Moderately growing solutions of third-order differential equations with a singular nonlinearity and regularly varying coefficients

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Abstract

This paper is concerned with asymptotic analysis of moderately growing solutions of the third-order differential equation with singular nonlinearity

$$((((x')^{\alpha_1})')^{\alpha_2})' + q(t)x^{-\beta} = 0, \quad (\text{A})$$

where α_1, α_2 and β are positive constants and $q : [a, \infty) \rightarrow (0, \infty)$ is a continuous regularly varying function of index σ , $a > 0$ and $u^{\gamma*} = |u|^{\gamma} \text{sgn} u$. An application of the theory of regular variation allows us to establish necessary and sufficient conditions for the existence of regularly varying solutions of (A) which are moderately growing and to acquire precise information about the asymptotic behavior at infinity of these solutions. The Schauder-Tychonoff fixed point technique is used.

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1. INTRODUCTION

In this paper we are interested in the existence and accurate asymptotic behavior near infinity of positive solutions of the third order nonlinear differential equation

$$((((x')^{\alpha_1})')^{\alpha_2})' + q(t)x^{-\beta} = 0, \quad (\text{A})$$

where $u^{\gamma*} = |u|^{\gamma} \text{sgn} u$ and

- (a) α_1, α_2 and β are positive constants;
- (b) $q : [a, \infty) \rightarrow (0, \infty)$ is a continuous function, $a > 0$.

By a proper solution of equation (A) we mean a function $x : [t_0, \infty) \rightarrow (0, \infty)$, $t_0 \geq a$, which is continuously differentiable together with $(x')^{\alpha_1^*}$ and $((x')^{\alpha_1^*})^{\alpha_2^*}$ and satisfies (A) on $[t_0, \infty)$. Due to singularity in equation (A) only positive solutions are acceptable. As we show these solutions may be classified into five disjoint subclasses.

Let x be a positive solution of (A) defined on $[t_0, \infty)$. Then $((x')^{\alpha_1^*})^{\alpha_2^*}(t) < 0$ for $t \geq t_0$, so $((x')^{\alpha_1^*})'$ decreases (to the positive constant or to zero) what leads to the fact $((x')^{\alpha_1^*})'(t) > 0$ (otherwise it would lead to the contradiction with the assumption of the positivity of the solution x). This means that x' increases and either $x'(t) > 0$ (and x increases to the positive constant or to ∞) or $x'(t) < 0$ (and x decreases to the positive constant or to zero). Using the L'Hospital rule we get that the limit $((x')^{\alpha_1^*})'(\infty) = \lim_{t \rightarrow \infty} ((x')^{\alpha_1^*})'(t) = \lim_{t \rightarrow \infty} x(t)/t^{1+\frac{1}{\alpha_1}}$ and $x'(\infty) = \lim_{x \rightarrow \infty} x'(t) = \lim_{t \rightarrow \infty} x(t)/t$. We conclude that each positive solution of (A) is one of the following types:

$$\lim_{t \rightarrow \infty} \frac{x(t)}{t^{1+\frac{1}{\alpha_1}}} = \text{const} > 0, \quad (\text{I})$$

$$\lim_{t \rightarrow \infty} \frac{x(t)}{t^{1+\frac{1}{\alpha_1}}} = 0, \lim_{t \rightarrow \infty} \frac{x(t)}{t} = \infty, \quad (\text{II})$$

$$\lim_{t \rightarrow \infty} \frac{x(t)}{t} = \text{const} > 0, \quad (\text{III})$$

$$\lim_{t \rightarrow \infty} x(t) = \text{const} > 0, \quad (\text{IV})$$

$$\lim_{t \rightarrow \infty} x(t) = 0. \quad (\text{V})$$

Let us introduce a relation $\sim$ defined by

$$f(t) \sim g(t) \quad \text{for } t \rightarrow \infty \iff \lim_{t \rightarrow \infty} \frac{f(t)}{g(t)} = 1,$$

where f and g are positive functions. The expression $f(t) \sim g(t)$ is read “ f is asymptotically equivalent to g as $t \rightarrow \infty$ ” or “ f behaves as g for $t \rightarrow \infty$ ”.

Using this symbol we can distinguish solutions $x(t) \sim c$, $x(t) \sim ct$, $x(t) \sim ct^{1+\frac{1}{\alpha_1}}$

for some constant $c > 0$ (solutions of type (IV), (III) and (I), respectively). These solutions are called *primitive* solutions of equation (A) and correspond to solutions 1 , t , $t^{1+\frac{1}{\alpha_1}}$ of the unperturbed differential equation

$$(((x')^{\alpha_1})')^{\alpha_2} = 0.$$

Solutions of types (II) and (V) are referred to as *nonprimitive* ones. In this paper we give detailed information about existence and asymptotic behavior at infinity of solutions of type (II) called *moderately growing*. Solutions of type (V) of Eq. (A) for $\alpha_1 = 1$ and $\alpha_2 = 1$ are analysed in [5].

The first step in our investigation will be to find a corresponding integral equation for this solution type. Let x be a solution of the type (II) on $[T, \infty)$, $T \geq a$. Since $((x')^{\alpha_1})'(\infty) = 0$, integrating (A) once on $[t, \infty)$, $t \geq T$ raising to the power $1/\alpha_2$, integrating on $[T, t]$, raising to the power $1/\alpha_1$ and integrating on $[T, t]$ again, we obtain for x and $t \geq T$

$$x(t) = x(T) + \int_T^t \left[x'(T)^{\alpha_1} + \int_T^s \left(\int_r^\infty q(u)x(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_2}} ds, \quad (1.1)$$

where $x(T) > 0$ and $x'(T) \geq 0$. Conversely, every continuous function x satisfying (1.1) on $[T, \infty)$ and $\int_T^t \left(\int_s^\infty q(r)x(r)^{-\beta} dr \right)^{\frac{1}{\alpha_2}} ds = \infty$ gives moderately growing solution of (A) on $[T, \infty)$.

Since $x'(\infty) = \infty$, the integral equation (1.1) can be approximated by the integral asymptotic relation

$$x(t) \sim \int_T^t \left[\int_T^s \left(\int_r^\infty q(u)x(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} ds, t \rightarrow \infty. \quad (\text{AR})$$

To get the more precise information about behavior of solution of either (A) or (1.1) is a very difficult task, if the coefficient q is a general continuous function. But restricting

ourselves to q which is *regularly varying* (see the definition bellow), the asymptotic analysis of (A) leading to important results about its solutions can be done. First, in Section 2, we find necessary and sufficient condition of existence of regularly varying solutions of (AR) and the formula of its asymptotic behavior. In Section 3, using the Schauder-Tychonoff fixed point and solutions of (AR) found in Section 2 we show that there exist solutions of (A) with certain asymptotic property at infinity and in Section 4 the precise asymptotic behavior of these solutions is specified. For this purpose the generalized L'Hospital rule is used. //

In what follows by *regularly varying function of index ρ* (with $\rho \in \mathbb{R}$) we mean A measurable function $f : (0, \infty) \rightarrow (0, \infty)$ if it satisfies

$$\lim_{t \rightarrow \infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho,$$

for all $\lambda \in \mathbb{R}^+$ or equivalently if it can be expressed in the form

$$f(t) = c(t) \exp \int_a^t \frac{\rho(s)}{s} ds, \quad t \geq a,$$

for some $a > 0$ and some measurable functions c and ρ fulfilling $\lim_{t \rightarrow \infty} c(t) = c_0 > 0$ and $\lim_{t \rightarrow \infty} \rho(t) = \rho$. [19]

Totality of regularly varying functions will be denoted by $RV(\rho)$. If in particular $\rho = 0$ we use SV instead of $RV(0)$ and refer to members of SV as *slowly varying* functions. It is clear that $RV(\rho)$ -functions can be expressed as

$$f(t) = t^\rho L(t), \quad L \in SV.$$

Some other basic properties used in the paper can be seen easily:

PROPOSITION 1.1[19]. *If L, L_1 and L_2 are slowly varying, then also L^α for every real α is slowly varying and $L_1.L_2$ is slowly varying, too.*

PROPOSITION 1.2[19]. *If some positive measurable function f behaves as a regularly varying function of index ρ , i.e. if $f(t) \sim t^\rho L(t)$, as $t \rightarrow \infty$, then f is a*

regularly varying function of index ρ , i.e. $f(t) = t^\rho L^*(t)$, where $L^* \in \text{SV}$ and in general $L^* \neq L$ but $L^*(t) \sim L(t)$, as $t \rightarrow \infty$.

Regularly varying function f will be called *trivial regularly varying function* (of index ρ) if it is possible to write it in the form $f(t) = t^\rho L(t)$, where $L(t)$ is slowly varying fulfilling

$$\lim_{t \rightarrow \infty} L(t) = \text{const} > 0.$$

Otherwise f is called *nontrivial regularly varying function* (of index ρ).

According to this definition a primitive solution x of (A) such that $x(t) \sim ct^j$, for some $c > 0$ and $j \in \{0, 1, 2\}$ is a trivial regularly varying solution of index j . As we will see later, moderately growing solutions of (A) can be trivial regularly varying function or nontrivial regularly varying function.

