f(t_1) + f(t_2)] \quad (2.3)$$

Putting $t_1 = t, t_2 = 1$ then

$$f\left(\frac{t+1}{2}\right) \leq \frac{1}{2}[f(t) + f(1)] \quad (2.4)$$

$$f\left(\frac{t+1}{2}\right) \leq \frac{1}{2}f(t) \quad , \quad \text{since } f(1) = 0$$

(2.5)

Putting $t = \frac{p_i}{q_i}$ in equation (2.5)

$$f\left(\frac{p_i + q_i}{2q_i}\right) \leq \frac{1}{2} f\left(\frac{p_i}{q_i}\right)$$

Multiplying by q_i and taking summation both side

$$\sum_{i=1}^n q_i f\left(\frac{p_i + q_i}{2q_i}\right) \leq \frac{1}{2} \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right)$$

$$S_f(P, Q) \leq \frac{1}{2} I_f(P, Q)$$

equation (2.1) is a generalized inequality relating the Csiszar's f-divergence measure and a new f-divergence measure.

3. Applications for some particular f-divergences:-

In this section we will show relationship in the form of inequality using the results of theorem (2.1) among the various known measures such as Kullback-Leibler, Hellinger, Chi-square, and J-divergence etc.

Result: 3.1- Let $P, Q \in \Gamma_n$, then we have the following relations

$$F(P, Q) \leq \frac{1}{2} D(P, Q) \tag{3.1}$$

$$F(Q, P) \leq \frac{1}{2} D(Q, P) \tag{3.2}$$

where $F(P, Q)$ and $D(P, Q)$ are Relative Jensen Shannon divergence measure and Kullback-Leibler divergence measure give in (1.5) and (1.2) respectively..

Proof: - Considering the mapping $f : (0, \infty) \rightarrow \mathbf{R}$

$$f(t) = -\log t, f''(t) = \frac{1}{t^2} > 0, \forall t > 0$$

$f''(t) \geq 0$ and $f(1) = 0$, So function f is convex and normalized.

then

$$S_f(P, Q) = \sum_{i=1}^n q_i f\left(\frac{p_i + q_i}{2q_i}\right) = \sum_{i=1}^n q_i \log\left(\frac{2q_i}{p_i + q_i}\right) = F(Q, P) \quad (3.3)$$

Interchange by $P \rightarrow Q$

$$S_f(Q, P) = \sum_{i=1}^n p_i f\left(\frac{p_i + q_i}{2p_i}\right) = \sum_{i=1}^n p_i \log\left(\frac{2p_i}{p_i + q_i}\right) = F(P, Q) \quad (3.4)$$

$$I_f(P, Q) = \sum_{i=1}^n q_i \log \frac{q_i}{p_i} = D(Q, P) \quad (3.5)$$

Interchange by $P \rightarrow Q$

$$I_f(Q, P) = \sum_{i=1}^n p_i \log \frac{p_i}{q_i} = D(P, Q) \quad (3.6)$$

equation (2.1) using (3.3), (3.4), (3.5) & (3.6) gives the results (3.1) & (3.2)

Results: 3.2- Let $P, Q \in \Gamma_n$, then we have the following relation

$$G(Q, P) \leq \frac{1}{2} D(Q, P) \quad (3.7)$$

where $G(Q, P)$ and $D(Q, P)$ are relative Arithmetic-Geometric and Kullback-Leibler divergence measure give in (1.6) and (1.2) respectively.

Proof: - Considering mapping $f : (0, \infty) \rightarrow \mathbf{R}$

$$f(t) = t \log t, f''(t) = \frac{1}{t}, \forall t > 0$$

$f''(t) \geq 0$ and $f(1) = 0$, So function f is convex and normalized.

$$S_f(P, Q) = \sum_{i=1}^n q_i \log \left(\frac{p_i + q_i}{2q_i} \right) = G(Q, P) \quad (3.8)$$

and

$$I_f(P, Q) = \sum_{i=1}^n q_i \left[\frac{p_i}{q_i} \log \frac{p_i}{q_i} \right] = \sum_{i=1}^n p_i \log \frac{p_i}{q_i} = D(P, Q) \quad (3.9)$$

equation (2.1) using (3.8) & (3.9) gives the result (3.7) .

