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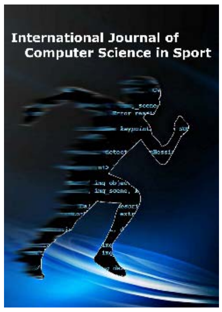

Performance Estimation using the Fitness-Fatigue Model with Kalman Filter Feedback

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Abstract

Tracking and predicting the performance of athletes is of great interest, not only in training science but also, increasingly, for serious hobbyists. The increasing availability and use of smart watches and fitness trackers means that abundant data is becoming available, and the interest to optimally use this data for performance tracking and training optimization is great. One competitive model in this domain is the 3-time-constant fitness-fatigue model by Busso based on the model by Banister and colleagues. In the following, we will show that this model can be written equivalently as a linear, time-variant state-space model. With this understanding, it becomes clear that all methods for optimum tracking in state-space models are also directly applicable here. As an example, we show how a Kalman filter can be combined with the fitness-fatigue model in a mathematically consistent fashion. This gives us the opportunity to optimally consider measurements of performance to adapt the fitness and fatigue estimates in a data-driven manner. Results show that this approach is capable of clearly improving performance tracking and prediction over a range of different scenarios.

KEYWORDS: PERFORMANCE MODELING, KALMAN FILTER, FITNESS, FATIGUE

Introduction

The relationship between training and performance is of particular significance in competitive as well as rehabilitation and recreational sports regarding the planning and design of training processes. In the past four decades, a number of attempts have been made to model training effects on performance by means of mathematical models, with the fitness-fatigue model (FF-model) and its extensions being the most popular approach (Busso, 2003; Calvert, T. W., Banister, E. W., Savage, M. V., & Bach, T., 1976). In this model, athletes are understood as a system with training load as the input, equally feeding two antagonistic effects - fitness and fatigue -, which compromise the performance as the output.

Several studies have been published to review the model and study its parameters (e.g. Chiu, L. Z., & Barnes, J. L., 2003; Clarke, D. C., & Skiba, P. F., 2013; Hellard, P., Avalos, M., Lacoste, L., Barale, F., Chatard, J. C., & Millet, G. P., 2006; Jobson, S. A., Passfield, L., Atkinson, G., Barton, G., & Scarf, P., 2009; Taha, T., & Thomas, S. G., 2003) and to compare different training loads and psychological markers as input and output variables (e.g. Millet, G. P., Groslambert, A., Barbier, B., Rouillon, J. D., & Candau, R. B., 2005; Wallace, L. K., Slattery, K. M., & Coutts, A. J., 2014). Further applications include the simulation of taper phases and training programs (e.g. Sanchez, A. M., Galbs, O., Fabre-Guery, F., Thomas, L., Douillard, A., Py, G., et. al., 2013; Thomas, L., Mujika, I., & Busso, T., 2008), though the prediction of performances in conjunction with verifications by means of performance measurements has been neglected almost entirely (Chalencon, S., Pichot, V., Roche, F., Lacour, J. R., Garet, M., Connes, P., et. al., 2015).

While the modified version by Busso (2003) incorporates a new variable to use past training loads to adapt the fatigue level, to our knowledge, in the literature there exist no online feedback mechanisms to use performance measurements for optimally improving fitness and fatigue estimates. This would be useful to account for unmodeled changes in fitness and fatigue (e.g. due to exhausting other activities, stress, health issues, vacation times) and for measurement errors (e.g. due to varying motivation or environmental conditions in all-out tests).

To allow for such online learning, we therefore propose a new solution incorporating feedback in a mathematically optimal fashion: A Kalman filter is used to better estimate the future performance levels based on training input while simultaneously improving the fitness and fatigue estimate at each point where a measurement is available, using optimal feedback of the prediction error. This is made possible by our reframing of the fitness-fatigue model as a linear, time-variant state-space model, which will be shown in section Methods. Experiments based on this model are shown in subsection Experimental Setup, with the results and conclusions in the according sections.

Methods

Our implementation is based on the 3-time-constant fitness-fatigue model, which was described by Busso in 2003, building upon Calvert et al. (1976), with a discretized version as follows

$$p(k) = p^* + c_1 \cdot \sum_{i=1}^{k-1} u(i) \cdot e^{\frac{-(k-i)}{\tau_1}} - \sum_{i=1}^{k-1} c_2(i) \cdot u(i) \cdot e^{\frac{-(k-i)}{\tau_2}}. \tag{1}$$

Here, the value of c_2 is estimated using a first-order filter by

$$c_2(i) = c_3 \sum_{j=1}^{i} u(j) \cdot e^{\frac{-(i-j)}{\tau_3}}. \tag{2}$$

The equations describe performance $p(k)$ as being dependent on the past training loads u until day k with exponentially decaying constituents of fitness and fatigue starting from initial point p^*. The free parameters in the model are the weighting factors[1] c_1 and c_3 as well as the three time constants τ_1, τ_2, and τ_3.