In Section 3 we use the following theorem about monotone equivalents:

PROPOSITION 1.3[12] *Let $f \in \text{RV}(\rho)$, choose $a \geq 0$ so that f is locally bounded on $[a, \infty)$. If $\rho > 0$ then*

$$(i) \bar{f}(t) := \sup\{f(s) : a \leq s \leq t\} \sim f(t), \quad t \rightarrow \infty,$$

$$(ii) \underline{f}(t) := \inf\{f(s) : s \geq t\} \sim f(t), \quad t \rightarrow \infty.$$

If $\rho < 0$ then

$$(i) \bar{f}(t) := \sup\{f(s) : s \geq t\} \sim f(t), \quad t \rightarrow \infty,$$

$$(ii) \underline{f}(t) := \inf\{f(s) : a \leq s \leq t\} \sim f(t), \quad t \rightarrow \infty.$$

The most crucial tool used in our analysis is the following proposition known as Karamata's integration theorem:

PROPOSITION 1.4[12]. *Let $L \in \text{SV}$. Then*

(i) *if $\alpha > -1$ and a is such that L is locally bounded on $[a, \infty)$, then*

$$\int_a^t s^\alpha L(s) ds \sim \frac{1}{\alpha + 1} t^{\alpha+1} L(t), \quad t \rightarrow \infty,$$

(ii) if $\alpha < -1$, then $\int^\infty t^\alpha L(t) dt < \infty$ and

$$\int_t^\infty s^\alpha L(s) ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \rightarrow \infty,$$

(iii) if $\alpha = -1$ and L is locally integrable on $[a, \infty)$, then

$$l(t) := \int_a^t \frac{L(s)}{s} ds \in \text{SV} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{l(s)}{L(s)} = \infty, \quad t \rightarrow \infty,$$

(iv) if $\alpha = -1$ and $\int^\infty \frac{L(t)}{t} dt < \infty$, then

$$m(t) := \int_t^\infty \frac{L(s)}{s} ds \in \text{SV} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{m(s)}{L(s)} = \infty.$$

For more about regular variation see Bingham et al [12]. For applications of the theory of regular variation in the study of ordinary differential equations see Marić [19]. For the literature devoted to differential equations (of the second order) with singular nonlinearities more or less related to present work, we refer to Tanigawa and Kusano [18], [20] and Kamo and Usami [15]. The Emden-Fowler-type equation of the second and higher order with a positive exponent in context of the regular variation was studied e.g. in [13], [14], [16], [17]. We partially extend these results to the singular equation (A).

2 ASYMPTOTIC ANALYSIS OF THE INTEGRAL ASYMPTOTIC RELATION

In this section we find necessary and sufficient conditions for the existence of solutions of the integral asymptotic relation

$$x(t) \sim \int_T^t \left[\int_T^s \left(\int_r^\infty q(u)x(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_2}} ds \quad (\text{AR})$$

satisfying (II) and we find a formula of their asymptotic behavior for $t \rightarrow \infty$.

THEOREM 2.1. *Let q be a regularly varying function of index σ . The integral asymptotic relation (AR) has regularly varying solutions of index $\rho = 1 + \frac{1}{\alpha_1}$ if and only if $\sigma = (1 + \frac{1}{\alpha_1})\beta - 1$ and*

$$\int_a^\infty t^{-(1+\frac{1}{\alpha_1})\beta} q(t) dt < \infty, \quad (2.1)$$

in which case any such solution x is nontrivial regularly varying and has the asymptotic behavior

$$x(t) \sim t^{1+\frac{1}{\alpha_1}} \left[\frac{(\alpha_1 \alpha_2 + \beta)}{\alpha_1 \alpha_2} \left(\frac{\alpha_1}{1 + \alpha_1} \right)^{\alpha_1 \alpha_2} \int_t^\infty s^{-(1+\frac{1}{\alpha_1})\beta} q(s) ds \right]^{\frac{1}{\alpha_1 \alpha_2 + \beta}}, \quad t \rightarrow \infty. \quad (2.2)$$

THEOREM 2.2. *Let q be a regularly varying function of index σ . The integral asymptotic relation (AR) has regularly varying solutions of index $\rho \in (1, 1 + \frac{1}{\alpha_1})$ if and only if $\sigma \in (\beta - \alpha_2 - 1, (1 + \frac{1}{\alpha_1})\beta - 1)$, in which case index ρ is given by*

$$\rho = \frac{\sigma + \alpha_2 + \alpha_1 \alpha_2 + 1}{\alpha_1 \alpha_2 + \beta} \quad (2.3)$$

and any such solution x has the asymptotic behavior

$$x(t) \sim \left[\frac{t^{\alpha_2 + \alpha_1 \alpha_2 + 1} q(t)}{\alpha_1^{(1+\alpha_2)} \alpha_2 \rho^{\alpha_1 \alpha_2} (\rho - 1)^{\alpha_2} (1 + \frac{1}{\alpha_1} - \rho)} \right]^{\frac{1}{\alpha_1 \alpha_2 + \beta}}, \quad t \rightarrow \infty. \quad (2.4)$$

THEOREM 2.3. *Let q be a regularly varying function of index σ . The integral asymptotic relation (AR) has nontrivial regularly varying solution of index $\rho = 1$ if and only if $\sigma = \beta - \alpha_2 - 1$ and*

$$\int_a^\infty \left(t^{1-\beta} q(t) \right)^{\frac{1}{\alpha_2}} dt = \infty, \quad (2.5)$$

in which case any such solution x has the asymptotic behavior

$$x(t) \sim t \left[\frac{\alpha_1 \alpha_2 + \beta}{\alpha_1 \alpha_2^{1+\frac{1}{\alpha_2}}} \int_a^t \left(s^{1-\beta} q(s) \right)^{\frac{1}{\alpha_2}} ds \right]^{\frac{\alpha_2}{\alpha_1 \alpha_2 + \beta}}, \quad t \rightarrow \infty. \quad (2.6)$$

PROOF OF THEOREMS 2.1, 2.2 AND 2.3. Let $q(t) = t^\sigma l(t)$, $l \in \text{SV}$.

(The “only if” part) Suppose that (AR) has a regularly varying solution $x \in \text{RV}(\rho)$, $\rho \in (1, 1 + \frac{1}{\alpha_1}]$, resp. nontrivial regularly varying solution of index $\rho = 1$ on $[t_0, \infty)$. It is obvious, that x satisfies (II). Put $x(t) = t^\rho \xi(t)$, $\xi \in \text{SV}$, then it is obvious also, that $\lim_{t \rightarrow \infty} \xi(t) = \infty$, resp. $\lim_{t \rightarrow \infty} \xi(t) = 0$, according as $\rho = 1$, resp. $\rho = 1 + \frac{1}{\alpha_1}$ (and so also for $\rho = 1 + \frac{1}{\alpha_1}$ x is nontrivial regularly varying solution).

We see that the integral

$$\int_t^\infty q(s) x(s)^{-\beta} ds = \int_t^\infty s^{\sigma - \rho \beta} l(s) \xi(s)^{-\beta} ds, \quad t \geq t_0 \quad (2.7)$$

have to converge what implies that $\sigma - \rho \beta \leq -1$.

(a) First consider the case $\sigma - \rho \beta = -1$. Then (2.7) reduces to

$$\int_t^\infty s^{-1} l(s) \xi(s)^{-\beta} ds. \quad (2.8)$$

Proposition 1.1 implies $l\xi^{-\beta} \in \text{SV}$, so we can use (iv) from Proposition 1.4 to see that also the integral in (2.8) is slowly varying function. Raising it to $1/\alpha_2$, integrating on $[t_0, t]$ and using Proposition 1.4 (i), we get for $t \rightarrow \infty$

$$\int_{t_0}^t \left(\int_s^\infty q(r)x(r)^{-\beta} dr \right)^{\frac{1}{\alpha_2}} ds \sim t \left(\int_t^\infty s^{-1}l(s)\xi(s)^{-\beta} ds \right)^{\frac{1}{\alpha_2}}. \quad (2.9)$$

Raising (2.9) to $1/\alpha_1$, integrating on $[t_0, t]$ and using (i) from Proposition 1.4 again, we see from (AR) that

$$x(t) \sim \frac{\alpha_1}{1 + \alpha_1} t^{1 + \frac{1}{\alpha_1}} \left[\int_t^\infty s^{-1}l(s)\xi(s)^{-\beta} ds \right]^{\frac{1}{\alpha_1 \alpha_2}}, \quad t \rightarrow \infty.$$

This means that x is asymptotically equivalent to regularly varying function of index $\rho = 1 + \frac{1}{\alpha_1}$, according to Proposition 1.2 x is regularly varying function of index $\rho = 1 + \frac{1}{\alpha_1}$ which implies that $\sigma = (1 + \frac{1}{\alpha_1})\beta - 1$ and $x(t) = t^{1 + \frac{1}{\alpha_1}} \xi(t)$ on $[t_0, \infty)$. So we can write

$$\xi(t)^{\alpha_1 \alpha_2} \sim \left(\frac{\alpha_1}{1 + \alpha_1} \right)^{\alpha_1 \alpha_2} \int_t^\infty s^{-1}l(s)\xi(s)^{-\beta} ds, \quad t \rightarrow \infty. \quad (2.10)$$

