On Some New Classes of Bi-univalent Functions

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Abstract

In the present paper, we introduce and investigate two new subclasses $\mathscr{Q}_{\Sigma}(n,\gamma,k)$ and $\mathscr{B}_{\Sigma}(n,\beta,k)$ of bi-valent functions in the unit disk $\mathbb{U}$. For functions belonging to the classes $\mathscr{Q}_{\Sigma}(n,\gamma,k)$ and $\mathscr{B}_{\Sigma}(n,\beta,k)$, we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$.

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathscr{A}$ be the class of analytic functions defined on the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ with the normalized condition $f(0) = 0 = f'(0) - 1$. Let $\mathscr{S}$ be the class of all functions $f \in \mathscr{A}$ which are univalent in Δ. So $f(z) \in \mathscr{S}$ has the form

$$f(z) = z + \sum_{k=2}^{\infty} a_n z^n, \quad z \in U. \tag{1}$$

Let $f^{-1}(z)$ be inverse of the function $f(z)$ and it is well known that every function $f \in \mathscr{S}$ has an inverse $f^{-1}(z)$, defined by

$$f^{-1}(f(z)) = z, \quad z \in U$$

and

$$f(f^{-1}(w)) = w, \quad for \quad |w| < r_0(f); r_0(f) \geq \frac{1}{4},$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{2}$$

A function $f \in \mathscr{A}$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(w)$ are univalent in U.
Let Σ denote the class of bi-univalent functions in U given by (1).
Many interesting examples of the functions of the class Σ, together with various other properties and characteristics associated with bi-univalent functions (including also

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several open problems and conjectures involving bounds of the coefficients of the functions in Σ), can be found in the earlier work studied by Lewin[17], Brannan and Clunie [16], Netanyahu[18] and others. They introduced subclasses of Σ, like class of bi-starlike and convex functions, bi-strongly starlike and convex functions similar to the well-known subclasses $\mathscr{S}^*(\alpha)$ and $\mathscr{K}^*(\alpha)$ of starlike and convex functions of order α $(0<\alpha<1)$, respectively (see [15]) and obtained non-sharp estimates on the initial coefficients in the Taylor-Maclaurin series expansion (1) see[16; 9; 10]. More recently, Srivastava et al. [8; 12; 13], Frasin and Aouf [11], R.M. Ali et al. [14] and Porwal and Darus [6] introduced some new subclasses of Σ and obtained bounds for the initial coefficients of the function given by (1).
Motivated by the work of Porwal and Darus [6], we introduce a new subclass $\mathscr{Q}_\Sigma(k,n,\alpha,\gamma)$.

DEFINITION 1.1. A function f given by (1) is said to be in the class $\mathscr{Q}_\Sigma(n,\gamma,k)$ if the following conditions are satisfied:
For $n\in Z$, $0\le\gamma<1$, $\alpha\ge 1, \lambda\ge 0$ we introduce the subclass $\mathscr{Q}_\Sigma(n,\gamma,k)$ of S of functions of the form (1) satisfying the condition

$$f\in\Sigma \quad and \quad \left|arg\left(\frac{(1-\alpha)I_\lambda^n f(z)+\alpha I_\lambda^{n+1}f(z)}{z}\right)\right|<\frac{\gamma\pi}{2} \quad z\in U, \tag{3}$$

$$f\in\Sigma \quad and \quad \left|arg\left(\frac{(1-\alpha)I_\lambda^n g(w)+\alpha I_\lambda^{n+1}g(w)}{w}\right)\right|<\frac{\gamma\pi}{2} \quad z\in U, \tag{4}$$

where

$$g(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots.$$

And

$$I_\lambda^n f(z)=z+\sum_{k=2}^{\infty}(1+\lambda(k-1))^n a_k z^k, \quad z\in\Delta, \quad \lambda\ge 0, \quad n\in\mathbb{Z}.$$

is generalized *Sălăgean* derivative defined by [2].

