A Class of Modified Ratio Estimators for Estimation of Population Variance

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Abstract

In this paper we have proposed a class of modified ratio type variance estimators for estimation of population variance of the study variable using the known parameters of the auxiliary variable. The bias and mean squared error of the proposed estimators are obtained and also derived the conditions for which the proposed estimators perform better than the traditional ratio type variance estimator and existing modified ratio type variance estimators. Further we have compared the proposed estimators with that of the traditional ratio type variance estimator and existing modified ratio type variance estimators for certain natural populations.

Mathematics Subject Classification 2000: 62 D05
General Terms: Auxiliary variable, Bias, Mean squared error
Additional Key Words and Phrases: Coefficient of variation, Kurtosis, Median, Natural populations, Simple random sampling, Skewness

1. INTRODUCTION

It is common practice to use the auxiliary variable for improving the precision of the estimate of a parameter. When the information on an auxiliary variable X is known, a number of estimators such as ratio, product and linear regression estimators are available in the literature. When the correlation between the study variable and the auxiliary variable is positive, ratio method of estimation is quite effective. Consider a finite population $U = \{U_1, U_2, \ldots, U_N\}$ of N distinct and identifiable units. Let Y be a real variable with value Y_i measured on $U_i, i = 1,2,3, \ldots, N$ giving a vector $Y = \{Y_1, Y_2, \ldots, Y_N\}$. The problem is to estimate the population mean $\overline{Y} = \frac{1}{N}\sum_{i=1}^{N} Y_i$ on the basis of a random sample selected from the population U or its variance $S_y^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(Y_i - \overline{Y})^2$. When there is no additional information on the auxiliary variable available, the simplest estimator of population mean or variance is the simple random sample mean or variance without replacement. As stated earlier, if an auxiliary variable X closely related to the study variable Y is available and X is easy

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to obtain then one can use ratio, product and regression estimators to improve the performance of the estimator of the study variable. Estimating the finite population variance has great significance in various fields such as Industry, Agriculture, Medical and Biological sciences. In this paper, we consider the problem of estimation of the population variance and use the auxiliary information to improve the efficiency of the estimator of population variance.

Before discussing further about the traditional ratio type variance estimator, existing modified ratio type variance estimators and the proposed modified ratio type variance estimators, the notations to be used in this paper are described below:

- $N -$ Population size
- $n -$ Sample size
- $\gamma = \frac{(1-f)}{n}$
- $Y -$ Study variable
- $X -$ Auxiliary variable
- $\overline{X}, \overline{Y} -$ Population means
- $\overline{x}, \overline{y} -$ Sample means
- $S_y^2, S_x^2 -$ Population variances
- $s_y^2, s_x^2 -$ Sample variances
- $C_X, C_y -$ Coefficient of variations
- $\lambda_{rs} = \frac{\mu_{rs}}{\mu_{20}^{r/2}\mu_{02}^{s/2}}$
- $\mu_{rs} = \frac{1}{N}\sum_{i=1}^{N}(y_i - \overline{Y})^r\,(x_i - \overline{X})^s$
- $\beta_{1(x)} = \frac{\mu_{03}^2}{\mu_{02}^3}$, Skewness of the auxiliary variable
- M_d –Median of the auxiliary variable
- Q_1 –First (lower) quartile of the auxiliary variable
- Q_3 –Third (upper) quartile of the auxiliary variable
- Q_r –Inter-quartile range of the auxiliary variable
- Q_d –Semi-quartile range of the auxiliary variable
- Q_a – Semi-quartile average of the auxiliary variable
- $D_i - i^{th}$ Decile of the auxiliary variable
- $B(.) -$ Bias of the estimator
- $MSE(.) -$ Mean squared error of the estimator
- $\hat{S}_R^2 -$ Traditional ratio type variance estimator of S_y^2
- $\hat{S}_i^2 -$ Existing modified ratio type variance estimator of S_y^2

- $\hat{S}^2_{p_i}$ − Proposed modified ratio type variance estimator of S^2_y

Isaki (1983) suggested a ratio type variance estimator for the population variance S^2_y when the population variance S^2_x of the auxiliary variable X is known together with its bias and mean squared error are given below:

$$\hat{S}^2_R = s^2_y \frac{S^2_x}{s^2_x} \quad (1.1)$$

$$B(\hat{S}^2_R) = \gamma S^2_y \left[\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right] \quad (1.2)$$

$$MSE(\hat{S}^2_R) = \gamma S^4_y \left[\left(\beta_{2(y)} - 1\right) + \left(\beta_{2(x)} - 1\right) - 2(\lambda_{22} - 1)\right] \quad (1.3)$$

where $\beta_{2(y)} = \frac{\mu_{40}}{\mu^2_{20}}, \beta_{2(x)} = \frac{\mu_{04}}{\mu^2_{02}}, \lambda_{22} = \frac{\mu_{22}}{\mu_{20}\mu_{02}}$

The ratio type variance estimator given in (1.1) is used to improve the precision of the estimate of the population variance compared to simple random sampling when there exists a positive correlation between X and Y. Further improvements are also achieved on the ratio estimator by introducing a number of modified ratio estimators with the use of known parameters like Coefficient of Variation, Coefficient of Kurtosis, Median, Quartiles and Deciles. The problem of constructing efficient estimators for the population variance has been widely discussed by various authors such as Agarwal and Sithapit (1995), Ahmed et al. (2000), Al-Jararha and Al-Haj Ebrahem (2012), Arcos et al (2005), Cochran (1977), Das and Tripathi (1978), Garcia and Cebrain (1997), Gupta and Shabbir (2008), Isaki (1983), Kadilar and Cingi (2006a,b), Murthy (1967), Prasad and Singh (1990), Reddy (1974), Shabbir and Gupta (2006), Singh and Solanki (2013), Singh and Chaudhary (1986), Singh et al. (1988), Subramani and Kumarapandiyan (2012a,b,c,2013), tailor and Shrama (2012), Upadhyaya and Singh (1999, 2001, 2006), Wolter (1985) and Yadav and Kadilar (2013a,b).

The following table contains all modified ratio type variance estimators using known population parameters of the auxiliary variable in which some of the estimators are already suggested in the literature, remaining estimators are introduced in this paper

Table 1: Modified ratio type estimators for estimating population variance with the bias and mean squared error

Estimator	**Bias - B(.)**	**Mean squared error MSE(.)**
$\hat{S}_1^2 = s_y^2\left[\frac{S_x^2 + C_x}{s_x^2 + C_x}\right]$ Kadilar and Cingi (2006b)	$\gamma S_y^2 \delta_1 \left[\delta_1\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_1^2\left(\beta_{2(x)} - 1\right) - 2\delta_1(\lambda_{22} - 1)\right]$
$\hat{S}_2^2 = s_y^2\left[\frac{S_x^2 + \beta_{2(x)}}{s_x^2 + \beta_{2(x)}}\right]$ Upadhyaya and Singh (1999)	$\gamma S_y^2 \delta_2 \left[\delta_2\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_2^2\left(\beta_{2(x)} - 1\right) - 2\delta_2(\lambda_{22} - 1)\right]$
$\hat{S}_3^2 = s_y^2\left[\frac{S_x^2 + \beta_{1(x)}}{s_x^2 + \beta_{1(x)}}\right]$	$\gamma S_y^2 \delta_3 \left[\delta_3\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_3^2\left(\beta_{2(x)} - 1\right) - 2\delta_3(\lambda_{22} - 1)\right]$
$\hat{S}_4^2 = s_y^2\left[\frac{S_x^2 + \rho}{s_x^2 + \rho}\right]$	$\gamma S_y^2 \delta_4 \left[\delta_4\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_4^2\left(\beta_{2(x)} - 1\right) - 2\delta_4(\lambda_{22} - 1)\right]$
$\hat{S}_5^2 = s_y^2\left[\frac{S_x^2 + S_x}{s_x^2 + S_x}\right]$	$\gamma S_y^2 \delta_5 \left[\delta_5\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_5^2\left(\beta_{2(x)} - 1\right) - 2\delta_5(\lambda_{22} - 1)\right]$
$\hat{S}_6^2 = s_y^2\left[\frac{S_x^2 + M_d}{s_x^2 + M_d}\right]$ Subramani and Kumarapandiyan (2012a)	$\gamma S_y^2 \delta_6 \left[\delta_6\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_6^2\left(\beta_{2(x)} - 1\right) - 2\delta_6(\lambda_{22} - 1)\right]$
$\hat{S}_7^2 = s_y^2\left[\frac{S_x^2 + Q_1}{s_x^2 + Q_1}\right]$ Subramani and Kumarapandiyan (2012b)	$\gamma S_y^2 \delta_7 \left[\delta_7\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \delta_7^2\left(\beta_{2(x)} - 1\right) - 2\delta_7(\lambda_{22} - 1)\right]$