Let denote the right-hand side of (2.10) by $\eta(t) (\Rightarrow \xi(t) \sim \eta(t)^{\alpha_1 \alpha_2})$ and rewrite it into the following differential asymptotic relation:

$$-\eta(t)^{\frac{\beta}{\alpha_1 \alpha_2}} \eta'(t) \sim \left(\frac{\alpha_1}{1 + \alpha_1} \right)^{\alpha_1 \alpha_2} t^{-1} l(t) = \left(\frac{\alpha_1}{1 + \alpha_1} \right)^{\alpha_1 \alpha_2} t^{-(1 + \frac{1}{\alpha_1})\beta} q(t), \quad t \rightarrow \infty. \quad (2.11)$$

Since $\eta(t) \rightarrow 0$ as $t \rightarrow \infty$ (because $\xi(t) \rightarrow 0$ as $t \rightarrow \infty$ and $\alpha_1 \alpha_2 > 0$) and $-1 < \frac{\beta}{\alpha_1 \alpha_2}$, the left-hand side of (2.11) is integrable on $[t_0, \infty)$, so the right-hand side is such, too what shows that the condition (2.1) has to be fulfilled. Integrating (2.11) from t to

∞ we get

$$\frac{\eta(t)^{1+\frac{\beta}{\alpha_1\alpha_2}}}{1+\frac{\beta}{\alpha_1\alpha_2}} \sim \left(\frac{\alpha_1}{1+\alpha_1}\right)^{\alpha_1\alpha_2} \int_t^\infty s^{-(1+\frac{1}{\alpha_1})\beta} q(s) ds, \quad t \rightarrow \infty$$

and finally

$$x(t) \sim t^{1+\frac{1}{\alpha_1}} \left[\frac{\alpha_1\alpha_2+\beta}{\alpha_1\alpha_2} \left(\frac{\alpha_1}{1+\alpha_1}\right)^{\alpha_1\alpha_2} \int_t^\infty s^{-(1+\frac{1}{\alpha_1})\beta} q(s) ds \right]^{\frac{1}{\alpha_1\alpha_2+\beta}}, \quad t \rightarrow \infty,$$

what shows that (2.2) holds.

(b) Next we consider the case $\sigma - \rho\beta < -1$. Then we can apply Proposition 1.4 (ii) on (2.7) and after raising to $1/\alpha_2$ we get

$$\left[\int_t^\infty q(s)x(s)^{-\beta} ds \right]^{\frac{1}{\alpha_2}} \sim \left[-\frac{1}{\sigma - \rho\beta + 1} \right]^{\frac{1}{\alpha_2}} t^{\frac{\sigma - \rho\beta + 1}{\alpha_2}} l(t)^{\frac{1}{\alpha_2}} \xi(t)^{\frac{-\beta}{\alpha_2}}, \quad t \rightarrow \infty. \quad (2.12)$$

From the second limit in (II) and the fact that $\alpha_1 > 0$ we see that (2.12) is not integrable on $[t_0, \infty)$. This means that

$$\frac{\sigma - \rho\beta + 1}{\alpha_2} \geq -1 \quad \text{or} \quad \text{equivalently} \quad \sigma - \rho\beta \geq -\alpha_2 - 1.$$

(b₁) If $-1 > \sigma - \rho\beta > -\alpha_2 - 1$ holds, we continue by integrating (2.12) on $[t_0, t]$, applying Proposition 1.4 (i), raising to $1/\alpha_1$ and integrating on $[t_0, t]$ again. Applying (i) from Proposition 1.4 lastly we get from (AR)

$$x(t) \sim \frac{t^{\frac{\sigma - \rho\beta + 1 + \alpha_2 + \alpha_1\alpha_2}{\alpha_1\alpha_2}} l(t)^{\frac{1}{\alpha_1\alpha_2}} \xi(t)^{\frac{-\beta}{\alpha_1\alpha_2}}}{[-(\sigma - \rho\beta + 1)]^{\frac{1}{\alpha_1\alpha_2}} \left(\frac{\sigma - \rho\beta + 1 + \alpha_2}{\alpha_2}\right)^{\frac{1}{\alpha_1}} \left(\frac{\sigma - \rho\beta + 1 + \alpha_2 + \alpha_1\alpha_2}{\alpha_1\alpha_2}\right)}, \quad t \rightarrow \infty. \quad (2.13)$$

Together with Proposition 1.2 the relation (2.13) shows that $x \in \text{RV}(\rho)$, where

$$\rho = \frac{\sigma - \rho\beta + 1 + \alpha_2 + \alpha_1\alpha_2}{\alpha_1\alpha_2} \quad (2.14)$$

and $\rho \in (1, 1 + \frac{1}{\alpha_1})$ or equivalently $\rho = \frac{\sigma + \alpha_2 + \alpha_1\alpha_2 + 1}{\alpha_1\alpha_2 + \beta}$, what means that (2.3) holds. From $\rho \in (1, 1 + \frac{1}{\alpha_1})$ we get $\sigma \in (\beta - \alpha_2 - 1, (1 + \frac{1}{\alpha_1})\beta - 1)$, what was needed to show. Finally we use (2.14) for the coefficients in (2.13), we return to q and x in the right-hand side of (2.13) and we see after short calculation that

$$x(t) \sim \frac{t^{\frac{1+\alpha_2+\alpha_1\alpha_2}{\alpha_1\alpha_2}} q(t)^{\frac{1}{\alpha_1\alpha_2}} x(t)^{\frac{-\beta}{\alpha_1\alpha_2}}}{\alpha_1^{\frac{1}{\alpha_1}} (\alpha_1\alpha_2)^{\frac{1}{\alpha_1\alpha_2}} \rho(\rho-1)^{\frac{1}{\alpha_1}} (1 + \frac{1}{\alpha_1} - \rho)^{\frac{1}{\alpha_1\alpha_2}}}, \quad t \rightarrow \infty,$$

or equivalently

$$x(t) \sim \left[\frac{t^{1+\alpha_2+\alpha_1\alpha_2} q(t)}{\alpha_1^{1+\alpha_2} \alpha_2 \rho \alpha_1 \alpha_2 (\rho-1) \alpha_2 (1 + \frac{1}{\alpha_1} - \rho)} \right]^{\frac{1}{\alpha_1\alpha_2 + \beta}}, \quad t \rightarrow \infty,$$

what means that (2.4) holds.

(b₂) If $\sigma - \rho\beta = -\alpha_2 - 1$ holds then (2.12) after integrating on $[t_0, t]$ reduces to

$$\int_{t_0}^t \left(\int_s^\infty q(r) x(r)^{-\beta} dr \right)^{\frac{1}{\alpha_2}} ds \sim \alpha_2^{-\frac{1}{\alpha_2}} \int_{t_0}^t s^{-1} l(s)^{\frac{1}{\alpha_2}} \xi(s)^{\frac{-\beta}{\alpha_2}} ds, \quad t \rightarrow \infty. \quad (2.15)$$

According to Proposition 1.4 (iii) the right-hand side of (2.15) is from SV, what remains true (using Proposition 1.1) also after raising to the power $1/\alpha_1$. Integrating asymptotically by Proposition 1.4 (iii) on $[t_0, t]$ again and remembering the relation (AR) we obtain

$$x(t) \sim \alpha_2^{-\frac{1}{\alpha_1\alpha_2}} t \left[\int_{t_0}^t s^{-1} l(s)^{\frac{1}{\alpha_2}} \xi(s)^{\frac{-\beta}{\alpha_2}} ds \right]^{\frac{1}{\alpha_1}} \in \text{RV}(1), \quad t \rightarrow \infty.$$

Using Proposition 1.2 we see that $x \in \text{RV}(\rho)$, where $\rho = 1$ what implies $\sigma = \beta - \alpha_2 - 1$ and

$$\xi(t)^{\alpha_1} \sim \alpha_2^{-\frac{1}{\alpha_2}} \int_{t_0}^t s^{-1} l(s)^{\frac{1}{\alpha_2}} \xi(s)^{\frac{-\beta}{\alpha_2}} ds, \quad t \rightarrow \infty. \quad (2.16)$$

Denote the right-hand side of (2.16) by $\eta(t)$ and rewrite (2.16) as the differential asymptotic relation

$$\eta(t)^{\frac{\beta}{\alpha_1 \alpha_2}} \eta'(t) \sim \alpha_2^{-\frac{1}{\alpha_2}} t^{-1} l(t)^{\frac{1}{\alpha_2}} = \alpha_2^{-\frac{1}{\alpha_2}} \left(t^{1-\beta} q(t) \right)^{\frac{1}{\alpha_2}}, \quad t \rightarrow \infty. \quad (2.17)$$

Since $\eta(t) \rightarrow \infty$ as $t \rightarrow \infty$ (because $\xi(t) \rightarrow \infty$ as $t \rightarrow \infty$ and $\alpha_1 > 0$) and $-1 < \frac{\beta}{\alpha_1 \alpha_2}$, the left-hand side of (2.11) is not integrable on $[t_0, \infty)$, so neither the right-hand side is such and the condition (2.5) holds. Next we continue in calculation to get for $t \rightarrow \infty$

$$\frac{\eta(t)^{\frac{\beta + \alpha_1 \alpha_2}{\alpha_1 \alpha_2}}}{\frac{\beta + \alpha_1 \alpha_2}{\alpha_1 \alpha_2}} - \frac{\eta(t_0)^{\frac{\beta + \alpha_1 \alpha_2}{\alpha_1 \alpha_2}}}{\frac{\beta + \alpha_1 \alpha_2}{\alpha_1 \alpha_2}} \sim \alpha_2^{-\frac{1}{\alpha_2}} \int_{t_0}^t \left(s^{1-\beta} q(s) \right)^{\frac{1}{\alpha_2}} ds$$

or equivalently

$$x(t) \sim t \left[\frac{\beta + \alpha_1 \alpha_2}{\alpha_1 \alpha_2} \int_{t_0}^t \left(s^{1-\beta} q(s) \right)^{\frac{1}{\alpha_2}} ds \right]^{\frac{\alpha_2}{\beta + \alpha_1 \alpha_2}}, \quad t \rightarrow \infty,$$

i.e. (2.6) holds.