This generalized operator is studied by many and mentioned again by [3]. For $k=1$, this class is introduced and investigated in [6]. For $n=0$ and $\lambda=1$ the class $\mathscr{Q}_\Sigma(n,\gamma,k)$ reduces to H_Σ^α introduced and studied by Srivastava et al. [8] and for $n=0$ the class $\mathscr{Q}_\Sigma(n,\gamma,k)$ reduces to $\mathscr{B}_\Sigma(\alpha,\lambda)$ introduced and studied by Frasin and Aouf

[11]. In this paper, we investigate the estimates for the initial coefficients a_2 and a_3 of bi-univalent functions belonging to the class $\mathscr{Q}_\Sigma(n,\gamma,k)$. Our results generalize several well-known results in [1; 4; 5; 10] and these are pointed out. In order to prove our main result we need the following lemma:

LEMMA 1.1. [3] If $p\in\mathscr{P}$, then $|c_k|\le 2$ for each k, where $\mathscr{P}$ is the family of all functions $p(z)$ analytic in U for which $\operatorname{Re}p(z)>0$, $p(z)=1+c_1z+c_2z^2+\cdots$ for $z\in U$.

2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\mathscr{Q}_\Sigma(N,\gamma,K)$

THEOREM 2.1. Let $f(z)$ given by (1) be in the class $\mathscr{Q}_\Sigma(n,\gamma,k)$, $k\in\mathbb{N}$, $n\in Z$, $0\le\gamma<1$, $\alpha\ge 1$,$\lambda\ge 0$. Then

$$|a_2|\le\frac{2\alpha}{\sqrt{(1+\lambda)^{2n}(1+\lambda\alpha)^2+\gamma[2(1+2\lambda)^n(1+2\lambda\alpha)-(1+\lambda)^{2n}(1+\lambda\alpha)^2]}},\tag{5}$$

and

$$|a_3|\le\frac{2\gamma}{2[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}]}+\frac{4\gamma^2}{2[(1-\alpha)(1+\lambda)^n+\alpha(1+\lambda)^{n+1}]}.\tag{6}$$

PROOF. It follows from (3) and (4) that

$$\frac{(1-\alpha)I_\lambda^n f(z)++\alpha I_\lambda^{n+1}f(z)}{z}=(p(z))^\gamma\tag{7}$$

$$\frac{(1-\alpha)I_\lambda^n g(w)++\alpha I_\lambda^{n+1}g(w)}{w}=(q(w))^\gamma\tag{8}$$

where $p(z)=1+p_1z+p_2z^2+\cdots$ and $q(w)=1+q_1w+q_2w^2+\cdots$ in $\mathscr{P}$. Now on equating the coefficients in (7) and (8), we have

$$[(1-\alpha)(1+\lambda)^n+\alpha(1+\lambda)^{n+1}]a_2=\gamma p_1\tag{9}$$

$$[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}]a_3=\gamma p_2+\frac{\gamma(\gamma-1)}{2}p_1^2\tag{10}$$

$$-[(1-\alpha)(1+\lambda)^n+\alpha(1+\lambda)^{n+1}]a_2=\gamma q_1\tag{11}$$

and

$$[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}](2a_2^2-a_3)=\gamma q_2+\frac{\gamma(\gamma-1)}{2}q_1^2. \qquad (12)$$

From (9) and (11) we get

$$p_1=-q_1 \qquad (13)$$

and

$$2[(1-\alpha)(1+\lambda)^n+\alpha(1+\lambda)^{n+1}]a_2^2=\gamma^2(p_1^2+q_1^2) \qquad (14)$$

From (10), (12) and (14), we get

$$2[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}]a_2^2$$

$$=(p_2+q_2)\gamma+\frac{\gamma(\gamma-1)}{2}(p_1^2+q_1^2)$$

$$=(p_2+q_2)\gamma+\frac{\gamma(\gamma-1)}{2}\frac{2[(1-\alpha)(1+\lambda)^n+\alpha(1+\lambda)^{n+1}]}{\alpha^2}a_2^2.$$

