Applying Fractional Calculus to Analyze Economic Growth Modelling

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Abstract

In this work, we apply fractional calculus to analyze a class of economic growth modelling (EGM) of the Spanish economy. More precisely, the Grünwald-Letnnikov and Caputo derivatives are used to simulate GDP by replacing the previous integer order derivatives with the help of Matlab, SPSS and R software. As a result, we find that the data raised from the Caputo derivative are better than the data raised from the Grünwald-Letnnikov derivative. We improve the previous result in [12].

Mathematics Subject Classification 2010: 26A33, 26A51, 26D15
Keywords: Grünwald-Letnnikov derivative, Caputo derivative, Economic growth modelling, Genetic algorithm, Method of least squares.

1. INTRODUCTION

It is well known that EGM is one of the most important models in studying the dynamics of finance behaviour. After reviewing the classical EGM in the literature, one can see that the integer order derivatives and integrals are always used to characterize such procedure in the development of economics. However, there exist some gaps by using the classical calculus to simulate the data from the real models. Recently, the basic theory including existence theory, stability and control theory for all kinds of fractional differential equations and inclusions [1; 2; 3; 4; 5; 6; 7; 8; 9] is studied extensively. In addition, one can see that fractional calculus [10] is also widely used to construct economic models involving the memory effect in the evolutionary process. It has been proved that fractional models [11] are better than integer models and provide an excellent tool for the description of memory of EGM, which has been taken into account in [12; 13; 14; 15; 16; 17; 18; 19].

In [12], the authors study GDP growth for the Spanish and Portuguese cases by applying Grünwald-Letnnikov fractional EGM via data between 1960 and 2012. By

This work was supported by the National Natural Science Foundation of China (grant number 11661016), Training Object of High Level and Innovative Talents of Guizhou Province (grant number (2016)4006), Unite Foundation of Guizhou Province (grant number [2015]7640), the Slovak Research and Development Agency (grant number APVV-14-0378) and the Slovak Grant Agency VEGA (grant numbers 2/0153/16 and 1/0078/17).

10.2478/jamsi-2018-0003

setting the mean absolute deviation as performance index and using Nelder-Mead's simplex search method, the coefficients and orders proposed in the fractional EGM are obtained. By comparing the coefficients of fractional EGM and integer EGM, a new hybrid model involving integer calculus and fractional calculus is established to remove low influence variables in the models. It is shown that fractional models have a better performance than the classical models.

In the present paper, we go on the study of GDP growth for the Spanish case to improve fractional EGM in [12] by using different computational methods. More precisely, we use four different EGMs, namely Grünwald-Letnnikov integer/fractional type and Caputo integer/fractional type models. Moreover, Nelder-Mead's simplex search method is replaced by genetic algorithm to give orders in the current work. The method of least squares is used to give the estimation of the coefficients. In spite of software of Matlab, SPSS and R are also used in linear regression analysis. We note that the Spanish case is used in this paper only for possibility to compare our achievement and proposed models with previous ones.

2. INTEGER AND FRACTIONAL EGMS

Throughout of this paper, we denote land area by LA (km^2), arable land by AL (km^2), population by P, school attendance by SA, gross capital formation by GCF, exports of goods and services by EGS, general government final consumption expenditure by GGFCE, money and quasi money by MQM, number of variables of the model by NVM and number of parameters of the model by NPM. We remark that all the data used here are taken from 1960 to 2012. We also denote the mean square error by MSE, the mean absolute deviation by MAD, the coefficient of determination by R^2, Akaike Information Criterion by AIC and the weight of AIC for the i-th model by ω_i.

Consider the following general formulation of EGM: $z = f(x_1, x_2, \cdots)$ where f is a given function. For simplify, we introduce the following notations:

x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
LA	AL	P	SA	GCF	EGS	GGFCE	MQM

and

z	n	k	t
GDP	NVM	NPM	Year

Define

$$\begin{aligned}
\mathrm{MSE} &= \frac{\sum_{i=1}^{n}(z_i-\tilde{z}_i)^2}{n},\\
\mathrm{MAD} &= \frac{\sum_{i=1}^{n}|z_i-\tilde{z}_i|}{n},\\
\mathrm{R}^2 &= 1-\frac{\sum_{i=1}^{n}(z_i-\tilde{z}_i)^2}{\sum_{i=1}^{n}(z_i-\bar{z})^2},\\
\mathrm{AIC} &= n\log\frac{\sum_{i=1}^{n}(z_i-\tilde{z}_i)^2}{n}+2k+\frac{2k(k+1)}{n-k-1},
\end{aligned}$$

if we have m model, then

$$\omega_i = \frac{\exp\left(-\frac{\mathrm{AIC}_i-\min_m \mathrm{AIC}}{2}\right)}{\sum_{j=1}^{m}\exp\left(-\frac{\mathrm{AIC}_j-\min_m \mathrm{AIC}}{2}\right)}.$$