This completes the proof of the "only if" parts of Theorems 2.1, 2.2 and 2.3.

(The "if" part) Let define the function X on $[a, \infty)$ in the following way

(a) if $\sigma = (1 + \frac{1}{\alpha_1})\beta - 1$ and (2.1) holds, then

$$X(t) = t^{1 + \frac{1}{\alpha_1}} \left[\frac{(\alpha_1 \alpha_2 + \beta)}{\alpha_1 \alpha_2} \left(\frac{\alpha_1}{1 + \alpha_1} \right)^{\alpha_1 \alpha_2} \int_t^\infty s^{-(1 + \frac{1}{\alpha_1})\beta} q(s) ds \right]^{\frac{1}{\alpha_1 \alpha_2 + \beta}}; \quad (2.18)$$

(b) if $\sigma \in (\beta - \alpha_2 - 1, (1 + \frac{1}{\alpha_1})\beta - 1)$, then

$$X(t) = \left[\frac{t^{\alpha_2 + \alpha_1 \alpha_2 + 1} q(t)}{\alpha_1^{(1 + \alpha_2)} \alpha_2 \rho^{\alpha_1 \alpha_2} (\rho - 1)^{\alpha_2} (1 + \frac{1}{\alpha_1} - \rho)} \right]^{\frac{1}{\alpha_1 \alpha_2 + \beta}}, \quad (2.19)$$

where for ρ holds (2.3) and

(c) if $\sigma = \beta - \alpha_2 - 1$ and (2.5) holds, then

$$X(t) = t \left[\frac{\alpha_1 \alpha_2 + \beta}{\alpha_1 \alpha_2^{1 + \frac{1}{\alpha_2}}} \int_a^t \left(s^{1 - \beta} q(s) \right)^{\frac{1}{\alpha_2}} ds \right]^{\frac{\alpha_2}{\alpha_1 \alpha_2 + \beta}}. \quad (2.20)$$

We show that X satisfies for any $b \geq a$ the integral asymptotic relation

$$X(t) \sim \int_b^t \left[\int_b^s \left(\int_r^\infty q(u) X(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} ds, t \rightarrow \infty. \quad (2.21)$$

(a) Let $\sigma = (1 + \frac{1}{\alpha_1})\beta - 1$ and (2.1) hold. Then we can write

$$\begin{aligned} & \int_t^\infty q(s) X(s)^{-\beta} ds \\ &= \int_t^\infty q(s) s^{-(1 + \frac{1}{\alpha_1})\beta} \left[\frac{(\alpha_1 \alpha_2 + \beta)}{\alpha_1 \alpha_2} \left(\frac{\alpha_1}{1 + \alpha_1} \right)^{\alpha_1 \alpha_2} \int_s^\infty r^{-(1 + \frac{1}{\alpha_1})\beta} q(r) dr \right]^{\frac{-\beta}{\alpha_1 \alpha_2 + \beta}} ds. \end{aligned}$$

Using the substitution

$$w_1(s) = \left[\frac{(\alpha_1 \alpha_2 + \beta)}{\alpha_1 \alpha_2} \left(\frac{\alpha_1}{1 + \alpha_1} \right)^{\alpha_1 \alpha_2} \int_s^\infty r^{-(1 + \frac{1}{\alpha_1})\beta} q(r) dr \right]^{\frac{\alpha_1 \alpha_2}{\alpha_1 \alpha_2 + \beta}},$$

for which $w_1(\infty) = 0$ thanks to (2.1), we get that right-hand side of (2.21) is equal to

$$\begin{aligned} & \int_b^t \left[\int_b^s \left(- \left(1 + \frac{1}{\alpha_1} \right)^{\alpha_1 \alpha_2} \int_r^\infty dw_1(u) \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} ds \\ &= \left(1 + \frac{1}{\alpha_1} \right) \int_b^t \left[\int_b^s w_1(r)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} ds. \end{aligned} \quad (2.22)$$

Since w_1 is from SV, $w_1^{\frac{1}{\alpha_2}}$ and $w_1^{\frac{1}{\alpha_1 \alpha_2}}$ are such, too (Proposition 1.1). So can we use Proposition 1.4 (i) twice to see that (2.22) is for $t \rightarrow \infty$ asymptotically equivalent to

$$\left(1 + \frac{1}{\alpha_1} \right) \int_b^t s^{\frac{1}{\alpha_1}} w_1(s)^{\frac{1}{\alpha_1 \alpha_2}} ds \sim t^{(1+\frac{1}{\alpha_1})} w_1(t)^{\frac{1}{\alpha_1 \alpha_2}} = X(t).$$

(b) Let $\sigma \in (\beta - \alpha_2 - 1, (1 + \frac{1}{\alpha_1})\beta - 1)$ and ρ be defined by (2.3). Using the expression of X in the form

$$X(t) = \frac{t^\rho l(t)^{\frac{1}{\alpha_1 \alpha_2 + \beta}}}{\left[\alpha_1^{(1+\alpha_2)} \alpha_2 \rho^{\alpha_1 \alpha_2} (\rho - 1)^{\alpha_2} \left(1 + \frac{1}{\alpha_1} - \rho \right) \right]^{\frac{1}{\alpha_1 \alpha_2 + \beta}}},$$

and the fact that $\sigma - \rho\beta = \alpha_1 \alpha_2 (\rho - 1) - \alpha_2 - 1$ we get that the right-hand side of (2.21) becomes

$$\frac{\int_b^t \left(\int_b^s \left(\int_r^\infty u^{\alpha_1 \alpha_2 (\rho - 1) - \alpha_2 - 1} l(u)^{\frac{\alpha_1 \alpha_2}{\alpha_1 \alpha_2 + \beta}} du \right)^{\frac{1}{\alpha_2}} dr \right)^{\frac{1}{\alpha_1}} ds}{\left[\alpha_1^{(1+\alpha_2)} \alpha_2 \rho^{\alpha_1 \alpha_2} (\rho - 1)^{\alpha_2} \left(1 + \frac{1}{\alpha_1} - \rho \right) \right]^{\frac{-\beta}{\alpha_1 \alpha_2 (\alpha_1 \alpha_2 + \beta)}}}. \quad (2.23)$$

We continue in calculation using Proposition 1.4 (ii) once and Proposition 1.4 (i)

twice to see that (2.23) is for $t \rightarrow \infty$ successively asymptotically equivalent to

$$\begin{aligned}
 & \frac{\int_b^t \left(\int_b^s r^{\alpha_1(\rho-1)-1} l(r)^{\frac{\alpha_1}{\alpha_1\alpha_2+\beta}} dr \right)^{\frac{1}{\alpha_1}} ds}{[-(\alpha_1\alpha_2(\rho-1)-\alpha_2)]^{\frac{1}{\alpha_1\alpha_2}} \left[\alpha_1^{(1+\alpha_2)} \alpha_2 \rho^{\alpha_1\alpha_2} (\rho-1)^{\alpha_2} \left(1 + \frac{1}{\alpha_1} - \rho\right) \right]^{\frac{-\beta}{\alpha_1\alpha_2(\alpha_1\alpha_2+\beta)}}} \\
 & \sim \frac{\int_b^t s^{\rho-1} l(s)^{\frac{1}{\alpha_1\alpha_2+\beta}} ds}{(\alpha_1(\rho-1))^{\frac{1}{\alpha_1}} \left[\alpha_1^{1-\frac{\beta}{\alpha_1}} \alpha_2 \left(1 + \frac{1}{\alpha_1} - \rho\right) \right]^{\frac{1}{\alpha_1\alpha_2+\beta}} \left[\rho(\rho-1)^{\frac{1}{\alpha_1}} \right]^{\frac{-\beta}{\alpha_1\alpha_2+\beta}}} \\
 & \sim \frac{t^\rho l(t)^{\frac{1}{\alpha_1\alpha_2+\beta}}}{\rho \left[\alpha_1^{(1+\alpha_2)} \alpha_2 (\rho-1)^{\alpha_2} \left(1 + \frac{1}{\alpha_1} - \rho\right) \right]^{\frac{1}{\alpha_1\alpha_2+\beta}} \rho^{\frac{-\beta}{\alpha_1\alpha_2+\beta}}} \\
 & = \frac{t^\rho l(t)^{\frac{1}{\alpha_1\alpha_2+\beta}}}{\left[\rho^{\alpha_1\alpha_2} \alpha_1^{1+\alpha_2} \alpha_2 (\rho-1)^{\alpha_2} \left(1 + \frac{1}{\alpha_1} - \rho\right) \right]^{\frac{1}{\alpha_1\alpha_2+\beta}}} = X(t).
 \end{aligned}$$