$\hat{S}_8^2 = s_y^2\left[\frac{S_x^2 + Q_3}{s_x^2 + Q_3}\right]$ Subramani and Kumarapandiyan (2012b)	$\gamma S_y^2\delta_8\left[\delta_8\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_8^2\left(\beta_{2(x)} - 1\right) - 2\delta_8(\lambda_{22} - 1)\right]$
$\hat{S}_9^2 = s_y^2\left[\frac{S_x^2 + Q_r}{s_x^2 + Q_r}\right]$ Subramani and Kumarapandiyan (2012b)	$\gamma S_y^2\delta_9\left[\delta_9\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_9^2\left(\beta_{2(x)} - 1\right) - 2\delta_9(\lambda_{22} - 1)\right]$
$\hat{S}_{10}^2 = s_y^2\left[\frac{S_x^2 + Q_d}{s_x^2 + Q_d}\right]$ Subramani and Kumarapandiyan (2012b)	$\gamma S_y^2\delta_{10}\left[\delta_{10}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{10}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{10}(\lambda_{22} - 1)\right]$
$\hat{S}_{11}^2 = s_y^2\left[\frac{S_x^2 + Q_a}{s_x^2 + Q_a}\right]$ Subramani and Kumarapandiyan (2012b)	$\gamma S_y^2\delta_{11}\left[\delta_{11}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{11}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{11}(\lambda_{22} - 1)\right]$
$\hat{S}_{12}^2 = s_y^2\left[\frac{S_x^2 + D_1}{s_x^2 + D_1}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S_y^2\delta_{12}\left[\delta_{12}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{12}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{12}(\lambda_{22} - 1)\right]$
$\hat{S}_{13}^2 = s_y^2\left[\frac{S_x^2 + D_2}{s_x^2 + D_2}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S_y^2\delta_{13}\left[\delta_{13}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{13}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{13}(\lambda_{22} - 1)\right]$
$\hat{S}_{14}^2 = s_y^2\left[\frac{S_x^2 + D_3}{s_x^2 + D_3}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S_y^2\delta_{14}\left[\delta_{14}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{14}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{14}(\lambda_{22} - 1)\right]$
$\hat{S}_{15}^2 = s_y^2\left[\frac{S_x^2 + D_4}{s_x^2 + D_4}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S_y^2\delta_{15}\left[\delta_{15}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{15}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{15}(\lambda_{22} - 1)\right]$
$\hat{S}_{16}^2 = s_y^2\left[\frac{S_x^2 + D_5}{s_x^2 + D_5}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S_y^2\delta_{16}\left[\delta_{16}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_{16}^2\left(\beta_{2(x)} - 1\right) - 2\delta_{16}(\lambda_{22} - 1)\right]$

$\hat{S}^2_{17} = s^2_y\left[\frac{S^2_x + D_6}{s^2_x + D_6}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S^2_y \delta_{17}\left[\delta_{17}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{17}\left(\beta_{2(x)} - 1\right) - 2\delta_{17}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{18} = s^2_y\left[\frac{S^2_x + D_7}{s^2_x + D_7}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S^2_y \delta_{18}\left[\delta_{18}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{18}\left(\beta_{2(x)} - 1\right) - 2\delta_{18}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{19} = s^2_y\left[\frac{S^2_x + D_8}{s^2_x + D_8}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S^2_y \delta_{19}\left[\delta_{19}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{19}\left(\beta_{2(x)} - 1\right) - 2\delta_{10}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{20} = s^2_y\left[\frac{S^2_x + D_9}{s^2_x + D_9}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S^2_y \delta_{20}\left[\delta_{20}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{20}\left(\beta_{2(x)} - 1\right) - 2\delta_{20}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{21} = s^2_y\left[\frac{S^2_x + D_{10}}{s^2_x + D_{10}}\right]$ Subramani and Kumarapandiyan (2012c)	$\gamma S^2_y \delta_{21}\left[\delta_{21}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{21}\left(\beta_{2(x)} - 1\right) - 2\delta_{21}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{22} = s^2_y\left[\frac{\beta_{2(x)}S^2_x + C_x}{\beta_{2(x)}s^2_x + C_x}\right]$ Kadilar and Cingi (2006b)	$\gamma S^2_y \delta_{22}\left[\delta_{22}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{22}\left(\beta_{2(x)} - 1\right) - 2\delta_{22}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{23} = s^2_y\left[\frac{C_x S^2_x + \beta_{2(x)}}{C_x s^2_x + \beta_{2(x)}}\right]$ Kadilar and Cingi (2006b)	$\gamma S^2_y \delta_{23}\left[\delta_{23}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{23}\left(\beta_{2(x)} - 1\right) - 2\delta_{23}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{24} = s^2_y\left[\frac{\beta_{1(x)}S^2_x + C_x}{\beta_{1(x)}s^2_x + C_x}\right]$	$\gamma S^2_y \delta_{24}\left[\delta_{24}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{24}\left(\beta_{2(x)} - 1\right) - 2\delta_{24}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{25} = s^2_y\left[\frac{C_x S^2_x + \beta_{1(x)}}{C_x s^2_x + \beta_{1(x)}}\right]$	$\gamma S^2_y \delta_{25}\left[\delta_{25}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{25}\left(\beta_{2(x)} - 1\right) - 2\delta_{25}(\lambda_{22} - 1)\right]$

$\hat{S}^2_{26} = s^2_y\left[\frac{\rho S^2_x + C_x}{\rho s^2_x + C_x}\right]$	$\gamma S^2_y \delta_{26}\left[\delta_{26}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{26}\left(\beta_{2(x)} - 1\right) - 2\delta_{26}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{27} = s^2_y\left[\frac{C_x S^2_x + \rho}{C_x s^2_x + \rho}\right]$	$\gamma S^2_y \delta_{27}\left[\delta_{27}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{27}\left(\beta_{2(x)} - 1\right) - 2\delta_{27}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{28} = s^2_y\left[\frac{S_x S^2_x + C_x}{S_x s^2_x + C_x}\right]$	$\gamma S^2_y \delta_{28}\left[\delta_{28}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{28}\left(\beta_{2(x)} - 1\right) - 2\delta_{28}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{29} = s^2_y\left[\frac{C_x S^2_x + S_x}{C_x s^2_x + S_x}\right]$	$\gamma S^2_y \delta_{29}\left[\delta_{29}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{29}\left(\beta_{2(x)} - 1\right) - 2\delta_{29}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{30} = s^2_y\left[\frac{M_d S^2_x + C_x}{M_d s^2_x + C_x}\right]$	$\gamma S^2_y \delta_{30}\left[\delta_{30}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{30}\left(\beta_{2(x)} - 1\right) - 2\delta_{30}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{31} = s^2_y\left[\frac{C_x S^2_x + M_d}{C_x s^2_x + M_d}\right]$ Subramani and Kumarapandiyan (2013)	$\gamma S^2_y \delta_{31}\left[\delta_{31}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{31}\left(\beta_{2(x)} - 1\right) - 2\delta_{31}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{32} = s^2_y\left[\frac{\beta_{1(x)} S^2_x + \beta_{2(x)}}{\beta_{1(x)} s^2_x + \beta_{2(x)}}\right]$	$\gamma S^2_y \delta_{32}\left[\delta_{32}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{32}\left(\beta_{2(x)} - 1\right) - 2\delta_{32}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{33} = s^2_y\left[\frac{\beta_{2(x)} S^2_x + \beta_{1(x)}}{\beta_{2(x)} s^2_x + \beta_{1(x)}}\right]$	$\gamma S^2_y \delta_{33}\left[\delta_{33}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{33}\left(\beta_{2(x)} - 1\right) - 2\delta_{33}(\lambda_{22} - 1)\right]$
$\hat{S}^2_{34} = s^2_y\left[\frac{\rho S^2_x + \beta_{2(x)}}{\rho s^2_x + \beta_{2(x)}}\right]$	$\gamma S^2_y \delta_{34}\left[\delta_{34}\left(\beta_{2(x)} - 1\right) - (\lambda_{22} - 1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)} - 1\right) + \delta^2_{34}\left(\beta_{2(x)} - 1\right) - 2\delta_{34}(\lambda_{22} - 1)\right]$