Therefore, we have

$$a_2^2=\frac{\gamma^2(p_2+q_2)}{(1+\lambda)^{2n}(1+\lambda\alpha)^2+\gamma[2(1+2\lambda)^n(1+2\lambda\alpha)-(1+\lambda)^{2n}(1+\lambda\alpha)^2]} \qquad (15)$$

Applying Lemma 1.1 for (15), we get

$$|a_2|\leq\frac{2\alpha}{\sqrt{(1+\lambda)^{2n}(1+\lambda\alpha)^2+\gamma[2(1+2\lambda)^n(1+2\lambda\alpha)-(1+\lambda)^{2n}(1+\lambda\alpha)^2]}}.$$

which gives us desired estimate on $|a_2|$ as asserted in (5).

Next in order to find the bound on $|a_3|$, by subtracting (12) from (10), we get

$$2[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}](a_3-a_2^2)=\gamma(p_2-q_2)+\frac{\gamma(\gamma-1)}{2}(p_1^2-q_1^2) \qquad (16)$$

It follows from (13), (14) and (16)

$$a_3=\frac{\gamma(p_2-q_2)}{2[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}]}+\frac{\gamma^2(p_1^2+q_1^2)}{2[(1-\alpha)(1+\lambda)^n+\alpha(1+\lambda)^{n+1}]} \qquad (17)$$

Applying Lemma 1.1 for (17), we get

$$|a_3| \leq \frac{2\gamma}{2[(1-\alpha)(1+2\lambda)^n + \alpha(1+2\lambda)^{n+1}]} + \frac{4\gamma^2}{2[(1-\alpha)(1+\lambda)^n + \alpha(1+\lambda)^{n+1}]}.$$

This completes the proof of Theorem 2.1.

3. COEFFICIENT BOUNDS FOR THE FUNCTION $\mathscr{B}_\Sigma(N,\beta,K)$

DEFINITION 3.1. A function f given by (1) is said to be in the class $\mathscr{B}_\Sigma(n,\beta,k)$ if the following conditions are satisfied:
For $n \in Z$, $0 \leq \beta < 1$, $\alpha \geq 1, \lambda \geq 0$, we introduce the subclass $\mathscr{B}_\Sigma(n,\beta,k)$ of S of functions of the form (1) satisfying the condition

$$f \in \Sigma \quad and \quad \mathscr{R}e\left(\frac{(1-\alpha)I_\lambda^n f(z) + \alpha I_\lambda^{n+1} f(z)}{z}\right) > \beta \quad z \in U, \tag{18}$$

$$f \in \Sigma \quad and \quad \mathscr{R}e\left(\frac{(1-\alpha)I_\lambda^n g(w) + \alpha I_\lambda^{n+1} g(w)}{w}\right) > \beta \quad z \in U, \tag{19}$$

where

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots.$$

And $I_\lambda^n f(z)$ is generalized *Sălăgean* derivative defined by [2].

For $k=1$ and $n=0$, the class $\mathscr{B}_\Sigma(n,\beta,k)$ reduces the class $\mathscr{H}_\Sigma(n,\beta,\lambda)$ and $\mathscr{H}_\Sigma(\beta,\lambda)$ studied by Porwal and Darus [6] and Frasin and Aouf [11], respectively. For $n=0$, $\lambda=1$, this class reduces to $\mathscr{H}_\Sigma(\lambda)$ studied by Srivastava et al. [8].