(c) Finally if $\sigma = \beta - \alpha_2 - 1$ and (2.5) holds, using Proposition 1.4 (ii) we see that

$$\begin{aligned}
 \left(\int_t^\infty q(s) X(s)^{-\beta} ds \right)^{\frac{1}{\alpha_2}} &= \left(\int_t^\infty l(s) s^{-\alpha_2-1} \left[\frac{\alpha_1\alpha_2+\beta}{\alpha_1\alpha_2^{1+\frac{1}{\alpha_2}}} \int_a^s r^{-1} l(r)^{\frac{1}{\alpha_2}} dr \right]^{\frac{-\alpha_2\beta}{\alpha_1\alpha_2+\beta}} ds \right)^{\frac{1}{\alpha_2}} \\
 &\sim \frac{t^{-1} l(t)^{\frac{1}{\alpha_2}}}{\alpha_2^{\frac{1}{\alpha_2}}} \left[\frac{\alpha_1\alpha_2+\beta}{\alpha_1\alpha_2^{1+\frac{1}{\alpha_2}}} \int_a^t s^{-1} l(s)^{\frac{1}{\alpha_2}} ds \right]^{\frac{-\beta}{\alpha_1\alpha_2+\beta}}, \quad t \rightarrow \infty.
 \end{aligned}$$

Next we use the substitution

$$w_2(t) = \left[\frac{\alpha_1\alpha_2+\beta}{\alpha_1\alpha_2^{1+\frac{1}{\alpha_2}}} \int_a^t s^{-1} l(s)^{\frac{1}{\alpha_2}} ds \right]^{\frac{\alpha_1\alpha_2}{\alpha_1\alpha_2+\beta}}.$$

Then we can write that the right-hand side of (2.21) is for $t \rightarrow \infty$ asymptotically equiv-

alent to

$$\int_b^t \left(\int_b^s dw_2(r) dr \right)^{\frac{1}{\alpha_1}} ds = \int_b^t (w_2(s) - w_2(b))^{\frac{1}{\alpha_1}} ds. \quad (2.24)$$

Since $w_2(t) \rightarrow \infty$ as $t \rightarrow \infty$ (due to (2.5)) and since $w_2 \in \text{SV}$ we use (i) of Proposition 1.4 and we continue to see that (2.24) is for $t \rightarrow \infty$ asymptotically equivalent to

$$\int_b^t w_2(s)^{\frac{1}{\alpha_1}} ds \sim t w_2(t)^{\frac{1}{\alpha_1}} = X(t).$$

This completes the "if" part of the proof of the Theorems 2.1, 2.2 and 2.3.

3 EXISTENCE OF NON-PRIMITIVE MODERATELY GROWING SOLUTIONS OF THE EQUATION (A)

Now we will relax the assumption about the coefficient q . For this purpose let us introduce the following relation. Let f and g be two positive functions defined on some $[T, \infty)$. We use the notation

$$f(t) \asymp g(t), \quad t \rightarrow \infty$$

to denote that there exist positive constants m and M such that

$$mg(t) \leq f(t) \leq Mg(t), \quad \text{for } t \in [T, \infty).$$

A positive function f defined on $[0, \infty)$ is called *f nearly regularly varying of index ρ* , if there exists $g \in \text{RV}(\rho)$ such that

$$f(t) \asymp g(t), \quad t \rightarrow \infty.$$

In what follows we will need an additional condition on nonlinearity in equation

(A), namely

$$\beta < \alpha_1 \alpha_2. \quad (3.1)$$

THEOREM 3.1. *Let (3.1) hold and q be nearly regularly varying of index σ , i.e. $q \asymp q_\sigma$ at infinity for some $q_\sigma \in \text{RV}(\sigma)$. Suppose that $\sigma = (1 + \frac{1}{\alpha_1})\beta - 1$ and (2.1) holds. Then Eq. (A) possesses a moderately growing solution x which is nearly regularly varying of index $\rho = 1 + \frac{1}{\alpha_1}$ and satisfies*

$$x(t) \asymp t^{1+\frac{1}{\alpha_1}} \left[\frac{(\alpha_1 \alpha_2 + \beta)}{\alpha_1 \alpha_2} \left(\frac{\alpha_1}{1 + \alpha_1} \right)^{\alpha_1 \alpha_2} \int_t^\infty s^{-(1+\frac{1}{\alpha_1})\beta} q(s) ds \right]^{\frac{1}{\alpha_1 \alpha_2 + \beta}}, \quad t \rightarrow \infty. \quad (3.2)$$

THEOREM 3.2. *Let (3.1) hold and q be nearly regularly varying of index σ , i.e. $q \asymp q_\sigma$ at infinity, for some $q_\sigma \in \text{RV}(\sigma)$. Suppose that $\sigma \in (\beta - \alpha_2 - 1, (1 + \frac{1}{\alpha_1})\beta - 1)$. Then equation (A) possesses a moderately growing solution x which is nearly regularly varying of index $\rho \in (1, 1 + \frac{1}{\alpha_1})$ and satisfies*

$$x(t) \asymp \left[\frac{t^{\alpha_2 + \alpha_1 \alpha_2 + 1} q(t)}{\alpha_1^{(1+\alpha_2)} \alpha_2 \rho^{\alpha_1 \alpha_2} (\rho - 1)^{\alpha_2} (1 + \frac{1}{\alpha_1} - \rho)} \right]^{\frac{1}{\alpha_1 \alpha_2 + \beta}}, \quad t \rightarrow \infty, \quad (3.3)$$

where ρ is given by (2.3).

THEOREM 3.3. *Let (3.1) hold and q be nearly regularly varying of index σ , i.e. $q \asymp q_\sigma$ at infinity for some $q_\sigma \in \text{RV}(\sigma)$. Suppose that $\sigma = \beta - \alpha_2 - 1$ and (2.5) holds. Then Eq. (A) possesses a moderately growing solution x which is nearly regularly varying of index 1 and satisfies*

$$x(t) \asymp t \left[\frac{\alpha_1 \alpha_2 + \beta}{\alpha_1 \alpha_2^{1+\frac{1}{\alpha_2}}} \int_a^t \left(s^{1-\beta} q(s) \right)^{\frac{1}{\alpha_2}} ds \right]^{\frac{\alpha_2}{\alpha_1 \alpha_2 + \beta}}, \quad t \rightarrow \infty. \quad (3.4)$$

PROOF OF THEOREMS 3.1, 3.2 AND 3.3 Suppose, that $\sigma = (1 + \frac{1}{\alpha_1})\beta - 1$ and (2.1) holds or $\sigma \in (\beta - \alpha_2 - 1, (1 + \frac{1}{\alpha_1})\beta - 1)$ or $\sigma = \beta - \alpha_2 - 1$ and (2.5) holds.

Since q is nearly regularly varying of index σ , there exist positive constants k and K ($k \leq K$) and $q_\sigma \in \text{RV}(\sigma)$ such that

$$kq_\sigma(t) \leq q(t) \leq Kq_\sigma(t), \quad t \in [T, \infty). \quad (3.5)$$

Let define function X on $[a, \infty)$ as follows:

if $\sigma = (1 + \frac{1}{\alpha_1})\beta - 1$ and (2.1) holds, then

$$X(t) = t^{1+\frac{1}{\alpha_1}} \left[\frac{(\alpha_1 \alpha_2 + \beta)}{\alpha_1 \alpha_2} \left(\frac{\alpha_1}{1 + \alpha_1} \right)^{\alpha_1 \alpha_2} \int_t^\infty s^{-(1+\frac{1}{\alpha_1})\beta} q_\sigma(s) ds \right]^{\frac{1}{\alpha_1 \alpha_2 + \beta}};$$

if $\sigma \in (\beta - \alpha_2 - 1, (1 + \frac{1}{\alpha_1})\beta - 1)$, then

$$X(t) = \left[\frac{t^{\alpha_2 + \alpha_1 \alpha_2 + 1} q_\sigma(t)}{\alpha_1^{(1+\alpha_2)} \alpha_2 \rho^{\alpha_1 \alpha_2} (\rho - 1)^{\alpha_2} (1 + \frac{1}{\alpha_1} - \rho)} \right]^{\frac{1}{\alpha_1 \alpha_2 + \beta}},$$

where for ρ holds (2.3) and

if $\sigma = \beta - \alpha_2 - 1$ and (2.5) holds, then

$$X(t) = t \left[\frac{\alpha_1 \alpha_2 + \beta}{\alpha_1 \alpha_2^{1+\frac{1}{\alpha_2}}} \int_a^t \left(s^{1-\beta} q_\sigma(s) \right)^{\frac{1}{\alpha_2}} ds \right]^{\frac{\alpha_2}{\alpha_1 \alpha_2 + \beta}}.$$

From the previous section we know that X satisfies the integral asymptotic relation

$$X(t) \sim \int_b^t \left[\int_b^s \left(\int_r^\infty q_\sigma(u) X(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} ds, \quad t \rightarrow \infty,$$

for all $b \geq a$. We know also that $X \in \text{RV}(\rho)$, where $\rho \in [1, 1 + \frac{1}{\alpha_1}]$, e.g. $\rho > 0$. Using Proposition 1.3 we know that if X is not nondecreasing, we are able to replace it by the function X_1 which is nondecreasing and simultaneously $X_1 \sim X$ and consequently also $X_1(t) \sim \int_b^t \left[\int_b^s \left(\int_r^\infty q_\sigma(u) X_1(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} ds$ for $t \rightarrow \infty$ ($X_1(t) := \sup\{X(s); a \leq$

$s \leq t\}$). So we can suppose, that X is nondecreasing.