$\hat{S}^2_{35} = s^2_y\left[\frac{\beta_{2(x)}S^2_x+\rho}{\beta_{2(x)}s^2_x+\rho}\right]$	$\gamma S^2_y\delta_{35}\left[\delta_{35}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)}-1\right)+\delta^2_{35}\left(\beta_{2(x)}-1\right)-2\delta_{35}(\lambda_{22}-1)\right]$
$\hat{S}^2_{36} = s^2_y\left[\frac{S_xS^2_x+\beta_{2(x)}}{S_xs^2_x+\beta_{2(x)}}\right]$	$\gamma S^2_y\delta_{36}\left[\delta_{36}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)}-1\right)+\delta^2_{36}\left(\beta_{2(x)}-1\right)-2\delta_{36}(\lambda_{22}-1)\right]$
$\hat{S}^2_{37} = s^2_y\left[\frac{\beta_{2(x)}S^2_x+S_x}{\beta_{2(x)}s^2_x+S_x}\right]$	$\gamma S^2_y\delta_{37}\left[\delta_{37}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)}-1\right)+\delta^2_{37}\left(\beta_{2(x)}-1\right)-2\delta_{37}(\lambda_{22}-1)\right]$
$\hat{S}^2_{38} = s^2_y\left[\frac{M_dS^2_x+\beta_{2(x)}}{M_ds^2_x+\beta_{2(x)}}\right]$	$\gamma S^2_y\delta_{38}\left[\delta_{38}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)}-1\right)+\delta^2_{38}\left(\beta_{2(x)}-1\right)-2\delta_{38}(\lambda_{22}-1)\right]$
$\hat{S}^2_{39} = s^2_y\left[\frac{\beta_{2(x)}S^2_x+M_d}{\beta_{2(x)}s^2_x+M_d}\right]$	$\gamma S^2_y\delta_{39}\left[\delta_{39}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)}-1\right)+\delta^2_{39}\left(\beta_{2(x)}-1\right)-2\delta_{39}(\lambda_{22}-1)\right]$
$\hat{S}^2_{40} = s^2_y\left[\frac{\rho S^2_x+\beta_{1(x)}}{\rho s^2_x+\beta_{1(x)}}\right]$	$\gamma S^2_y\delta_{40}\left[\delta_{40}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)}-1\right)+\delta^2_{40}\left(\beta_{2(x)}-1\right)-2\delta_{40}(\lambda_{22}-1)\right]$
$\hat{S}^2_{41} = s^2_y\left[\frac{\beta_{1(x)}S^2_x+\rho}{\beta_{1(x)}s^2_x+\rho}\right]$	$\gamma S^2_y\delta_{41}\left[\delta_{41}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)}-1\right)+\delta^2_{41}\left(\beta_{2(x)}-1\right)-2\delta_{41}(\lambda_{22}-1)\right]$
$\hat{S}^2_{42} = s^2_y\left[\frac{S_xS^2_x+\beta_{1(x)}}{S_xs^2_x+\beta_{1(x)}}\right]$	$\gamma S^2_y\delta_{42}\left[\delta_{42}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)}-1\right)+\delta^2_{42}\left(\beta_{2(x)}-1\right)-2\delta_{42}(\lambda_{22}-1)\right]$
$\hat{S}^2_{43} = s^2_y\left[\frac{\beta_{1(x)}S^2_x+S_x}{\beta_{1(x)}s^2_x+S_x}\right]$	$\gamma S^2_y\delta_{43}\left[\delta_{43}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S^4_y\left[\left(\beta_{2(y)}-1\right)+\delta^2_{43}\left(\beta_{2(x)}-1\right)-2\delta_{43}(\lambda_{22}-1)\right]$

$\hat{S}^2_{44} = s_y^2\left[\frac{M_d S_x^2+\beta_{1(x)}}{M_d s_x^2+\beta_{1(x)}}\right]$	$\gamma S_y^2\delta_{44}\left[\delta_{44}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)}-1\right)+\delta_{44}^2\left(\beta_{2(x)}-1\right)-2\delta_{44}(\lambda_{22}-1)\right]$
$\hat{S}^2_{45} = s_y^2\left[\frac{\beta_{1(x)} S_x^2+M_d}{\beta_{1(x)} s_x^2+M_d}\right]$	$\gamma S_y^2\delta_{45}\left[\delta_{45}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)}-1\right)+\delta_{45}^2\left(\beta_{2(x)}-1\right)-2\delta_{45}(\lambda_{22}-1)\right]$
$\hat{S}^2_{46} = s_y^2\left[\frac{S_x S_x^2+\rho}{S_x s_x^2+\rho}\right]$	$\gamma S_y^2\delta_{46}\left[\delta_{46}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)}-1\right)+\delta_{46}^2\left(\beta_{2(x)}-1\right)-2\delta_{46}(\lambda_{22}-1)\right]$
$\hat{S}^2_{47} = s_y^2\left[\frac{\rho S_x^2+S_x}{\rho s_x^2+S_x}\right]$	$\gamma S_y^2\delta_{47}\left[\delta_{47}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)}-1\right)+\delta_{47}^2\left(\beta_{2(x)}-1\right)-2\delta_{47}(\lambda_{22}-1)\right]$
$\hat{S}^2_{48} = s_y^2\left[\frac{M_d S_x^2+\rho}{S_x s_x^2+\rho}\right]$	$\gamma S_y^2\delta_{48}\left[\delta_{48}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)}-1\right)+\delta_{48}^2\left(\beta_{2(x)}-1\right)-2\delta_{48}(\lambda_{22}-1)\right]$
$\hat{S}^2_{49} = s_y^2\left[\frac{\rho S_x^2+M_d}{\rho s_x^2+M_d}\right]$	$\gamma S_y^2\delta_{49}\left[\delta_{49}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)}-1\right)+\delta_{49}^2\left(\beta_{2(x)}-1\right)-2\delta_{49}(\lambda_{22}-1)\right]$
$\hat{S}^2_{50} = s_y^2\left[\frac{M_d S_x^2+S_x}{M_d s_x^2+S_x}\right]$	$\gamma S_y^2\delta_{50}\left[\delta_{50}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)}-1\right)+\delta_{50}^2\left(\beta_{2(x)}-1\right)-2\delta_{50}(\lambda_{22}-1)\right]$
$\hat{S}^2_{51} = s_y^2\left[\frac{S_x S_x^2+M_d}{S_x s_x^2+M_d}\right]$	$\gamma S_y^2\delta_{51}\left[\delta_{51}\left(\beta_{2(x)}-1\right)-(\lambda_{22}-1)\right]$	$\gamma S_y^4\left[\left(\beta_{2(y)}-1\right)+\delta_{51}^2\left(\beta_{2(x)}-1\right)-2\delta_{51}(\lambda_{22}-1)\right]$

where $\delta_i = \frac{S_x^2}{S_x^2+\omega_i}; i = 1,2,3,\ldots,51; \omega_1 = C_x, \omega_2 = \beta_{2(x)}, \omega_3 = \beta_{1(x)}, \omega_4 = \rho,$
$\omega_5 = S_x, \omega_6 = M_d, \omega_7 = Q_1, \omega_8 = Q_3, \omega_9 = Q_r, \omega_{10} = Q_d, \omega_{11} = Q_a, \omega_{12} = D_1,$
$\omega_{13} = D_2, \omega_{14} = D_3, \omega_{15} = D_4, \omega_{16} = D_5, \omega_{17} = D_6, \omega_{18} = D_7, \omega_{19} = D_8, \omega_{20} = D_9,$

$\omega_{21}=D_{10}, \omega_{22}=\frac{C_x}{\beta_{2(x)}}, \omega_{23}=\frac{\beta_{2(x)}}{C_x}, \omega_{24}=\frac{C_x}{\beta_{1(x)}}, \omega_{25}=\frac{\beta_{1(x)}}{C_x}, \omega_{26}=\frac{C_x}{\rho}, \omega_{27}=\frac{\rho}{C_x}$

$\omega_{28}=\frac{C_x}{S_x}, \omega_{29}=\frac{S_x}{C_x}, \omega_{30}=\frac{C_x}{M_d}, \omega_{31}=\frac{M_d}{C_x}, \omega_{32}=\frac{\beta_{2(x)}}{\beta_{1(x)}}, \omega_{33}=\frac{\beta_{1(x)}}{\beta_{2(x)}}, \omega_{34}=\frac{\beta_{2(x)}}{\rho}$

$\omega_{35}=\frac{\rho}{\beta_{2(x)}}, \omega_{36}=\frac{\beta_{2(x)}}{S_x}, \omega_{37}=\frac{S_x}{\beta_{2(x)}}, \omega_{38}=\frac{\beta_{2(x)}}{M_d}, \omega_{39}=\frac{M_d}{\beta_{2(x)}}, \omega_{40}=\frac{\beta_{1(x)}}{\rho}, \omega_{41}=\frac{\rho}{\beta_{1(x)}}$

$\omega_{42}=\frac{\beta_{1(x)}}{S_x}, \omega_{43}=\frac{S_x}{\beta_{1(x)}}, \omega_{44}=\frac{\beta_{1(x)}}{M_d}, \omega_{45}=\frac{M_d}{\beta_{1(x)}}, \omega_{46}=\frac{\rho}{S_x}, \omega_{47}=\frac{S_x}{\rho}, \omega_{48}=\frac{\beta_{1(x)}}{M_d},$

$\omega_{49}=\frac{M_d}{\beta_{1(x)}}, \omega_{50}=\frac{S_x}{M_d}$ and $\omega_{51}=\frac{M_d}{S_x}$

Upadhyaya and Singh (2001) have suggested the following modified ratio type variance estimator using the population mean of the auxiliary variable together with its bias and mean squared error are given below:

$$\hat{S}_{52}^2=s_y^2\left[\frac{\bar{X}}{\bar{x}}\right] \tag{1.4}$$

$$B(\hat{S}_{52}^2)=\gamma S_y^2[C_x^2-\lambda_{21}C_x] \tag{1.5}$$

$$MSE(\hat{S}_{52}^2)=\gamma S_y^4\left[\left(\beta_{2(y)}-1\right)+C_x^2-2\lambda_{21}C_x\right] \text{ where } \lambda_{21}=\frac{\mu_{21}}{\mu_{20}\sqrt{\mu_{02}}} \tag{1.6}$$

The modified ratio type variance estimators discussed above are biased but have smaller mean squared errors compared to the traditional ratio type variance estimator suggested by Isaki (1983) under certain conditions. The list of estimators given in Table 1 uses the known values of the parameters and their linear combinations and improved the traditional ratio type estimator. In this paper an attempt has been made to modify the ratio type variance estimator suggested by Upadhyaya and Singh (2001) using known parameters of the auxiliary variable and its linear combination.