Results: 3.3- Let $P, Q \in \Gamma_n$, then we have the following relations

$$J_R(P, Q) \leq J(P, Q) \quad (3.10)$$

$$I(P, Q) + T(P, Q) \leq \frac{1}{2} J(P, Q) \quad (3.11)$$

where $J(P, Q)$, $J_R(P, Q)$, $T(P, Q)$ and $I(P, Q)$ are J-divergence, Relative J-divergence, Arithmetic-geometric divergence, and Jensen-Shannon divergence measure gives in (1.11), (1.12), (1.13) and (1.16) respectively.

Proof: - Considering the mapping $f : (0, \infty) \rightarrow \mathbf{R}$

$$f(t) = (t-1)\log t, f''(t) = \frac{t+1}{t^2}, \forall t > 0$$

$f''(t) \geq 0$ and $f(1) = 0$, So function f is convex and normalized.

then

$$S_f(P, Q) = \sum_{i=1}^n q_i f\left(\frac{p_i + q_i}{2q_i}\right) = \frac{1}{2} \sum_{i=1}^n (p_i - q_i) \log\left(\frac{p_i + q_i}{2q_i}\right) = \frac{1}{2} J_R(P, Q) \quad (3.12)$$

$$I_f(P, Q) = \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right) = \sum_{i=1}^n (p_i - q_i) \log\left(\frac{p_i}{q_i}\right) = J(P, Q) \quad (3.13)$$

equation (2.1) using (3.12) & (3.13) gives (3.10)

Now again we get

$$\begin{aligned} S_f(P, Q) &= \frac{1}{2} \sum_{i=1}^n \left[p_i \log\left(\frac{p_i + q_i}{2q_i}\right) - q_i \log\left(\frac{p_i + q_i}{2q_i}\right) \right] \\ &\Rightarrow \\ &= \frac{1}{2} \sum_{i=1}^n \left[p_i \log\left(\frac{2p_i}{p_i + q_i}\right) + q_i \log\left(\frac{2q_i}{p_i + q_i}\right) \right] + \frac{1}{2} \sum_{i=1}^n p_i \log\left(\frac{p_i + q_i}{2q_i} \cdot \frac{p_i + q_i}{2p_i}\right) \\ &\Rightarrow S_f(P, Q) = I(P, Q) + \sum_{i=1}^n p_i \log\left(\frac{p_i + q_i}{2\sqrt{p_i q_i}}\right) \end{aligned} \quad (3.14)$$

using equation (2.1), (3.13) & (3.14), we get

$$I(P, Q) + \sum_{i=1}^n p_i \log\left(\frac{p_i + q_i}{2\sqrt{p_i q_i}}\right) \leq \frac{1}{2} J(P, Q) \quad (3.15)$$

Interchanging P & Q we get

$$I(Q, P) + \sum_{i=1}^n q_i \log \left(\frac{p_i + q_i}{2\sqrt{p_i q_i}} \right) \leq \frac{1}{2} J(Q, P) \quad (3.16)$$

adding inequalities (3.15) & (3.16) both side

$$I(P, Q) + I(Q, P) + \sum_{i=1}^n (p_i + q_i) \log \left(\frac{p_i + q_i}{2\sqrt{p_i q_i}} \right) \leq \frac{1}{2} J(P, Q) + \frac{1}{2} J(Q, P)$$

$$2I(P, Q) + 2T(P, Q) \leq \sum_{i=1}^n (p_i - q_i) \log \frac{p_i}{q_i}$$

$$\therefore I(P, Q) = I(Q, P)$$

$$2I(P, Q) + 2T(P, Q) \leq J(P, Q)$$

$$I(P, Q) + T(P, Q) \leq \frac{1}{2} J(P, Q) \text{ which is required result ,}$$

Note: - From equation (3.11) we get

$$(a) \quad I(P, Q) \leq \frac{1}{2} J(P, Q)$$

$$(b) \quad T(P, Q) \leq \frac{1}{2} J(P, Q) , \text{ with equality if } P = Q$$

Since $I(P, Q)$ & $T(P, Q)$ both are positive.

Results: 3.4- Let $P, Q \in \Gamma_n$, then we have the following relation

$$\Delta(P, Q) < E_1^*(P, Q) \quad (3.17)$$

where $\Delta(P, Q)$ and $E_1^*(P, Q)$ are Triangular discrimination and Symmetric Chi-square, Arithmetic and Geometric mean divergence measure give in (1.9) and (1.14) respectively.