Next, we recall the following standard integer order model (IOM)

$$z(t)=\sum_{i=1,2,3,4,6,7} c_i x_i(t)+c_5(I^1_{t_0,t}x_5)(t)+\sum_{i=8,9} c_i x_i'(t).$$

We also need the following modified models:

- IOM1 (Grünwald-Letnnikov integer type)

$$z(t)=\sum_{i=1,2,3,4,6,7} c_i({}^{GL}D^0_{t_0,t}x_i)(t)+c_5({}^{GL}D^{-1}_{t_0,t}x_5)(t)+\sum_{i=8,9} c_i({}^{GL}D^1_{t_0,t}x_i)(t),$$

- IOM2 (Caputo integer type)

$$z(t)=\sum_{i=1,2,3,4,6,7} c_i({}^{C}D^0_{t_0,t}x_i)(t)+c_5(I^1_{t_0,t}x_5)(t)+\sum_{i=8,9} c_i({}^{C}D^1_{t_0,t}x_i)(t),$$

- FOM1 (Grünwald-Letnnikov fractional type)

$$z(t)=\sum_{i=1}^{9} c_i({}^{GL}D^{\alpha_k}_{t_0,t}x_i)(t),$$

- FOM2 (Caputo fractional type)

$$z(t)=\sum_{i=1}^{9} c_i({}^{C}D^{\alpha_i}_{t_0,t}x_i)(t),$$

where t_0 denotes the first year and the fractional calculus [10] is given by

$$(I_{a,t}^{\alpha}u)(t) := \frac{1}{\Gamma(\alpha)}\int_a^t \frac{u(\tau)}{(t-\tau)^{1-\alpha}}d\tau,\ 0<\alpha\leq 1,$$

and the Grünwald-Letnnikov (GL) derivative

$${}^{GL}D_{a,t}^{\alpha}u(t) = \lim_{h\to 0}\frac{\sum_{j=0}^{[(t-a)/h]}}{h^{\alpha}}(-1)^j C_{\alpha}^j u(t-jh),$$

$$C_{\alpha}^j = \frac{(-1)^j\Gamma(\alpha-j)}{\Gamma(j+1)\Gamma(-\alpha-j+1)},\ 0<\alpha\leq 1,$$

$$C_{\alpha}^j = \frac{\Gamma(\alpha+1)}{\Gamma(j+1)\Gamma(\alpha-j+1)},\ -1\leq\alpha<0,$$

$$C_{\alpha}^j = 1,\ \alpha = 0,$$

and the Caputo derivative

$${}^{C}D_{a,t}^{\alpha}u(t) = \frac{1}{\Gamma(1-\alpha)}\int_a^t \frac{u'(s)}{(t-s)^{\alpha}}ds,\ t>a,\ 0<\alpha\leq 1.$$

3. MAIN RESULTS

3.1. Economic data for Spanish economy

By using the Spanish data from 1960 to 2012 in [12, Table 5], we apply Matlab to obtain the following figures (see Figure 1).

3.2. The coefficients and orders

By using genetic algorithm in the Matlab, we obtain the following data (see Tables I and II). Here we remark that the coefficients are estimated by using the method of least squares.