The materials of the present study are arranged as given below. The proposed estimators using known parameters of the auxiliary variable are presented in section 2 where as the conditions in which the proposed estimators perform better than the traditional and existing modified estimators are derived in section 3. The performance of the proposed estimators with that of the traditional and existing modified estimators are assessed for certain natural populations in section 4 and the conclusion is presented in section 5.

2. PROPOSED ESTIMATORS

In this section we have suggested a class of modified ratio type variance estimators using the known parameters of the auxiliary variable for estimating the population variance of the study variable Y. The proposed class of modified ratio type variance estimators $\hat{S}^2_{p_i}$, $i = 1,2, \ldots ,51$ for estimating the population variance S^2_y is given below:

$$\hat{S}^2_{p_i} = s^2_y \left[\frac{\bar{X}+\omega_i}{\bar{x}+\omega_i}\right]; i = 1,2,3, \ldots ,51 \tag{2.1}$$

The bias and mean squared error of the proposed estimators $\hat{S}^2_{p_i}$, $i = 1,2, \ldots ,51$ have been derived (see Appendix A) and are given below

$$\text{Bias}\left(\hat{S}^2_{p_i}\right) = \frac{(1-f)}{n} S^2_y\left(\theta^2_{p_i} C^2_x - \theta_{p_i}\lambda_{21} C_x\right) \tag{2.2}$$

$$\text{MSE}\left(\hat{S}^2_{p_i}\right) = \frac{(1-f)}{n} S^4_y \left(\left(\beta_{2(y)} - 1\right) + \theta^2_{p_i} C^2_x - 2\theta_{p_i}\lambda_{21} C_x\right); i = 1, 2, 3, \ldots ,51 \tag{2.3}$$

where $\theta_{p_i} = \frac{\bar{X}}{\bar{X}+\omega_i}$

REMARK 2.1 When the study variable Y and auxiliary variable X are negatively correlated and the population parameters of the auxiliary variable are known, the following modified product type variance estimators can be proposed:

$$\hat{S}^2_{p_i} = s^2_y \left[\frac{\bar{x}+\omega_i}{\bar{X}+\omega_i}\right] i = 1,2,3, \ldots ,51 \tag{2.4}$$

3. EFFICIENCY OF THE PROPOSED ESTIMATORS

As we mentioned earlier the mean squared error of the traditional ratio type variance estimator $\hat{\mathbf{S}}^2_{\mathbf{r}}$ is given below:

$$\text{MSE}\left(\hat{S}^2_R\right) = \gamma S^4_y \left[\left(\beta_{2(y)} - 1\right) + \left(\beta_{2(x)} - 1\right) - 2(\lambda_{22} - 1)\right] \tag{3.1}$$

The mean squared error of the modified ratio type variance estimators $\hat{\mathbf{S}}^2_{\mathbf{i}}$ given in table 1 are represented in single class as given below:

$$MSE(\hat{S}_i^2) = \gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + \delta_i^2\left(\beta_{2(x)} - 1\right) - 2\delta_i(\lambda_{22} - 1)\right]; i = 1,2,3,\dots,51 \quad (3.2)$$

The mean squared error of the modified ratio type variance estimators $\hat{\mathbf{S}}_{52}^2$ suggested by upadhyaya and singh (2001) is given below:

$$MSE(\hat{S}_{52}^2) = \gamma S_y^4\left[\left(\beta_{2(y)} - 1\right) + C_x^2 - 2\lambda_{21}C_x\right] \quad (3.3)$$

The mean squared errors of the proposed modified ratio type variance estimators are given below:

$$MSE(\hat{S}_{p_i}^2) = \frac{(1-f)}{n}S_y^4\left(\left(\beta_{2(y)} - 1\right) + \theta_{p_i}^2C_x^2 - 2\theta_{p_i}\lambda_{21}C_x\right); i = 1, 2, 3, \dots, 51 \quad (3.4)$$

From the expressions given in (3.1) and (3.4) we have derived (see Appendix B) the condition for which the proposed estimators $\hat{\mathbf{S}}_{\mathbf{p_i}}^2$, $\mathbf{i = 1, 2, 3, \dots, 51}$ are more efficient than the traditional ratio type variance estimator and it is given below:

$$MSE(\hat{S}_{p_i}^2) < MSE(\hat{S}_R^2) \text{ if } \theta_{p_i} \le \frac{\lambda_{21} + \left(\left(\beta_{2(x)} - 1\right) - 2(\lambda_{22} - 1) + \lambda_{21}^2\right)^{\frac{1}{2}}}{C_x} \quad (3.5)$$

From the expressions given in (3.2) and (3.4) we have derived (see Appendix C) the conditions for which the proposed estimators $\mathbf{S}_{\mathbf{p_i}}^2$, $\mathbf{i = 1, 2, 3, \dots, 51}$ are more efficient than the modified ratio type variance estimators given in table 1, $\hat{\mathbf{S}}_{\mathbf{i}}^2; \mathbf{i = 1, 2, 3, \dots, 51}$ and are given below:

$$MSE(\hat{S}_{p_i}^2) < MSE(\hat{S}_i^2) \text{ if } \theta_{p_i} \le \frac{\lambda_{21} + \left(\delta_i^2\left(\beta_{2(x)} - 1\right) - 2\delta_i(\lambda_{22} - 1) + \lambda_{21}^2\right)^{\frac{1}{2}}}{C_x} \quad (3.6)$$

From the expressions given in (3.3) and (3.4) we have derived (see Appendix D) the conditions for which the proposed estimators $\hat{\mathbf{S}}_{\mathbf{p_i}}^2$, $\mathbf{i = 1, 2, 3, \dots, 51}$ are more efficient than the modified ratio type variance estimator $\hat{\mathbf{S}}_{52}^2$ and are given below:

$$MSE(\hat{S}_{p_i}^2) \le MSE(\hat{S}_{52}^2) \text{ either } 2\frac{\lambda_{21}}{C_x} - 1 \le \theta_{p_i} \le 1 \,(\mathbf{or})\, 1 \le \theta_{p_i} \le 2\frac{\lambda_{21}}{C_x} - 1 \quad (3.7)$$

4. NUMERICAL STUDY

The performance of the proposed modified ratio type variance estimators listed in table 2 are assessed with that of traditional ratio type estimator and existing modified ratio type variance estimators listed for certain natural populations. The populations 1 and 2 are taken from Singh and Chaudhary (1986) given in page 177. The population parameters and the constants computed from the above populations are given below:

Population 1: Singh and Chaudhary (1986)

$N = 34$	$n = 20$	$\bar{Y} = 85.6412$	$\bar{X} = 20.8882$	$S_y = 73.3141$
$\lambda_{22} = 1.1525$	$S_x = 15.0506$	$C_x = 0.7205$	$\beta_{2(y)} = 13.3666$	$\beta_{2(x)} = 2.9123$
$\beta_{1(x)} = 0.8732$	$\lambda_{21} = -0.3104$	$M_d = 15$	$Q_1 = 9.425$	$Q_3 = 25.475$
$Q_r = 16.05$	$Q_d = 16.05$	$Q_a = 17.45$	$D_1 = 7.03$	$D_2 = 7.68$
$D_3 = 10.82$	$D_4 = 12.94$	$D_5 = 15$	$D_6 = 22.72$	$D_7 = 25.04$
$D_8 = 33.56$	$D_9 = 43.61$	$D_{10} = 56.4$		

Population 2: Singh and Chaudhary (1986)

$N = 34$	$n = 20$	$\bar{Y} = 85.6412$	$\bar{X} = 19.9441$	$S_y = 73.3141$
$\lambda_{22} = 1.2244$	$S_x = 15.0215$	$C_x = 0.7532$	$\beta_{2(y)} = 13.3666$	$\beta_{2(x)} = 3.7257$
$\beta_{1(x)} = 1.2758$	$\lambda_{21} = -0.2946$	$M_d = 14.25$	$Q_1 = 9.925$	$Q_3 = 27.8$
$Q_r = 17.875$	$Q_d = 8.9375$	$Q_a = 18.8625$	$D_1 = 6.06$	$D_2 = 8.3$
$D_3 = 10.27$	$D_4 = 11.12$	$D_5 = 14.25$	$D_6 = 21.02$	$D_7 = 26.45$
$D_8 = 30.44$	$D_9 = 37.32$	$D_{10} = 63.4$		

5.