THEOREM 3.1. Let $f(z)$ given by (1) be in the class $\mathscr{B}_\Sigma(n,\beta,k)$, $n \in Z$, $0 \leq \beta < 1$, $\alpha \geq 1$, $\lambda \geq 0$. Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta))}{2[(1-\alpha)(1+2\lambda)^n + \alpha(1+2\lambda)^{n+1}]^2}} \tag{20}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{[(1-\alpha)(1+\lambda)^n + \alpha(1+\lambda)^{n+1}]^2} + \frac{2(1-\beta)}{[(1-\alpha)(1+2\lambda)^n + \alpha(1+2\lambda)^{n+1}]}. \tag{21}$$

PROOF. It follows from (18) and (19) that there exists $p(z) \in P$ and $q(z) \in P$

$$\frac{(1-\alpha)I_\lambda^n f(z) + +\alpha I_\lambda^{n+1} f(z)}{z} = \beta + (1-\beta)p(z) \tag{22}$$

$$\frac{(1-\alpha)I_\lambda^n g(w) + +\alpha I_\lambda^{n+1} g(w)}{w} = \beta + (1-\beta)q(w) \tag{23}$$

where $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ and $q(w) = 1 + q_1 w + q_2 w^2 + \cdots$ in $\mathscr{P}$. Now on equating the coefficients in (22) and (23), we have

$$[(1-\alpha)(1+\lambda)^n + \alpha(1+\lambda)^{n+1}]a_2 = (1-\beta)p_1 \tag{24}$$

$$([(1-\alpha)(1+2\lambda)^n + \alpha(1+2\lambda)^{n+1}]a_3 = (1-\beta)p_2 \tag{25}$$

$$-[(1-\alpha)(1+\lambda)^n + \alpha(1+\lambda)^{n+1}]a_2 = (1-\beta)q_1 \tag{26}$$

and

$$[(1-\alpha)(1+2\lambda)^n + \alpha(1+2\lambda)^{n+1}](2a_2^2 - a_3) = (1-\beta)q_2 \tag{27}$$

From (24) and (26) we get

$$p_1 = -q_1 \tag{28}$$

and

$$2[(1-\alpha)(1+\lambda)^n + \alpha(1+\lambda)^{n+1}]a_2^2 = (1-\beta)^2(p_1^2 + q_1^2) \tag{29}$$

From (25) and (27), we get

$$2[(1-\alpha)(1+2\lambda)^n + \alpha(1+2\lambda)^{n+1}]^2 a_2^2 = (p_2 + q_2)(1-\beta) \tag{30}$$

From (29) and (30), we get

$$|a_2|^2 \leq \frac{(1-\beta)(|p_2|^2 + |q_2|^2)}{2[(1-\alpha)(1+2\lambda)^n + \alpha(1+2\lambda)^{n+1}]^2} \tag{31}$$

and

$$|a_2^2| \leq \frac{2(1-\beta))}{2[(1-\alpha)(1+2\lambda)^n + \alpha(1+2\lambda)^{n+1}]^2}. \tag{32}$$

Which is the bound on $|a_2|$ as given in (20).

Next, in order to find the bound on $|a_3|$ by subtracting (29) from (25), we obtain

$$2[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}](a_3-2a_2^2)=(1-\beta)(p_2-q_2) \tag{33}$$

$$a_3=a_2^2+\frac{(1-\beta)(p_2-q_2)}{2[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}]} \tag{34}$$

On substituting the value $|a_2^2|$ from (31), we have

$$a_3=\frac{(1-\beta)^2(|p_2|^2+|q_2|^2)}{2[(1-\alpha)(1+\lambda)^n+\alpha(1+\lambda)^{n+1}]^2}+\frac{(1-\beta)(p_2-q_2)}{2[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}]} \tag{35}$$

On applying Lemma 1.1 for the coefficients p_1, q_1, p_2 and q_2, we obtain

$$|a_3|\leq\frac{4(1-\beta)^2}{[(1-\alpha)(1+\lambda)^n+\alpha(1+\lambda)^{n+1}]^2}+\frac{2(1-\beta)}{[(1-\alpha)(1+2\lambda)^n+\alpha(1+2\lambda)^{n+1}]}. \tag{36}$$

This completes the proof of Theorem 3.1.

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