Proof: - Considering the mapping $f : (0, \infty) \rightarrow \mathbf{R}$

$$f(t) = \frac{(t-1)^2}{\sqrt{t}}, \quad f''(t) = \frac{3}{4}t^{-\frac{1}{2}} + \frac{3}{4}t^{-\frac{5}{2}} + \frac{1}{2}t^{-\frac{3}{2}} > 0, \forall t > 0$$

$f''(t) \geq 0$ and $f(1) = 0$, So function f is convex and normalized.

then

$$S_f(P, Q) = \sum_{i=1}^n q_i f\left(\frac{p_i + q_i}{2q_i}\right) = \sum_{i=1}^n \frac{(p_i - q_i)^2}{\sqrt{2q_i(p_i + q_i)}} \quad (3.18)$$

$$I_f(P, Q) = \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right) = \sum_{i=1}^n \frac{(p_i - q_i)^2}{\sqrt{p_i q_i}} = E_1^*(P, Q) \quad (3.19)$$

from equation (2.1), (3.18) & (3.19)

$$\sum_{i=1}^n \frac{(p_i - q_i)^2}{\sqrt{2q_i(p_i + q_i)}} \leq \frac{1}{2} E_1^*(P, Q)$$

$$2 \sum_{i=1}^n \frac{(p_i - q_i)^2}{\sqrt{2q_i(p_i + q_i)}} \leq E_1^*(P, Q)$$

$$\Delta(P, Q) = \sum_{i=1}^n \frac{(p_i - q_i)^2}{(p_i + q_i)} < \sqrt{2} \sum_{i=1}^n \frac{(p_i - q_i)^2}{\sqrt{(p_i^2 + q_i^2 + 2p_i q_i)}} < \sqrt{2} \sum_{i=1}^n \frac{(p_i - q_i)^2}{\sqrt{q_i(p_i + q_i)}} < E_1^*(P, Q)$$

$\Delta(P, Q) < E_1^*(P, Q)$ hence the relation (3.17) is proved.

Results: 3.5- Let $P, Q \in \Gamma_n$, then we have the following relation

$$\frac{1}{2} \chi^2(P, Q) + \Delta(P, Q) \leq \Psi(P, Q)$$

where $\chi^2(P, Q)$, $\Delta(P, Q)$ and $\Psi(P, Q)$ are Chi-square divergence, (3.20)

Triangular discrimination and Symmetric Chi-square divergence measure give in (1.4), (1.9) and (1.10) respectively.

Proof: - Considering the mapping $f : (0, \infty) \rightarrow \mathbf{R}$

$$f(t) = \frac{(t-1)^2(t+1)}{t}, \quad f''(t) = 2 + \frac{2}{t^3}, \quad \forall t > 0$$

$f''(t) \geq 0$ and $f(1) = 0$, So function f is convex and normalized.

$$\begin{aligned} S_f(P, Q) &= \sum_{i=1}^n q_i f\left(\frac{p_i + q_i}{2q_i}\right) = \sum_{i=1}^n \frac{(p_i - q_i)^2 (p_i + 3q_i)}{4q_i(p_i + q_i)} \\ &= \frac{1}{4} \sum_{i=1}^n \frac{(p_i - q_i)^2}{q_i} + \frac{1}{2} \sum_{i=1}^n \frac{(p_i - q_i)^2}{(p_i + q_i)} \\ S_f(P, Q) &= \sum_{i=1}^n q_i f\left(\frac{p_i + q_i}{2q_i}\right) = \frac{1}{4} \chi^2(P, Q) + \frac{1}{2} \Delta(P, Q) \end{aligned}$$

(3.21)

$$I_f(P, Q) = \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right) = \Psi(P, Q)$$

(3.22)

equation (2.1) using (3.21) & (3.22) gives the result (3.20).

Results: 3.6- Let $P, Q \in \Gamma_n$, then we have the following relation

$$FG_s(Q, P) \leq \frac{1}{2} {}^2K_s(P, Q)$$

(3.23)

where $FG_s(P, Q)$ and $K_s(P, Q)$ Unified relative Jensen-Shannon and Arithmetic-Geometric Divergence of type s and Relative information of type s give in (1.7) and (1.3) respectively.