The mean squared error of the existing and proposed modified ratio type variance estimators for the above population given below:

Table 2 MSE(.) of the existing and proposed modified ratio type variance estimators for population 1

Existing Estimator	Bias(.)	MSE(.)	Proposed Estimator	Bias(.)	MSE(.)
$\hat{S}_{1}^{2}$	193.4567	8305040.9247	$\hat{S}_{p_{1}}^{2}$	77.6142	7901409.5289
$\hat{S}_{2}^{2}$	189.6191	8285277.5233	$\hat{S}_{p_{2}}^{2}$	65.9785	7827023.2643
$\hat{S}_{3}^{2}$	193.1857	8303644.9272	$\hat{S}_{p_{3}}^{2}$	76.6955	7895569.2207
$\hat{S}_{4}^{2}$	193.9398	8307529.5562	$\hat{S}_{p_{4}}^{2}$	79.2931	7912069.8376
$\hat{S}_{5}^{2}$	170.2472	8185653.1155	$\hat{S}_{p_{5}}^{2}$	33.7963	7614602.8707
$\hat{S}_{6}^{2}$	170.3219	8186036.5850	$\hat{S}_{p_{6}}^{2}$	33.8714	7615115.3167
$\hat{S}_{7}^{2}$	194.2999	8309384.7128	$\hat{S}_{p_{7}}^{2}$	80.5802	7920230.8059
$\hat{S}_{8}^{2}$	187.6845	8275317.6357	$\hat{S}_{p_{8}}^{2}$	61.0748	7795375.0862
$\hat{S}_{9}^{2}$	193.2710	8304084.3017	$\hat{S}_{p_{9}}^{2}$	76.9829	7897396.7792
$\hat{S}_{10}^{2}$	192.5865	8300558.7975	$\hat{S}_{p_{10}}^{2}$	74.7213	7883000.6155
$\hat{S}_{11}^{2}$	191.8956	8297000.4445	$\hat{S}_{p_{11}}^{2}$	72.5377	7869069.1599
$\hat{S}_{12}^{2}$	193.6296	8305931.4127	$\hat{S}_{p_{12}}^{2}$	78.2088	7905186.9248
$\hat{S}_{13}^{2}$	194.6572	8311225.1070	$\hat{S}_{p_{13}}^{2}$	81.8881	7928514.0560
$\hat{S}_{14}^{2}$	161.9435	8143027.5287	$\hat{S}_{p_{14}}^{2}$	26.7408	7565872.9971
$\hat{S}_{15}^{2}$	194.6569	8311223.6155	$\hat{S}_{p_{15}}^{2}$	81.8870	7928507.2658
$\hat{S}_{16}^{2}$	162.0399	8143521.8877	$\hat{S}_{p_{16}}^{2}$	26.8101	7566357.2506
$\hat{S}_{17}^{2}$	188.8916	8281531.8967	$\hat{S}_{p_{17}}^{2}$	64.0680	7814716.2766
$\hat{S}_{18}^{2}$	194.2062	8308901.9409	$\hat{S}_{p_{18}}^{2}$	80.2422	7918089.0587
$\hat{S}_{19}^{2}$	183.5978	8254285.7975	$\hat{S}_{p_{19}}^{2}$	52.3462	7738509.3354
$\hat{S}_{20}^{2}$	194.4667	8310243.7205	$\hat{S}_{p_{20}}^{2}$	81.1867	7924073.4637
$\hat{S}_{21}^{2}$	194.3964	8309881.4741	$\hat{S}_{p_{21}}^{2}$	80.9301	7922448.0103
$\hat{S}_{22}^{2}$	185.7842	8265536.8722	$\hat{S}_{p_{22}}^{2}$	56.7674	7767404.1126
$\hat{S}_{23}^{2}$	194.3952	8309875.4575	$\hat{S}_{p_{23}}^{2}$	80.9259	7922421.0747
$\hat{S}_{24}^{2}$	185.8133	8265686.6605	$\hat{S}_{p_{24}}^{2}$	56.8299	7767811.3938
$\hat{S}_{25}^{2}$	191.3000	8293932.9140	$\hat{S}_{p_{25}}^{2}$	70.7307	7857516.3340
$\hat{S}_{26}^{2}$	193.8236	8306930.9379	$\hat{S}_{p_{26}}^{2}$	78.8843	7909475.9332
$\hat{S}_{27}^{2}$	194.6390	8311131.4101	$\hat{S}_{p_{27}}^{2}$	81.8207	7928087.7466
$\hat{S}_{28}^{2}$	167.0682	8169328.3846	$\hat{S}_{p_{28}}^{2}$	30.8108	7594123.3077
$\hat{S}_{29}^{2}$	194.6386	8311129.6027	$\hat{S}_{p_{29}}^{2}$	81.8194	7928079.5282
$\hat{S}_{30}^{2}$	167.1514	8169755.4807	$\hat{S}_{p_{30}}^{2}$	30.8840	7594627.7126
$\hat{S}_{31}^{2}$	194.6895	8311391.6909	$\hat{S}_{p_{31}}^{2}$	82.0080	7929273.2247
$\hat{S}_{32}^{2}$	145.8871	8060755.6421	$\hat{S}_{p_{32}}^{2}$	17.9756	7503320.8713
$\hat{S}_{33}^{2}$	194.6893	8311390.7610	$\hat{S}_{p_{33}}^{2}$	82.0073	7929268.9825
$\hat{S}_{34}^{2}$	146.0200	8061435.6153	$\hat{S}_{p_{34}}^{2}$	18.0305	7503722.1084

$\hat{S}_{35}^2$	192.9551	8302457.3932	$\hat{S}_{p_{35}}^2$	75.9268	7890677.8152
$\hat{S}_{36}^2$	192.9670	8302518.7994	$\hat{S}_{p_{36}}^2$	75.9662	7890929.0386
$\hat{S}_{37}^2$	178.8485	8229857.3273	$\hat{S}_{p_{37}}^2$	44.3384	7685618.9421
$\hat{S}_{38}^2$	155.8258	8111656.3586	$\hat{S}_{p_{38}}^2$	22.8150	7538189.7352
$\hat{S}_{39}^2$	168.7823	8178129.8387	$\hat{S}_{p_{39}}^2$	32.3710	7604849.6035
$\hat{S}_{40}^2$	181.0872	8241370.6859	$\hat{S}_{p_{40}}^2$	47.8702	7709041.6847
$\hat{S}_{41}^2$	166.7609	8167750.7848	$\hat{S}_{p_{41}}^2$	30.5426	7592273.8712
$\hat{S}_{42}^2$	182.7032	8249683.3804	$\hat{S}_{p_{42}}^2$	50.6830	7727586.6369
$\hat{S}_{43}^2$	181.6452	8244240.6215	$\hat{S}_{p_{43}}^2$	48.8148	7715279.6791
$\hat{S}_{44}^2$	176.6576	8218593.1860	$\hat{S}_{p_{44}}^2$	41.2404	7664933.2877
$\hat{S}_{45}^2$	173.4022	8201861.6358	$\hat{S}_{p_{45}}^2$	37.1913	7637676.2776
$\hat{S}_{46}^2$	170.3219	8186036.5850	$\hat{S}_{p_{46}}^2$	33.8714	7615115.3167
$\hat{S}_{47}^2$	159.4592	8130284.6462	$\hat{S}_{p_{47}}^2$	25.0396	7553934.3985
$\hat{S}_{48}^2$	156.3915	8114555.7223	$\hat{S}_{p_{48}}^2$	23.1425	7540518.2755
$\hat{S}_{49}^2$	145.8315	8060471.0773	$\hat{S}_{p_{49}}^2$	17.9527	7503153.4053
$\hat{S}_{50}^2$	134.6559	8003341.2091	$\hat{S}_{p_{50}}^2$	14.0431	7474185.5247
$\hat{S}_{51}^2$	122.1457	7939536.8330	$\hat{S}_{p_{51}}^2$	10.8871	7450090.5703
$\hat{S}_{52}^2$	82.2071	7930533.2085			
$\hat{S}_{R}^2$	152.5580	8311667.4056			

Table 3 $\mathbf{MSE}(.)$ of the existing and proposed modified ratio type variance estimators for population 2