Fig. 1. Data for Spanish

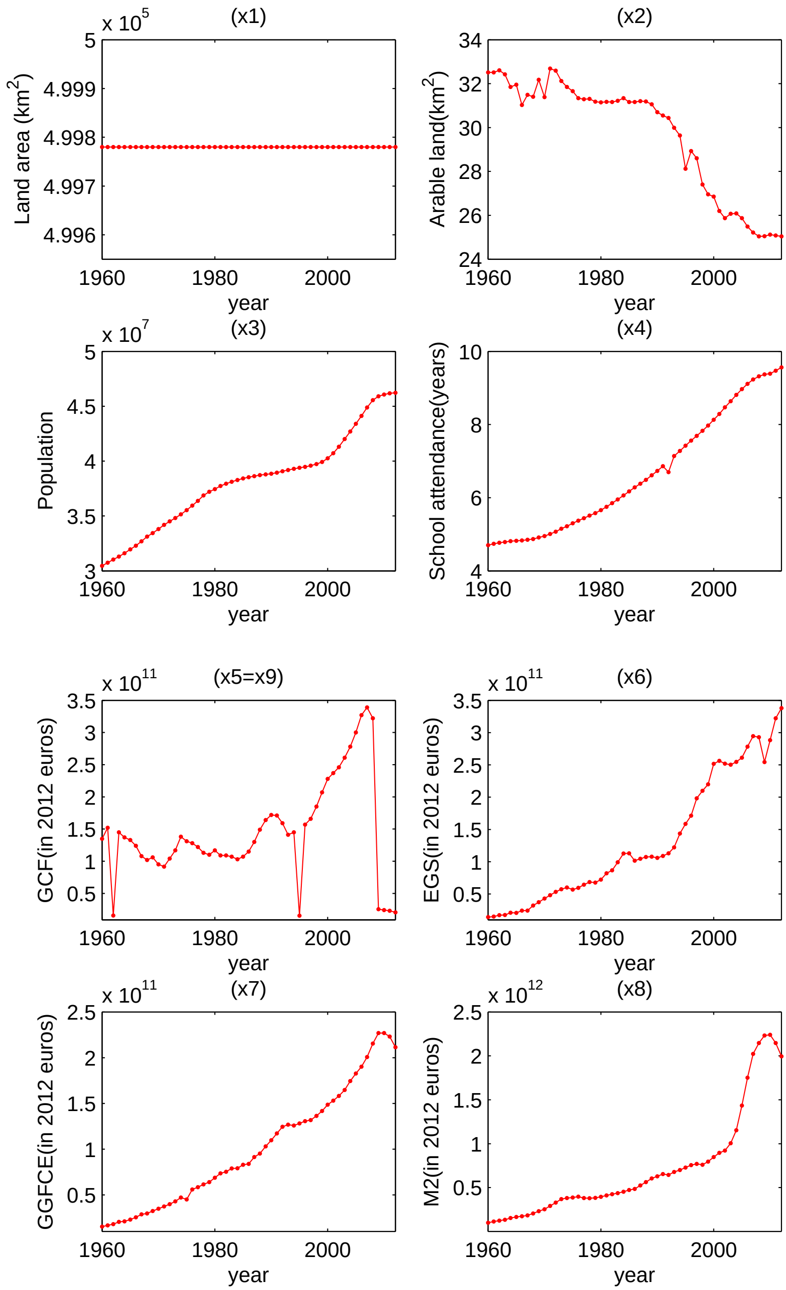


Table I. The orders of the fractional operators

	IOM1	FOM1	IOM2	FOM2
α_1	0	0.31072	0	0.26735
α_2	0	-0.75424	0	-0.69281
α_3	0	-0.73633	0	-0.70304
α_4	0	-0.99999	0	-0.67233
α_5	-1	-1	-1	-0.60068
α_6	0	-0.83616	0	-0.95969
α_7	0	0.31073	0	0.43180
α_8	1	-0.13985	1	0.31072
α_9	1	-0.34465	1	-0.94727

Table II. The coefficients of the fractional operators

	IOM1	FOM1	IOM2	FOM2
$c_1(\times 10^{05})$	9.903	1.954	10.393	227.211
$c_2(\times 10^{09})$	7.531	11.371	-0.872	9.732
$c_3(\times 10^{04})$	-1.416	-1.150	-0.851	-1.009
$c_4(\times 10^{10})$	-2.455	0.141	-0.207	1.576
$c_5(\times 10^{-1})$	2.887	0.296	1.658	1.963
$c_6(\times 10^{-1})$	-2.123	4.707	-0.269	2.464
$c_7(\times 10^{00})$	-3.845	1.513	-1.16	1.072
$c_8(\times 10^{-2})$	9.582	4.236	16.556	5.358
$c_9(\times 10^{-2})$	1.759	9.542	12.459	-6.060

Now we are ready to give analysis by virtue of estimated value from Matlab via true value.