Proof: - Considering the mapping $f : (0, \infty) \rightarrow \mathbf{R}$

$$f_s(t) = \begin{cases} [s(s-1)]^{-1} [t^s - 1], & s \neq 0, 1 \\ -\log t & \text{if } s = 0 \\ t \log t & \text{if } s = 1 \end{cases} \quad (3.24)$$

$$f''(t) = t^{s-2} > 0, \forall t > 0$$

$f''(t) \geq 0$ and $f(1) = 0$, So function f is convex and normalized.

then

$$S_f(P, Q) = \sum_{i=1}^n q_i f\left(\frac{p_i + q_i}{2q_i}\right) = [s(s-1)]^{-1} \left[\sum_{i=1}^n q_i \left(\frac{p_i + q_i}{2q_i}\right)^s - 1 \right] = FG_s(Q, P) \quad (3.25)$$

$$I_f(P, Q) = \frac{1}{2} \sum_{i=1}^n q_i f\left(\frac{p_i}{q_i}\right) = [s(s-1)]^{-1} \left[\sum_{i=1}^n p_i^s q_i^{1-s} - 1 \right] = {}^2K_s(P, Q) \quad (3.26)$$

equation (2.1) using (3.24), (3.25) & (3.26) gives the result (3.23)

Particular Cases of Result 3.6:-

(i) If $s = 0, 1$ then we have proved in equation (3.1) & (3.7) of results 3.1 & 3.2 respectively.

(ii) If $s = -1$ and let $P, Q \in \Gamma_n$, then we have the following relation

$$\Delta(P, Q) \leq \chi^2(Q, P) \quad (3.27)$$

Proof: - Put $s = -1$ in equation (3.23), then

$$FG_{-1}(Q, P) \leq \frac{1}{2} {}^2K_{-1}(P, Q) \quad (3.28)$$

where

$$FG_{-1}(Q, P) = \frac{1}{2} \left[\sum_{i=1}^n \frac{2q_i^2}{p_i + q_i} - 1 \right] = \frac{1}{2} \left[\sum_{i=1}^n \left(2q_i - \frac{2p_i q_i}{p_i + q_i} \right) - 1 \right] = \frac{1}{2} \left[2 - \sum_{i=1}^n \frac{2p_i q_i}{p_i + q_i} - 1 \right]$$

$$FG_{-1}(Q, P) = \frac{1}{2} \left[1 - \sum_{i=1}^n \frac{2p_i q_i}{p_i + q_i} \right]$$

$$FG_{-1}(Q, P) = \frac{1}{2} [1 - W(P, Q)]$$

$$FG_{-1}(Q, P) = \frac{1}{4} \Delta(P, Q) \quad [\text{using (1.9) and (3.25)}]$$

(3.29)

and

$${}^2K_{-1}(P, Q) = \frac{1}{2} \chi^2(Q, P) \quad [\text{using (1.4) and (3.26)}]$$

(3.30)

using equation (3.28), (3.29) & (3.30) gives the result (3.27)

(iii) If $s = \frac{1}{2}$ and let $P, Q \in \Gamma_n$, then we have the following relation

$$4 \left[1 - B \left(\frac{P+Q}{2}, Q \right) \right] \leq h(P, Q)$$

(3.31)

Proof: - Put $s = \frac{1}{2}$ in equation (3.23), then

$$FG_{1/2}(Q, P) \leq \frac{1}{2} {}^2K_{1/2}(P, Q)$$

(3.32)

where

$$F_{G_{\frac{1}{2}}}(Q, P) = -4 \left[\sum_{i=1}^n q_i \left(\frac{p_i + q_i}{2q_i} \right)^{\frac{1}{2}} - 1 \right] = 4 \left[1 - \sum_{i=1}^n \sqrt{\frac{p_i q_i + q_i^2}{2}} \right] = 4 \left[1 - B \left(\frac{P+Q}{2}, Q \right) \right]$$

$$\Rightarrow F_{G_{\frac{1}{2}}}(Q, P) = 4 \left[1 - B \left(\frac{P+Q}{2}, Q \right) \right] \quad [\text{using (1.8) and (3.25)}]$$
(3.33)

and

$${}^2K_{\frac{1}{2}}(P, Q) = -4 \left[\sum_{i=1}^n \sqrt{p_i q_i} - 1 \right] = 4 \left[1 - \sum_{i=1}^n \sqrt{p_i q_i} \right] = 4 [1 - B(P, Q)] = 2h(P, Q)$$

$${}^2K_{\frac{1}{2}}(P, Q) = 2h(P, Q) \quad [\text{using (1.8) and (3.26)}]$$
(3.34)

using equation (3.32), (3.33) & (3.34) gives the result (3.31)

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