Existing Estimator	$\mathbf{Bias}(.)$	$\mathbf{MSE}(.)$	Proposed Estimator	$\mathbf{Bias}(.)$	$\mathbf{MSE}(.)$
$\hat{S}_1^2$	274.8748	8700066.3664	$\hat{S}_{p_1}^2$	81.9543	7923335.0831
$\hat{S}_2^2$	267.4800	8662043.5448	$\hat{S}_{p_2}^2$	65.2611	7817638.5301
$\hat{S}_3^2$	273.5533	8693269.2999	$\hat{S}_{p_3}^2$	78.5355	7901826.3797
$\hat{S}_4^2$	275.6578	8704093.4323	$\hat{S}_{p_4}^2$	84.0855	7936710.5220
$\hat{S}_5^2$	241.8679	8530542.5925	$\hat{S}_{p_5}^2$	34.4309	7616001.5114
$\hat{S}_6^2$	243.5012	8538919.0760	$\hat{S}_{p_6}^2$	35.6790	7624408.3942
$\hat{S}_7^2$	276.2783	8707285.4442	$\hat{S}_{p_7}^2$	85.8340	7947666.4656
$\hat{S}_8^2$	264.5267	8646864.7198	$\hat{S}_{p_8}^2$	59.9815	7783806.2406
$\hat{S}_9^2$	275.2885	8702194.1132	$\hat{S}_{p_9}^2$	83.0702	7930341.3699
$\hat{S}_{10}^2$	272.5028	8687867.1404	$\hat{S}_{p_{10}}^2$	75.9674	7885626.4956
$\hat{S}_{11}^2$	272.5089	8687898.6943	$\hat{S}_{p_{11}}^2$	75.9820	7885718.9285
$\hat{S}_{12}^2$	275.2863	8702182.9335	$\hat{S}_{p_{12}}^2$	83.0643	7930304.2040
$\hat{S}_{13}^2$	276.6674	8709286.6597	$\hat{S}_{p_{13}}^2$	86.9579	7954701.1868
$\hat{S}_{14}^2$	231.8128	8479004.9265	$\hat{S}_{p_{14}}^2$	27.9728	7571999.8359
$\hat{S}_{15}^2$	276.6604	8709250.8872	$\hat{S}_{p_{15}}^2$	86.9377	7954574.2911
$\hat{S}_{16}^2$	233.8546	8489465.8908	$\hat{S}_{p_{16}}^2$	29.1347	7579983.9267
$\hat{S}_{17}^2$	269.4550	8672196.4026	$\hat{S}_{p_{17}}^2$	69.1858	7842651.4287
$\hat{S}_{18}^2$	275.9200	8705442.4919	$\hat{S}_{p_{18}}^2$	84.8179	7941301.8389
$\hat{S}_{19}^2$	256.5008	8605634.9323	$\hat{S}_{p_{19}}^2$	48.4540	7709070.5888
$\hat{S}_{20}^2$	276.4897	8708372.8292	$\hat{S}_{p_{20}}^2$	86.4420	7951472.7129
$\hat{S}_{21}^2$	276.1611	8706682.5452	$\hat{S}_{p_{21}}^2$	85.4996	7945572.4829
$\hat{S}_{22}^2$	266.7349	8658213.4456	$\hat{S}_{p_{22}}^2$	63.8659	7808719.1725
$\hat{S}_{23}^2$	276.1268	8706506.0880	$\hat{S}_{p_{23}}^2$	85.4021	7944961.8077
$\hat{S}_{24}^2$	267.2385	8660802.2542	$\hat{S}_{p_{24}}^2$	64.8041	7814718.1590
$\hat{S}_{25}^2$	269.5908	8672894.5544	$\hat{S}_{p_{25}}^2$	69.4685	7844448.5663
$\hat{S}_{26}^2$	275.9032	8705356.1073	$\hat{S}_{p_{26}}^2$	84.7708	7941006.1426
$\hat{S}_{27}^2$	276.5782	8708828.2800	$\hat{S}_{p_{27}}^2$	86.6986	7953078.3599
$\hat{S}_{28}^2$	248.8531	8566375.7830	$\hat{S}_{p_{28}}^2$	40.2619	7655051.1228
$\hat{S}_{29}^2$	276.5665	8708767.7148	$\hat{S}_{p_{29}}^2$	86.6644	7952864.4529
$\hat{S}_{30}^2$	250.1860	8573216.2487	$\hat{S}_{p_{30}}^2$	41.5362	7663513.4206
$\hat{S}_{31}^2$	276.7199	8709556.8046	$\hat{S}_{p_{31}}^2$	87.1113	7955660.8324
$\hat{S}_{32}^2$	206.6709	8350388.1091	$\hat{S}_{p_{32}}^2$	17.7903	7500315.7683
$\hat{S}_{33}^2$	276.7158	8709535.6494	$\hat{S}_{p_{33}}^2$	87.0993	7955585.5952
$\hat{S}_{34}^2$	209.6060	8365383.3370	$\hat{S}_{p_{34}}^2$	18.6824	7506746.2177

$\hat{S}^2_{35}$	274.1127	8696146.6470	$\hat{S}^2_{p_{35}}$	79.9563	7910773.1951
$\hat{S}^2_{36}$	274.3795	8697518.9442	$\hat{S}^2_{p_{36}}$	80.6475	7915121.2084
$\hat{S}^2_{37}$	252.9612	8587461.1638	$\hat{S}^2_{p_{37}}$	44.3855	7682352.0994
$\hat{S}^2_{38}$	216.9786	8403072.8621	$\hat{S}^2_{p_{38}}$	21.2121	7524802.2329
$\hat{S}^2_{39}$	235.9633	8500271.5781	$\hat{S}^2_{p_{39}}$	30.4083	7588699.7565
$\hat{S}^2_{40}$	255.1960	8598934.7234	$\hat{S}^2_{p_{40}}$	46.8929	7698842.0318
$\hat{S}^2_{41}$	233.9686	8490049.7272	$\hat{S}^2_{p_{41}}$	29.2015	7580442.3507
$\hat{S}^2_{42}$	261.8741	8633234.9291	$\hat{S}^2_{p_{42}}$	55.7610	7756593.1089
$\hat{S}^2_{43}$	256.6540	8606421.4282	$\hat{S}^2_{p_{43}}$	48.6422	7710301.9522
$\hat{S}^2_{44}$	252.1872	8583487.6588	$\hat{S}^2_{p_{44}}$	43.5628	7676923.4831
$\hat{S}^2_{45}$	250.2947	8573774.1884	$\hat{S}^2_{p_{45}}$	41.6428	7664219.6974
$\hat{S}^2_{46}$	243.5012	8538919.0760	$\hat{S}^2_{p_{46}}$	35.6790	7624408.3942
$\hat{S}^2_{47}$	229.6949	8468156.2743	$\hat{S}^2_{p_{47}}$	26.8362	7564156.3652
$\hat{S}^2_{48}$	219.4277	8415600.5132	$\hat{S}^2_{p_{48}}$	22.1574	7531487.3776
$\hat{S}^2_{49}$	212.3024	8379163.5680	$\hat{S}^2_{p_{49}}$	19.5567	7513015.7817
$\hat{S}^2_{50}$	200.7812	8320314.8836	$\hat{S}^2_{p_{50}}$	16.1672	7488520.5228
$\hat{S}^2_{51}$	164.4345	8135286.8030	$\hat{S}^2_{p_{51}}$	9.4709	7438144.1980
$\hat{S}^2_{52}$	87.3340	7957053.4638			
$\hat{S}^2_{R}$	210.6359	8709947.6355			

From the values of table 2 and table 3, it is observed that the bias of the proposed modified ratio type variance estimators are less than the bias of the traditional and existing modified ratio type variance estimators. Similarly, it is observed that the mean squared error of the proposed modified ratio type variance estimators are less than the mean squared error of the traditional and existing modified ratio type variance estimators.

6. CONCLUSION

In this paper a class of modified ratio type variance estimators has been proposed using the known parameters of the auxiliary variable. The bias and mean squared error of the proposed modified ratio type variance estimators are derived. Further we have derived the conditions for which the proposed estimators are more efficient than the traditional and existing modified ratio type variance estimators. We have also assessed the performances of the proposed estimators with that of the existing estimators for two natural populations. It is observed from the numerical comparison that the bias and mean squared error of the proposed estimators are less than the bias and mean squared error of the traditional and existing estimators. Hence we strongly recommend that the proposed modified ratio type variance estimators may be preferred over the traditional ratio type variance estimator and modified ratio type variance estimators for the use of practical applications.