Choose $T_0 \geq a$ so that

$$\int_{T_0}^t \left[\int_{T_0}^s \left(\int_r^\infty q_\sigma(u) X(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} ds \leq 2X(t), \quad t \geq T_0. \quad (3.6)$$

Since $T_0 \geq a$ also

$$\int_{T_0}^t \left[\int_{T_0}^s \left(\int_r^\infty q_\sigma(u) X(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} ds \sim X(t), \quad t \geq T_0,$$

holds we can choose $T_1 > T_0$ such, that

$$\frac{X(t)}{2} \leq \int_{T_0}^t \left[\int_{T_0}^s \left(\int_r^\infty q_\sigma(u) X(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} ds, \quad t \geq T_1. \quad (3.7)$$

Let choose positive constants m and M ($m < M$) so that

$$m^{\alpha_1 \alpha_2} M^\beta \leq \frac{k}{2^{\alpha_1 \alpha_2}}, \quad M^{\alpha_1 \alpha_2} m^\beta \geq 4^{\alpha_1 \alpha_2} K, \quad mX(T_1) \leq \frac{1}{2} MX(T_0) \quad (3.8)$$

hold. Notice that $k < K$, first two relations from (3.8) imply $2^{3\alpha_1 \alpha_2} m^{\alpha_1 \alpha_2 - \beta} < M^{\alpha_1 \alpha_2 - \beta}$ and this implies necessity of inequality

$$m^{\alpha_1 \alpha_2 - \beta} < M^{\alpha_1 \alpha_2 - \beta}.$$

Since $\alpha_1 > 0$ and $\alpha_2 > 0$, this is possible only if $\beta < \alpha_1 \alpha_2$, what leads to the mentioned additional condition (3.1).

Later let define the set

$$\mathcal{X}_0 = \{x \in C[T_0, \infty) : mX(t) \leq x(t) \leq MX(t), \quad t \geq T_0\} \quad (3.9)$$

and the operator

$$\mathcal{F}x(t) = x_0 + \int_{T_0}^t \left[\int_{T_0}^s \left(\int_r^\infty q(u)x(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} ds, \quad t \geq T_0,$$

where

$$mX(T_1) \leq x_0 \leq \frac{M}{2}X(T_0). \quad (3.10)$$

It is clear that $\mathcal{X}_0$ is a closed convex subset of the locally convex space $C[T_0, \infty)$.

Let show that $\mathcal{F}$ fulfils conditions of the Schauder-Tychonoff fixed point theorem

(i) $\mathcal{F}(\mathcal{X}_0) \subset \mathcal{X}_0$.

Using (3.10) and monotonicity of X we know that for an arbitrary but fixed $x \in \mathcal{X}_0$

$$\mathcal{F}x(t) \geq x_0 \geq mX(T_1) \geq mX(t), \quad t \in [T_0, T_1], \quad (3.11)$$

and using (3.5), (3.7) - (3.9) and bearing in mind that $x^{-\beta}$ is a nonincreasing function of x , we see that

$$\begin{aligned} \mathcal{F}x(t) &\geq \int_{T_0}^t \left[\int_{T_0}^s \left(\int_r^\infty kq_\sigma(u)(MX(u))^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} ds \\ &\geq k^{\frac{1}{\alpha_1\alpha_2}} M^{\frac{-\beta}{\alpha_1\alpha_2}} \frac{X(t)}{2} \geq mX(t), \quad t \geq T_1. \end{aligned} \quad (3.12)$$

Inequalities (3.11) and (3.12) imply that $\mathcal{F}x(t) \geq mX(t)$, $t \geq T_0$. Using (3.5), (3.6) and (3.8) - (3.10) and monotonicity of X respectively, we get

$$\begin{aligned} \mathcal{F}x(t) &\leq \frac{M}{2}X(T_0) + \int_{T_0}^t \left[\int_{T_0}^s \left(\int_r^\infty Kq_\sigma(u)(mX(u))^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} ds \\ &\leq \frac{M}{2}X(t) + K^{\frac{1}{\alpha_1\alpha_2}} m^{\frac{-\beta}{\alpha_1\alpha_2}} 2X(t) \leq \frac{M}{2}X(t) + \frac{M}{2}X(t) = MX(t), \quad t \geq T_0. \end{aligned}$$

We see that $mX(t) \leq \mathcal{F}x(t) \leq MX(t)$ for all $t \in [T_0, \infty)$, i.e. $\mathcal{F}x(t) \in \mathcal{X}_0$ for any $x \in \mathcal{X}_0$, i.e. $\mathcal{F}(\mathcal{X}_0) \subset \mathcal{X}_0$.

(ii) $\mathcal{F}(\mathcal{X}_0)$ is relatively compact.

From (i) we see that $mX(t) \leq \mathcal{F}x(t) \leq MX(t)$, since neither upper bound nor lower bound depends on $x \in \mathcal{X}_0$, $\mathcal{F}(\mathcal{X}_0)$ is uniformly bounded on every $[T_*, T^*] \subset [T_0, \infty)$, i.e. is locally uniformly bounded on $[T_0, \infty)$. The local equicontinuity of $\mathcal{F}(\mathcal{X}_0)$ is implied by the following inequalities holding for all $x \in \mathcal{X}_0$

$$\begin{aligned} 0 \leq (\mathcal{F}x)'(t) &= \left[\int_{T_0}^s \left(\int_r^\infty q(u)x(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} \\ &\leq K^{\frac{1}{\alpha_1 \alpha_2}} m^{\frac{-\beta}{\alpha_1 \alpha_2}} \left[\int_{T_0}^s \left(\int_r^\infty q_\sigma(u)X(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} \\ &\leq K^{\frac{1}{\alpha_1 \alpha_2}} m^{\frac{-\beta}{\alpha_1 \alpha_2}} \left[\int_{T_0}^{T^*} \left(\int_r^\infty q_\sigma(u)X(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}}, \quad t \in [T_0, T^*], \quad T^* \geq T_0. \end{aligned}$$

Using The Arzela-Ascoli lemma we see that $\mathcal{F}(\mathcal{X}_0)$ is relatively compact.

(iii) $\mathcal{F}$ is continuous.

Let $\{x_n\}_{n=1}^\infty \subset \mathcal{X}_0$ be a sequence converging to $x \in \mathcal{X}_0$ in a topology on $C[T_0, \infty)$, i.e. $x_n(t) \rightarrow x(t)$ uniformly on every $[T_*, T^*] \subset [T_0, \infty)$. Let show that also $\mathcal{F}x_n \rightarrow \mathcal{F}x$ in topology on $C[T_0, \infty)$.

First we show that

$$g_n(t) := \left| \left(\int_t^\infty q(s)x(s)^{-\beta} ds \right)^{\frac{1}{\alpha_2}} - \left(\int_t^\infty q(s)x_n(s)^{-\beta} ds \right)^{\frac{1}{\alpha_2}} \right|$$

$$\leq L \left| \int_t^\infty q(s)x(s)^{-\beta} ds - \int_t^\infty q(s)x_n(s)^{-\beta} ds \right|, \quad t \geq T_0, \quad (3.13)$$

where $L > 0$ is a constant.

For $\alpha_2 \geq 1$ this follows from the inequalities $|A^\gamma - B^\gamma| \leq |A - B|^\gamma \leq |A - B|$, where $A > 0, B > 0, 0 < \gamma < 1$. In this case we can put $L = 1$.

If $\alpha_2 < 1$ we can use the mean value theorem to see that for $t \geq T_0$

$$g_n(t) \leq \frac{1}{\alpha_2} \left(\int_t^\infty q(s)(mX(s))^{-\beta} ds \right)^{\frac{1-\alpha_2}{\alpha_2}} \left| \int_t^\infty q(s)x(s)^{-\beta} ds - \int_t^\infty q(s)x_n(s)^{-\beta} ds \right|.$$

Since $\frac{1-\alpha_2}{\alpha_2} > 0$ we can put $L = \frac{m^{-\beta}}{\alpha_2} \left(\int_{T_0}^\infty q(t)X(t)^{-\beta} dt \right)^{\frac{1-\alpha_2}{\alpha_2}}$ to see that (3.13) holds.