Table III: Performance indices for the Spanish economy

index	Variable	IOM1	FOM1	IOM2	FOM2
MSE($\times 10^{20}$)		4.042	0.638	3.075	0.479
R^2		0.9947	0.9992	0.9960	0.9994
MAD($\times 10^{10}$)		1.666	0.642	1.540	0.569
AIC		2537.0	2439.1	2522.5	2423.9
AIC without one variable	x_1	2536.8	2477.2	2523.6	2463.5
	x_2	2535.3	2521.2	2519.6	2496.1
	x_3	2537.4	2508.4	2521.1	2496.3
	x_4	2534.7	2439.7	2519.5	2446.4
	x_5	2565.1	2437.0	2528.9	2440.3
	x_6	2534.6	2514.6	2519.6	2485.1
	x_7	2550.9	2445.4	2520.8	2450.6
	x_8	2538.2	2445.9	2537.4	2428.8
	x_9	2534.1	2441.8	2536.7	2422.7
ω found from the AIC without one variable	x_1	7%	0%	3%	0%
	x_2	15%	0%	24%	0%
	x_3	5%	0%	11%	0%
	x_4	20%	18%	24%	0%
	x_5	0%	73%	0%	0%
	x_6	21%	0%	24%	0%
	x_7	0%	1%	13%	0%
	x_8	4%	1%	0%	5%
	x_9	27%	7%	0%	95%

REMARK 3.1. The value 0% implies that the i-th variable for the corresponding models cannot be removed in the simulation from Table III.

3.3. Analysis of significance level

Now we apply Matlab, SPSS and R software to give the analysis of significance level (see Tables IV and V)

Table IV. Significance level of GL model

	Variable	IOM1			FOM1		
		Matlab	SPSS	R	Matlab	SPSS	R
t value	x_1	-0.765	0.557	1.637	3.326	5.378	6.841
	x_2	0.898	0.078	1.058	11.35	6.250	12.436
	x_3	-1.451	-1.711	-1.760	-10.479	-11.511	-10.623
	x_4	-0.656	0.368	-0.901	1.663	-3.215	1.736
	x_5	4.869	8.353	6.001	0.787	-1.048	0.631
	x_6	-0.530	-0.791	-0.584	9.760	3.949	11.473
	x_7	-3.156	-3.637	-4.035	2.855	.531	2.546
	x_8	2.006	2.149	2.049	2.755	.343	3.143
	x_9	0.512	0.510	0.391	2.172	7.868	2.541
P value	x_1	0.449	0.580	0.108699	0.001	0.000	1.96e-08
	x_2	0.374	0.938	0.295763	1.597e-14	0.000	5.33e-16
	x_3	0.154	0.094	0.085435	2.047e-13	0.000	1.00e-13
	x_4	0.515	0.715	0.372647	0.104	0.002	0.08954
	x_5	1.550e-5	0.000	3.37e-7	0.436	0.300	0.53117
	x_6	0.599	0.433	0.562351	1.793e-12	0.000	8.16e-15
	x_7	0.003	0.001	0.000215	0.007	0.598	0.01448
	x_8	0.051	0.037	0.046428	0.009	0.733	0.00299
	x_9	0.612	0.613	0.698030	0.035	0.000	0.01467

REMARK 3.2. The red data in Table IV denote 5% significance level.

Table V. Significance level of Caputo model

	Variable	IOM2			FOM2		
		Matlab	SPSS	R	Matlab	SPSS	R
t value	x_1	0.257	1.463	2.012	3.039	-6.802	7.020
	x_2	-0.130	-1.103	-0.126	4.478	7.638	11.312
	x_3	-1.161	-1.159	-1.241	-5.981	5.878	-11.178
	x_4	-0.043	.890	-0.248	4.338	8.019	4.798
	x_5	2.879	3.159	3.021	3.722	2.982	4.632
	x_6	-0.111	-.094	0.001	4.812	-5.245	10.100
	x_7	-1.013	-.638	-1.032	5.552	-4.452	5.638
	x_8	4.038	4.622	4.376	2.575	-2.736	2.783
	x_9	3.873	4.489	4.282	-1.311	7.173	-1.478
P value	x_1	0.798	0.150	0.05041	0.004	0.000	1.38e-08
	x_2	0.897	0.276	0.90010	5.479e-05	0.000	2.340e-14
	x_3	0.252	0.253	0.22103	3.901e-07	0.000	2.78e-14
	x_4	0.966	0.378	0.80550	8.554e-05	0.000	1.83e-05
	x_5	0.006	0.003	0.00418	0.001	0.005	4.39e-05
	x_6	0.912	0.925	0.99934	1.867e-05	0.000	8.69e-13
	x_7	0.317	0.527	0.30772	1.634e-06	0.000	1.35e-06
	x_8	2.181e-4	0.000	7.33e-05	0.014	0.009	0.0103
	x_9	3.616e-4	0.000	9.88e-05	0.197	0.000	0.1778

REMARK 3.3. The red data in Table V denote 5% significance level.