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Appendix-A

We have derived the expression for the bias and mean squared error of the proposed estimators $\hat{S}^2_{p_i}: i = 1,2,3, \dots ,51$ to first order of approximation with the following notations:

Let $e_0 = \frac{s_y^2 - S_y^2}{S_y^2}$ and $e_1 = \frac{\bar{x} - \bar{X}}{\bar{X}}$. Further we can write $s_y^2 = S_y^2(1 + e_0)$

and $\bar{x} = \bar{X}(1 + e_1)$

and from the definition of e_0 and e_1 we obtain:

$E[e_0] = E[e_1] = 0$

$E[e_0^2] = \frac{(1-f)}{n}\left(\beta_{2(y)} - 1\right)$

$E[e_1^2] = \frac{(1-f)}{n}C_x^2$

$E[e_0e_1] = \frac{(1-f)}{n}\lambda_{21}C_x$

The bias of the proposed estimators $\hat{S}^2_{p_i}: i = 1,2,3, \dots ,51$ is derived as given below:

$$\hat{S}^2_{p_i} = s_y^2\left[\frac{\bar{X} + \eta_i}{\bar{x} + \eta_i}\right]; i = 1,2,3, \dots ,51$$

$$\Rightarrow \hat{S}^2_{p_i} = \frac{s_y^2}{\left(\bar{X} + e_1\bar{X} + \eta_i\right)}\left(\bar{X} + \eta_i\right)$$

$$\Rightarrow \hat{S}^2_{p_i} = \frac{s_y^2}{\left(\bar{X} + \eta_i\right)\left(1 + \frac{e_1\bar{X}}{\bar{X} + \eta_i}\right)}\left(\bar{X} + \eta_i\right)$$

$$\Rightarrow \hat{S}^2_{p_i} = \frac{s_y^2}{\left(1 + \theta_{p_i}e_1\right)} \text{ where } \theta_{p_i} = \frac{\bar{X}}{\bar{X} + \eta_i}$$

$$\Rightarrow \hat{S}^2_{p_i} = s_y^2\left(1 + \theta_{p_i}e_1\right)^{-1}$$

$$\Rightarrow \hat{S}^2_{p_i} = s_y^2\left(1 - \theta_{p_i}e_1 + \theta_{p_i}^2e_1^2 - \theta_{p_i}^3e_1^3 + \cdots\right)$$

Neglecting the terms more than 2nd order, we will get

$$\hat{S}^2_{p_i} = s_y^2\left(1 - \theta_{p_i}e_1 + \theta_{p_i}^2e_1^2\right)$$

$$\Rightarrow \hat{S}^2_{p_i} = \left(S_y^2(1 + e_0)\right)\left(1 - \theta_{p_i}e_1 + \theta_{p_i}^2e_1^2\right)$$

$$\Rightarrow \hat{S}^2_{p_i} = \left(S_y^2 + S_y^2e_0\right)\left(1 - \theta_{p_i}e_1 + \theta_{p_i}^2e_1^2\right)$$

$$\Rightarrow \hat{S}^2_{p_i} = S_y^2 + S_y^2e_0 - S_y^2\theta_{p_i}e_1 - S_y^2\theta_{p_i}e_0e_1 + S_y^2\theta_{p_i}^2e_1^2 + S_y^2\theta_{p_i}^2e_0e_1^2$$

Neglecting the terms more than 3rd order, we will get

$$\hat{S}_{p_i}^2 = S_y^2 + S_y^2 e_0 - S_y^2\theta_{p_i}e_1 - S_y^2\theta_{p_i}e_0e_1 + S_y^2\theta_{p_i}^2 e_1^2$$

$$\Rightarrow \hat{S}_{p_i}^2 - S_y^2 = S_y^2 e_0 - S_y^2\theta_{p_i}e_1 - S_y^2\theta_{p_i}e_0e_1 + S_y^2\theta_{p_i}^2 e_1^2$$

Taking expectation on both sides, we will get

$$E\left(\hat{S}_{p_i}^2 - S_y^2\right) = S_y^2 E(e_0) - S_y^2\theta_{p_i}E(e_1) - S_y^2\theta_{p_i}E(e_0e_1) + S_y^2\theta_{p_i}^2 E(e_1^2)$$

$$\Rightarrow \mathrm{Bias}\left(\hat{S}_{p_i}^2\right) = S_y^2\theta_{p_i}^2 E(e_1^2) - S_y^2\theta_{p_i}E(e_0e_1)$$

$$\Rightarrow \mathrm{Bias}\left(\hat{S}_{p_i}^2\right) = S_y^2\theta_{p_i}^2\frac{(1-f)}{n}C_x^2 - S_y^2\frac{(1-f)}{n}\lambda_{21}C_x\theta_{p_i}$$

$$\Rightarrow \mathrm{Bias}\left(\hat{S}_{p_i}^2\right) = \frac{(1-f)}{n}\left(S_y^2\theta_{p_i}^2 C_x^2 - S_y^2\lambda_{21}C_x\theta_{p_i}\right)$$

$$\Rightarrow \mathrm{Bias}\left(\hat{S}_{p_i}^2\right) = \gamma S_y^2\left(\theta_{p_i}^2 C_x^2 - \theta_{p_i}\lambda_{21}C_x\right) \text{ where } \theta_{p_i} = \frac{\bar{X}}{\bar{X}+\eta_i}$$

The mean squared error of the class of proposed estimators $\hat{S}_{p_i}^2 : i = 1, 2, 3, \ldots, 51$ to first order of approximation is derived as given below:

$$\hat{S}_{p_i}^2 = s_y^2\left[\frac{\bar{X}+\eta_i}{\bar{x}+\eta_i}\right]; i = 1,2,3,\ldots,51$$

$$\Rightarrow \hat{S}_{p_i}^2 = \frac{s_y^2}{\left(\bar{X}+e_1\bar{X}+\eta_i\right)}\left(\bar{X}+\eta_i\right)$$

$$\Rightarrow \hat{S}_{p_i}^2 = \frac{s_y^2}{\left(\bar{X}+\eta_i\right)\left(1+\frac{e_1\bar{X}}{\bar{X}+\eta_i}\right)}\left(\bar{X}+\eta_i\right)$$

$$\Rightarrow \hat{S}_{p_i}^2 = \frac{s_y^2}{\left(1+\theta_{p_i}e_1\right)} \text{ where } \theta_{p_i} = \frac{\bar{X}}{\bar{X}+\eta_i}$$

$$\Rightarrow \hat{S}_{p_i}^2 = s_y^2\left(1+\theta_{p_i}e_1\right)^{-1}$$

$$\Rightarrow \hat{S}_{p_i}^2 = s_y^2\left(1-\theta_{p_i}e_1 + \theta_{p_i}^2 e_1^2 - \theta_{p_i}^3 e_1^3 + \cdots\right)$$

Neglecting the terms more than 1st order, we will get

$$\Rightarrow \hat{S}_{p_i}^2 = s_y^2\left(1-\theta_{p_i}e_1\right)$$

$$\Rightarrow \hat{S}_{p_i}^2 = \left(S_y^2(1+e_0)\right)\left(1-\theta_{p_i}e_1\right)$$

$$\Rightarrow \hat{S}_{p_i}^2 = \left(S_y^2 + S_y^2 e_0\right)\left(1-\theta_{p_i}e_1\right)$$

$$\Rightarrow \hat{S}_{p_i}^2 = S_y^2 + S_y^2 e_0 - S_y^2\theta_{p_i}e_1 - S_y^2\theta_{p_i}e_0e_1$$

$$\Rightarrow \hat{S}_{p_i}^2 - S_y^2 = S_y^2 e_0 - S_y^2\theta_{p_i}e_1 - S_y^2\theta_{p_i}e_0e_1$$

Squaring both sides

$$\Rightarrow \left(\hat{S}_{p_i}^2 - S_y^2\right)^2 = \left(S_y^2 e_0 - S_y^2 \theta_{p_i} e_1 - S_y^2\, \theta_{p_i} e_0 e_1\right)^2$$

Neglecting the terms more than 2nd order, we will get

$$\left(\hat{S}_{p_i}^2 - S_y^2\right)^2 = S_y^4 {e_0}^2 + S_y^4 \theta_{p_i}^2 e_1^2 - 2 S_y^4 \theta_{p_i} e_0 e_1$$

Taking expectation on both sides we will get:

$$E\left(\hat{S}_{p_i}^2 - S_y^2\right)^2 = S_y^4 E(e_0^2) + S_y^4 \theta_{p_i}^2 E(e_1^2) - 2 S_y^4 \theta_{p_i} E(e_0 e_1)$$

$$\Rightarrow MSE\left(\hat{S}_{p_i}^2\right) = \frac{(1-f)}{n}\left(S_y^4\left(\beta_{2(y)} - 1\right) + S_y^4 \theta_{p_i}^2 C_x^2 - 2 S_y^4\, \theta_{p_i} \lambda_{21} C_x\right)$$

$$\Rightarrow MSE\left(\hat{S}_{p_i}^2\right) = \frac{(1-f)}{n} S_y^4 \left(\left(\beta_{2(y)} - 1\right) + \theta_{p_i}^2 C_x^2 - 2\theta_{p_i} \lambda_{21} C_x\right); i = 1, 2, 3, \ldots, 51$$

$$\Rightarrow MSE\left(\hat{S}_{p_i}^2\right) = \gamma S_y^4 \left(\left(\beta_{2(y)} - 1\right) + \theta_{p_i}^2 C_x^2 - 2\theta_{p_i} \lambda_{21} C_x\right); i = 1, 2, 3, \ldots, 51$$

Appendix-B

The conditions for which proposed estimators $\hat{S}_{p_i}^2$ perform better than the traditional ratio type variance estimator $\hat{S}_R^2$ are derived and are given below:

For $MSE\left(\hat{S}_{p_i}^2\right) \leq MSE\left(\hat{S}_R^2\right)$

$$\gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \theta_{p_i}^2 C_x^2 - 2\theta_{p_i} \lambda_{21} C_x\right] \leq \gamma S_y^4 \left[\left(\beta_{2(y)} - 1\right) + \left(\beta_{2(x)} - 1\right) - 2(\lambda_{22} - 1)\right]$$

$$\Rightarrow \left(\left(\beta_{2(y)} - 1\right) + \theta_{p_i}^2 C_x^2 - 2\theta_{p_i} \lambda_{21} C_x\right) \leq \left[\left(\beta_{2(y)} - 1\right) + \left(\beta_{2(x)} - 1\right) - 2(\lambda_{22} - 1)\right]$$

$$\Rightarrow \theta_{p_i}^2 C_x^2 - 2\theta_{p_i} \lambda_{21} C_x \leq \left(\beta_{2(x)} - 1\right) - 2(\lambda_{22} - 1)$$

$$\Rightarrow \theta_{p_i}^2 C_x^2 - 2\theta_{p_i} \lambda_{21} C_x + \lambda_{21}^2 - \lambda_{21}^2 \leq \left(\beta_{2(x)} - 1\right) - 2(\lambda_{22} - 1)$$

$$\Rightarrow \left(\theta_{p_i} C_x - \lambda_{21}\right)^2 \leq \left(\beta_{2(x)} - 1\right) - 2(\lambda_{22} - 1) + \lambda_{21}^2$$

$$\Rightarrow \left(\theta_{p_i} C_x - \lambda_{21}\right) \leq \left(\left(\beta_{2(x)} - 1\right) - 2(\lambda_{22} - 1) + \lambda_{21}^2\right)^{\frac{1}{2}}$$

$$\Rightarrow \theta_{p_i} C_x \leq \lambda_{21} + \left(\left(\beta_{2(x)} - 1\right) - 2(\lambda_{22} - 1) + \lambda_{21}^2\right)^{\frac{1}{2}}$$

$$\Rightarrow \theta_{p_i} \leq \frac{\lambda_{21} + \left(\left(\beta_{2(x)} - 1\right) - 2(\lambda_{22} - 1) + \lambda_{21}^2\right)^{\frac{1}{2}}}{C_x}$$

That is, $MSE\left(\hat{S}_{p_i}^2\right) \leq MSE\left(\hat{S}_R^2\right)$ if $\theta_{p_i} \leq \frac{\lambda_{21} + \left(\left(\beta_{2(x)} - 1\right) - 2(\lambda_{22} - 1) + \lambda_{21}^2\right)^{\frac{1}{2}}}{C_x}$

Appendix-C

The conditions for which proposed estimators $\hat{S}^2_{p_i}$ perform better than the existing modified ratio type variance estimators $\hat{S}^2_i$ are derived and are given below:

For $\mathbf{MSE}(\hat{S}^2_{p_i}) \leq \mathbf{MSE}(\hat{S}^2_i)$

$$\gamma S_y^4\left[\left(\beta_{2(y)}-1\right)+\theta_{p_i}^2 C_x^2-2\theta_{p_i}\lambda_{21}C_x\right] \leq \gamma S_y^4\left[\left(\beta_{2(y)}-1\right)+\delta_i^2\left(\beta_{2(x)}-1\right)-2\delta_i(\lambda_{22}-1)\right]$$

$$\Rightarrow \left(\left(\beta_{2(y)}-1\right)+\theta_{p_i}^2 C_x^2-2\theta_{p_i}\lambda_{21}C_x\right) \leq \left[\left(\beta_{2(y)}-1\right)+\delta_i^2\left(\beta_{2(x)}-1\right)-2\delta_i(\lambda_{22}-1)\right]$$

$$\Rightarrow \theta_{p_i}^2 C_x^2-2\theta_{p_i}\lambda_{21}C_x \leq \delta_i^2\left(\beta_{2(x)}-1\right)-2\delta_i(\lambda_{22}-1)$$

$$\Rightarrow \theta_{p_i}^2 C_x^2-2\theta_{p_i}\lambda_{21}C_x+\lambda_{21}^2-\lambda_{21}^2 \leq \delta_i^2\left(\beta_{2(x)}-1\right)-2\delta_i(\lambda_{22}-1)$$

$$\Rightarrow \left(\theta_{p_i}C_x-\lambda_{21}\right)^2 \leq \delta_i^2\left(\beta_{2(x)}-1\right)-2\delta_i(\lambda_{22}-1)+\lambda_{21}^2$$

$$\Rightarrow \left(\theta_{p_i}C_x-\lambda_{21}\right) \leq \left(\delta_i^2\left(\beta_{2(x)}-1\right)-2\delta_i(\lambda_{22}-1)+\lambda_{21}^2\right)^{\frac{1}{2}}$$

$$\Rightarrow \theta_{p_i}C_x \leq \lambda_{21}+\left(\delta_i^2\left(\beta_{2(x)}-1\right)-2\delta_i(\lambda_{22}-1)+\lambda_{21}^2\right)^{\frac{1}{2}}$$

$$\Rightarrow \theta_{p_i} \leq \frac{\lambda_{21}+\left(\delta_i^2\left(\beta_{2(x)}-1\right)-2\delta_i(\lambda_{22}-1)+\lambda_{21}^2\right)^{\frac{1}{2}}}{C_x}$$

That is, $\text{MSE}(\hat{S}^2_{p_i}) \leq \text{MSE}(\hat{S}^2_i)$ if $\theta_{p_i} \leq \dfrac{\lambda_{21}+\left(\delta_i^2\left(\beta_{2(x)}-1\right)-2\delta_i(\lambda_{22}-1)+\lambda_{21}^2\right)^{\frac{1}{2}}}{C_x}$

Appendix-D

The conditions for which proposed estimators $\hat{S}^2_{p_i}$ perform better than the existing modified ratio type variance estimator $\hat{S}^2_{52}$ are derived and are given below:

For $\mathbf{MSE}(\hat{S}^2_{p_i}) \leq \mathbf{MSE}(\hat{S}^2_{52})$

$$\gamma S_y^4\left[\left(\beta_{2(y)}-1\right)+\theta_{p_i}^2 C_x^2-2\theta_{p_i}\lambda_{21}C_x\right] \leq \gamma S_y^4\left[\left(\beta_{2(y)}-1\right)+C_x^2-2\lambda_{21}C_x\right]$$

$$\Rightarrow \left(\left(\beta_{2(y)}-1\right)+\theta_{p_i}^2 C_x^2-2\theta_{p_i}\lambda_{21}C_x\right) \leq \left[\left(\beta_{2(y)}-1\right)+C_x^2-2\lambda_{21}C_x\right]$$

$$\Rightarrow \theta_{p_i}^2 C_x^2-2\theta_{p_i}\lambda_{21}C_x \leq C_x^2-2\lambda_{21}C_x$$

$$\Rightarrow \theta_{p_i}^2 C_x^2-2\theta_{p_i}\lambda_{21}C_x-C_x^2+2\lambda_{21}C_x \leq 0$$

$$\Rightarrow (\theta_{p_i}^2-1)C_x^2-2\lambda_{21}C_x(\theta_{p_i}-1) \leq 0$$

$$\Rightarrow (\theta_{p_i}+1)(\theta_{p_i}-1)C_x^2-2\lambda_{21}C_x(\theta_{p_i}-1) \leq 0$$

$$\Rightarrow (\theta_{p_i}-1)\left[(\theta_{p_i}+1)C_x^2-2\lambda_{21}C_x\right] \leq 0$$

Condition 1: $(\theta_{p_i}-1) \leq 0$ and $\left[(\theta_{p_i}+1)C_x^2-2\lambda_{21}C_x\right] \geq 0$

$$\Rightarrow \theta_{p_i} \leq 1 \text{ and } (\theta_{p_i}+1)C_x^2 \geq 2\lambda_{21}C_x$$

$$\Rightarrow \theta_{p_i} \le 1 \text{ and } \theta_{p_i} \ge 2\frac{\lambda_{21}}{C_x} - 1$$

$$\Rightarrow 2\frac{\lambda_{21}}{C_x} - 1 \le \theta_{p_i} \le 1$$

Condition 2: $(\theta_{p_i} - 1) \ge 0$ and $[(\theta_{p_i} + 1)C_x^2 - 2\lambda_{21}C_x] \le 0$

$$\Rightarrow \theta_{p_i} \ge 1 \text{ and } (\theta_{p_i} + 1)C_x^2 \le 2\lambda_{21}C_x$$

$$\Rightarrow \theta_{p_i} \ge 1 \text{ and } \theta_{p_i} \le 2\frac{\lambda_{21}}{C_x} - 1$$

$$\Rightarrow 1 \le \theta_{p_i} \le 2\frac{\lambda_{21}}{C_x} - 1$$

That is, $MSE(\hat{S}_{p_i}^2) \le MSE(\hat{S}_{52}^2)$ either $2\frac{\lambda_{21}}{C_x} - 1 \le \theta_{p_i} \le 1$ (**or**) $1 \le \theta_{p_i} \le 2\frac{\lambda_{21}}{C_x} - 1$

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