From (3.13) it is easy to see that

$$g_n(t) \leq L \int_t^\infty q(s) |x(s)^{-\beta} - x_n(s)^{-\beta}| ds. \quad (3.14)$$

Analogically distinguishing cases $\alpha_1 \geq 1$ and $\alpha_1 < 1$ we get that

$$\begin{aligned} & \left| \left[\int_{T_0}^t \left(\int_s^\infty q(r)x(r)^{-\beta} dr \right)^{\frac{1}{\alpha_2}} ds \right]^{\frac{1}{\alpha_1}} - \left[\int_{T_0}^t \left(\int_s^\infty q(r)x_n(r)^{-\beta} dr \right)^{\frac{1}{\alpha_2}} ds \right]^{\frac{1}{\alpha_1}} \right| \\ & \leq N \int_{T_0}^t g_n(s) ds, \quad t \in [T_0, T^*], \quad T^* \geq T_0. \end{aligned} \quad (3.15)$$

In the case $\alpha_1 < 1$ we put $N = \frac{m^{-\beta}}{\alpha_1} \left[\int_{T_0}^{T^*} \left(\int_r^\infty q(u)X(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1-\alpha_1}{\alpha_1}}$.

Next using (3.15) and (3.14) we show that

$$|Fx(t) - Fx_n(t)| \leq \int_{T_0}^t N \int_{T_0}^s g_n(r) dr ds \leq NL \int_{T_0}^t \int_{T_0}^s \int_r^\infty q(u) |x(u)^{-\beta} - x_n(u)^{-\beta}| du dr ds$$

$$\leq \frac{(T^* - T_0)^2}{2} \int_{T_0}^{\infty} f_n(t) dt, \quad t \in [T_0, T^*], \quad T^* \geq T_0,$$

where $f_n(t) \rightarrow 0$ as $t \rightarrow \infty$ and

$$|f_n(t)| \leq 2Km^{-\beta} q_{\sigma}(t) X(t)^{-\beta}. \quad (3.15)$$

The right-hand side of (3.15) is integrable on $[T_0, \infty)$, so we can use the Lebesgue dominated convergence theorem which ensures that $\mathcal{F}x_n \rightarrow \mathcal{F}x$ in topology on $C[T_0, \infty)$, and this means that $\mathcal{F}$ is continuous.

Using results of (i), (ii), (iii) from the Schauder-Tychonoff fixed point theorem we know that there is $x \in \mathcal{X}$ such that $x(t) = \mathcal{F}x(t)$, i.e.

$$x(t) = x_0 + \int_{T_0}^t \left[\int_{T_0}^s \left(\int_r^{\infty} q(u)x(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} ds, \quad t \geq T_0,$$

which after differentiating three times gives that x is a solution of Eq. (A) and simultaneously $mX(t) \leq x(t) \leq MX(t)$, $t \geq T$, so x is nearly regularly varying of index $\rho = 1 + \frac{1}{\alpha_1}$, $\rho = \frac{\sigma + \alpha_1 \alpha_2 + \alpha_2 + 1}{\alpha_1 \alpha_2 + \gamma}$ or $\rho = 1$ satisfying (3.2), (3.3) or (3.4) when $\sigma = (1 + \frac{1}{\alpha_1})\beta - 1$, $\sigma \in (\beta - \alpha_2 - 1, (1 + \frac{1}{\alpha_1})\beta - 1)$, or $\sigma = \beta - \alpha_2 - 1$. This completes the proof of Theorems 3.1, 3.2 and 3.3.

4 REGULARLY VARYING SOLUTIONS

Returning to the property of regular variation of q , we show that there are also moderately growing solutions of Eq. (A) which are regularly varying. The following lemma which is a generalization of the L'Hospital rule is useful for this purpose.

Lemma 4.1[21]. *Let $f, g \in C^1[T, \infty)$, g and g' are not zero in the neighbourhood of infinity and suppose that either*

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} g(t) = \infty, \quad t \in [T, \infty),$$

or

$$\lim_{t \rightarrow \infty} f(t) = \lim_{t \rightarrow \infty} g(t) = 0, \quad t \in [T, \infty).$$

Then

$$\liminf_{t \rightarrow \infty} \frac{f'(t)}{g'(t)} \leq \liminf_{t \rightarrow \infty} \frac{f(t)}{g(t)}; \quad \limsup_{t \rightarrow \infty} \frac{f(t)}{g(t)} \leq \limsup_{t \rightarrow \infty} \frac{f'(t)}{g'(t)}.$$

THEOREM 4.1. *Let $\beta < \alpha_1 \alpha_2$ and q be a regularly varying function of index σ . Eq. (A) has nontrivial regularly varying solution of index $\rho = 1 + \frac{1}{\alpha_1}$ if and only if $\sigma = (1 + \frac{1}{\alpha_1})\beta - 1$ and (2.1) holds. Any such solution x has the asymptotic behavior*

$$x(t) \sim t^{1 + \frac{1}{\alpha_1}} \left[\frac{(\alpha_1 \alpha_2 + \beta)}{\alpha_1 \alpha_2} \left(\frac{\alpha_1}{1 + \alpha_1} \right)^{\alpha_1 \alpha_2} \int_t^\infty s^{-(1 + \frac{1}{\alpha_1})\beta} q(s) ds \right]^{\frac{1}{\alpha_1 \alpha_2 + \beta}}, \quad t \rightarrow \infty. \quad (4.1)$$

THEOREM 4.2. *Let $\beta < \alpha_1 \alpha_2$ and q be a regularly varying function of index σ . The integral asymptotic relation (AR) has regularly varying solutions of index $\rho \in (1, 1 + \frac{1}{\alpha_1})$ if and only if $\sigma \in (\beta - \alpha_2 - 1, (1 + \frac{1}{\alpha_1})\beta - 1)$. In this case index ρ is given by (2.3) and any such solution x has the asymptotic behavior*

$$x(t) \sim \left[\frac{t^{\alpha_2 + \alpha_1 \alpha_2 + 1} q(t)}{\alpha_1^{(1 + \alpha_2)} \alpha_2 \rho^{\alpha_1 \alpha_2} (\rho - 1)^{\alpha_2} (1 + \frac{1}{\alpha_1} - \rho)} \right]^{\frac{1}{\alpha_1 \alpha_2 + \beta}}, \quad t \rightarrow \infty. \quad (4.2)$$

THEOREM 4.3. *Let $\beta < \alpha_1 \alpha_2$ and q be a regularly varying function of index σ . The integral asymptotic relation (AR) has nontrivial regularly varying solution of index $\rho = 1$ if and only if $\sigma = \beta - \alpha_2 - 1$ and (2.5) holds. In this case any such solution x has the asymptotic behavior*

$$x(t) \sim t \left[\frac{\alpha_1 \alpha_2 + \beta}{\alpha_1 \alpha_2^{1 + \frac{1}{\alpha_2}}} \int_a^t \left(s^{1 - \beta} q(s) \right)^{\frac{1}{\alpha_2}} ds \right]^{\frac{\alpha_2}{\alpha_1 \alpha_2 + \beta}}, \quad t \rightarrow \infty. \quad (4.3)$$

PROOF OF THEOREMS 4.1, 4.2 AND 4.3.

(The “only if” part) The classification of solutions of Eq. (A) in Section 1 and the

assumption of nontriviality of regular variation of solutions from $RV(1)$ and $RV(1 + \frac{1}{\alpha_1})$ ensures that the Theorems 4.1-4.3 concern moderately growing solutions satisfying the integral equation (1.1) and so the asymptotic integral relation (AR), too. So the "only if" parts of Theorems 4.1-4.3 are implied by "only if" parts of Theorems 2.1-2.3.

(The "if" part) Let σ and q fulfil required conditions. From Theorem 3.1-3.3 we know that there exists a nearly regularly varying solution of Eq. (A) obtained as the solution of the integral equation

$$x(t) = x_0 + \int_{T_0}^t \left[\int_{T_0}^s \left(\int_r^\infty q(u)x(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} ds.$$

Let define again X as in (2.18)-(2.20) and let show that $x(t) \sim X(t)$. Put $J(t) := \int_{T_0}^t \left[\int_{T_0}^s \left(\int_r^\infty q(u)X(u)^{-\beta} du \right)^{\frac{1}{\alpha_2}} dr \right]^{\frac{1}{\alpha_1}} ds$. From the proof of Theorems 2.1.-2.3 we know that

$$J(t) \sim X(t), \quad t \rightarrow \infty \quad (4.3)$$

and Theorems 3.1-3.3 ensures that

$$x(t) \asymp X(t), \quad t \rightarrow \infty. \quad (4.4)$$

We put

$$l := \liminf_{t \rightarrow \infty} \frac{x(t)}{J(t)}, \quad L := \limsup_{t \rightarrow \infty} \frac{x(t)}{J(t)}. \quad (4.4)$$

Combining (4.3) and (4.4) we see that $x(t) \asymp J(t)$ for $t \rightarrow \infty$, so the constants l and L defined by (4.4) satisfy

$$0 < l \leq L < \infty. \quad (4.5)$$

Since $\lim_{t \rightarrow \infty} J^{(i)}(t) = \infty$ and $J^{(i+1)}(t) > 0$ on $[T_0, \infty)$ for $i = 0, 1$ and $\lim_{t \rightarrow \infty} J''(t) =$