3.4. Fitting results

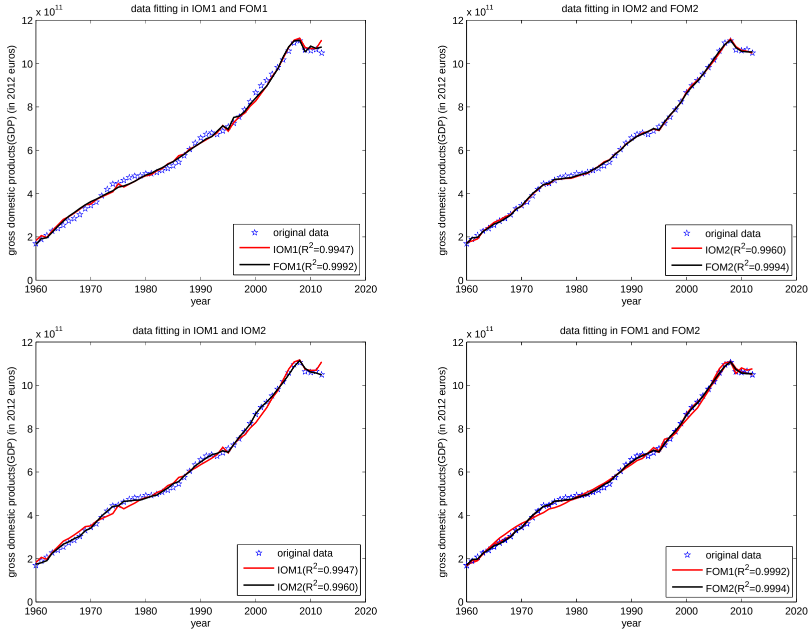
Now we are ready to give the fitting results for IOM, FOM1 and FOM2 (see Fig. 2).

REMARK 3.4. (i) From the figure of data fitting in IOM1 and FOM1, one can see that the simulation result of FOM1 is better than the simulation result of IOM1.

(ii) From the figure of data fitting in IOM2 and FOM2, one can see that the simulation results of IOM2 and FOM2 are very close to original data. However, R^2 of FOM2 is closer to 1 than R^2 of IOM2. Thus, FOM2 is better than IOM2.

(iii) From the figure of data fitting in IOM1 and IOM2 , one can see that the simulation result of IOM2 is better than the simulation result of IOM1.

Fig. 2. Data fitting



(iv) From the figure of data fitting in FOM1 and FOM2, one can see that FOM2 is closer to original data than FOM1 although the value of R^2 for both FOM1 and FOM2 tend to 1.

From above, one can deduce that FOM2 is the most suitable model for this case.

3.5. Comparison of models

Finally, we present the following tables (see Table VI and Table VII) to compare the current results with the previous ones in [12], which show that our results derived by genetic algorithm are much better than the results derived by Nelder-Mead's simplex search method [12].

Table VI. The model of [12]

	Integer (5)	Fractional (6)	Fractional (12)	Integer (13)	Fractional (14)
AIC	2554.3	2473.8	2474.4	2552.9	2472
ω_i	0%	0%	0%	0%	0%

Table VII. Our results

	IOM1	FOM1	IOM2	FOM2
AIC	2537	2439.1	2522.5	2423.9
ω_i	0%	5%	0%	95%

3.6. Conclusions

This paper studies a class of economic growth modelling for the Spanish case. Based on our results, four models of fractional calculus (IMO1, IMO2, FOM1 and FOM2) are proposed. It is shown that the date of GDP raised from the Caputo derivative are better than the Grünwald-Letnnikov derivative. They are not identical in the significance level of models (t value and P value) via Matlab, SPSS and R software. In addition, the data of FOM2 are derived via genetic algorithm and the method of least squares, which is better than IOM1, IOM2, FOM1 and the reference [12].

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Received June 1, 2017