$\lim_{t \rightarrow \infty} x(t) = 0$ and $J'''(t) < 0$, we can apply Lemma 4.1 on l three times to get

$$\begin{aligned}
 l = \liminf_{t \rightarrow \infty} \frac{x(t)}{J(t)} &\geq \liminf_{t \rightarrow \infty} \frac{x'(t)}{J'(t)} = \liminf_{t \rightarrow \infty} \frac{\left[\int_{T_0}^t \left(\int_s^\infty q(r) x(r)^{-\beta} dr \right)^{\frac{1}{\alpha_2}} ds \right]^{\frac{1}{\alpha_1}}}{\left[\int_{T_0}^t \left(\int_s^\infty q(r) X(r)^{-\beta} dr \right)^{\frac{1}{\alpha_2}} ds \right]^{\frac{1}{\alpha_1}}} \\
 &\geq \left[\liminf_{t \rightarrow \infty} \frac{\left(\int_t^\infty q(s) x(s)^{-\beta} ds \right)^{\frac{1}{\alpha_2}}}{\left(\int_t^\infty q(s) X(s)^{-\beta} ds \right)^{\frac{1}{\alpha_2}}} \right]^{\frac{1}{\alpha_1}} \geq \left[\liminf_{t \rightarrow \infty} \frac{q(t) x(t)^{-\beta}}{q(t) X(t)^{-\beta}} \right]^{\frac{1}{\alpha_1 \alpha_2}} \\
 &= \left[\liminf_{t \rightarrow \infty} \frac{x(t)}{X(t)} \right]^{\frac{-\beta}{\alpha_1 \alpha_2}} = \left[\liminf_{t \rightarrow \infty} \frac{x(t)}{J(t)} \right]^{\frac{-\beta}{\alpha_1 \alpha_2}} = l^{\frac{-\beta}{\alpha_1 \alpha_2}},
 \end{aligned}$$

using (4.3) in the last step. Since $\frac{-\beta}{\alpha_1 \alpha_2} < 0$ and (4.5) holds, the inequality $l \geq l^{\frac{-\beta}{\alpha_1 \alpha_2}}$ implies $l \geq 1$. Analogically we get, that $L \leq L^{\frac{-\beta}{\alpha_1 \alpha_2}}$ and $L \leq 1$. Finally we get the equality $L = l = 1$, which ensures the existence of $\lim_{t \rightarrow \infty} \frac{x(t)}{J(t)} = 1$. From this follows $x(t) \sim J(t) \sim X(t)$ for $t \rightarrow \infty$, i.e. (4.1)-(4.3) hold, as well as nontriviality of regular variation of the solution x in the cases $\sigma = (1 + \frac{1}{\alpha_1})\beta - 1$ and $\sigma = \beta - \alpha_2 - 1$.

EXAMPLE 4.1. Let α_1 , α_2 and β be positive constants with $\beta < \alpha_1 \alpha_2$ and consider the equation

$$(((((x')^{\alpha_1})')^{\alpha_2})')' + q_1(t)x^{-\beta} = 0, \quad (\text{A1})$$

where

$$q_1(t) \sim \left(1 + \frac{1}{\alpha_1}\right)^{\alpha_1 \alpha_2} \alpha_1 \alpha_2 \frac{t^{\beta(1 + \frac{1}{\alpha_1}) - 1}}{(\log t)^{\beta + \alpha_1 \alpha_2 + 1}}, \quad t \rightarrow \infty.$$

The function $q_1 \in \text{RV}(\beta(1 + \frac{1}{\alpha_1}) - 1)$ and satisfies the condition (2.1) (due to a need of the positivity of the function q on $[a, \infty)$, we consider $a > 1$)

$$\begin{aligned}
 &\int_a^\infty \left(1 + \frac{1}{\alpha_1}\right)^{\alpha_1 \alpha_2} \alpha_1 \alpha_2 \frac{1}{t(\log t)^{\beta + \alpha_1 \alpha_2 + 1}} dt \\
 &= \frac{\left(1 + \frac{1}{\alpha_1}\right)^{\alpha_1 \alpha_2} \alpha_1 \alpha_2}{\alpha_1 \alpha_2 + \beta} \frac{1}{\log a}.
 \end{aligned}$$

Consequently by Theorem 4.1 Eq. (A1) possesses moderately growing solutions which are nontrivial regularly varying of index $1 + \frac{1}{\alpha_1}$ and all such solutions have the unique asymptotic behavior

$$x(t) \sim t^{1+\frac{1}{\alpha_1}} \left[\frac{\alpha_1 \alpha_2 + \beta}{\alpha_1 \alpha_2} \left(\frac{\alpha_1}{1 + \alpha_1} \right)^{\alpha_1 \alpha_2} \int_t^\infty s^{-(1+\frac{1}{\alpha_1})\beta} q_1(s) ds \right]^{\frac{1}{\alpha_1 \alpha_2 + \beta}} \\ \sim \frac{t^{1+\frac{1}{\alpha_1}}}{\log t}, \quad t \rightarrow \infty.$$

EXAMPLE 4.2. Let α_1 , α_2 and β be positive constants with $\beta < \alpha_1 \alpha_2$ and consider the equation

$$((((x')^{\alpha_1 *})')^{\alpha_2 *})' + q_2(t)x^{-\beta} = 0, \quad (\text{A2})$$

where

$$q_2(t) \sim \left(1 + \frac{1}{\alpha_1 + 1} \right)^{\alpha_1 \alpha_2} \left(\frac{\alpha_1}{\alpha_1 + 1} \right)^{\alpha_2} \frac{\alpha_2}{\alpha_1 + 1} t^{\frac{-\alpha_2 - \alpha_1 - 1}{\alpha_1 + 1} + (1 + \frac{1}{\alpha_1 + 1})\beta} \\ \times \exp((\alpha_1 \alpha_2 + \beta)(\log t)^{\frac{1}{3}} \cos(\log t)^{\frac{1}{3}}) \quad t \rightarrow \infty.$$

Notice that $q_2 \in \text{RV}(\sigma)$, with $\sigma = \frac{-\alpha_2 - \alpha_1 - 1}{\alpha_1 + 1} + (1 + \frac{1}{\alpha_1 + 1})\beta$, i.e. $\sigma \in (\beta - \alpha_2 - 1, (1 + \frac{1}{\alpha_1})\beta - 1)$. By Theorem 4.2 Eq. (A2) possesses moderately growing solutions which are regularly varying of index $\rho = 1 + \frac{1}{\alpha_1 + 1}$ and each such solution behaves like

$$x(t) \sim \left[\frac{t^{\alpha_2 + \alpha_1 \alpha_2 + 1} q_2(t)}{\alpha_1^{(1+\alpha_2)} \alpha_2 \rho^{\alpha_1 \alpha_2} (\rho - 1)^{\alpha_2} (1 + \frac{1}{\alpha_1} - \rho)} \right]^{\frac{1}{\alpha_1 \alpha_2 + \beta}} \\ \sim t^{1+\frac{1}{\alpha_1+1}} \exp((\log t)^{\frac{1}{3}} \cos(\log t)^{\frac{1}{3}}), \quad t \rightarrow \infty.$$

EXAMPLE 4.3. Let α_1 , α_2 and β be positive constants with $\beta < \alpha_1 \alpha_2$ and consider the equation

$$((((x')^{\alpha_1 *})')^{\alpha_2 *})' + q_3(t)x^{-\beta} = 0, \quad (\text{A3})$$

where

$$q_3(t) \sim \frac{\alpha_1^{\alpha_2} \alpha_2 (\log t)^{\beta + \alpha_1 \alpha_2 - \alpha_2}}{t^{\alpha_2 + 1 - \beta}}, \quad t \rightarrow \infty.$$

The function $q_3 \in \text{RV}(\beta - \alpha_2 - 1)$ and satisfies the condition (2.5) ($a > 1$)

$$\begin{aligned} & \int_a^\infty \frac{\alpha_1 \alpha_2^{\frac{1}{\alpha_2}} (\log t)^{\frac{\beta + \alpha_1 \alpha_2}{\alpha_2} - 1}}{t} dt \\ &= \frac{\alpha_1 \alpha_2^{1 + \frac{1}{\alpha_2}}}{\alpha_1 \alpha_2 + \beta} [\log t]_a^\infty = \infty. \end{aligned}$$

Consequently by Theorem 4.3 Eq. (A3) possesses moderately growing solutions which are nontrivial regularly varying of index 1 and all such solutions have the unique asymptotic behavior

$$x(t) \sim t \left[\frac{\alpha_1 \alpha_2 + \beta}{\alpha_1 \alpha_2^{1 + \frac{1}{\alpha_2}}} \int_a^t \left(s^{1-\beta} q_3(s) \right)^{\frac{1}{\alpha_2}} ds \right]^{\frac{\alpha_2}{\alpha_1 \alpha_2 + \beta}} \sim t \log t, \quad t \rightarrow \infty.$$

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