The two summation terms in Eq. (1) can be treated as states in a linear system model of the athlete, as will be shown below. The variability of training effectiveness, inaccuracies in measurement and to some extent a change in the physiological response of the athlete can then be catered for by using a feedback mechanism in the model.

Linear System Model of Fitness and Fatigue

A linear system is often described (Ludyk, G., 1995) in the following form:

$$\mathbf{x}_{k+1} = \boldsymbol{A}_k \mathbf{x}_k + \boldsymbol{B}_k u_k + \boldsymbol{v}_k. \tag{3}$$

Here, the vector $\boldsymbol{x}_k$ describes the d-dimensional state of the system, the system matrix $\boldsymbol{A}_k$ shows how the state at one time instant changes to the next point in time, u_k is the system input, influencing the system state via the input matrix $\boldsymbol{B}_k$ and the system noise term $\boldsymbol{v}_k$ describes random changes in the state. It is typically assumed that the states of the system are not observable directly, but are only accessible by means of indirect measurements y_k, which are again a linear function of the state, in accordance with

$$y_k = \boldsymbol{C}_k \mathbf{x}_k + n_k. \tag{4}$$

The output matrix $\boldsymbol{C}_k$ shows the influence of each state component on the measurement, and the observation noise n_k is often assumed to be Gaussian distributed.

A linear system that corresponds with these equations possesses exponential dynamics, when the noise is neglected. For a time-invariant state matrix $\boldsymbol{A} = \boldsymbol{A}_k \ \forall k$, and for a zero initial state, it can be shown (Dahleh, M., Dahleh, M. A., & Verghese, G., 1999) that the system state will evolve via

$$\mathbf{x}_k = \sum_{i=1}^{k-1} \boldsymbol{A}^{k-i-1} \boldsymbol{B}_i u_i. \tag{5}$$

To convert this model into the form of the fitness-fatigue model, we have used the following definitions:

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \boldsymbol{A_k} = \boldsymbol{A} = \begin{pmatrix} e^{\frac{-1}{\tau_1}} & 0 \\ 0 & e^{\frac{-1}{\tau_2}} \end{pmatrix}, \boldsymbol{B_k} = \begin{pmatrix} e^{\frac{-1}{\tau_1}} \\ c_2(k) \cdot e^{\frac{-1}{\tau_2}} \end{pmatrix}, \boldsymbol{C_k} = \boldsymbol{C} = \begin{pmatrix} c_1 & -1 \end{pmatrix} \tag{6}$$

where $\boldsymbol{x} = [x_1, x_2]^T$ is the state vector composed of fitness as x_1 and fatigue as x_2. $\boldsymbol{A}$ is the system matrix containing exponential decay rates for both states in the diagonal. The time varying input matrix $\boldsymbol{B}_k$ contains the same two exponential decays with time constants τ_1 and

[1] We denote these factors by c, rather than k, to be consistent with engineering literature on state-space models and Kalman filtering. The same reason leads us to name the training load u instead of w, as would be more usual in training science publications.

τ_2. In addition, there is a weighting factor for the influence of the training on the fatigue component, which is defined as $c_2(k) = c_3 \sum_{j=1}^{k} u(j)\ e^{\frac{-(k-j)}{\tau_3}}$.

Inserting these definitions into Eq. (5) and considering x_1 and x_2 separately leads to

$$x_1 = \sum_{i=1}^{k-1} \boldsymbol{A}_{11}^{k-i-1} B_{i,1} u_i \quad (7)$$

$$= \sum_{i=1}^{k-1} e^{\frac{-(k-i-1)}{\tau 1}} e^{\frac{-1}{\tau 1}} u_i \quad (8)$$

$$= \sum_{i=1}^{k-1} e^{\frac{-(k-i)}{\tau 1}} u_i. \quad (9)$$

and

$$x_2 = \sum_{i=1}^{k-1} \boldsymbol{A}_{22}^{k-i-1} B_{i,2} u_i \quad (10)$$

$$= \sum_{i=1}^{k-1} c_2(i) \cdot e^{\frac{-(k-i)}{\tau 2}} u_i. \quad (11)$$

Combining x_1 and x_2 according to Eq. (4), while neglecting the observation noise n_k, gives the final output

$$y(k) = \boldsymbol{C}\mathbf{x}_k \quad (12)$$

$$= c_1 x_1 - x_2 \quad (13)$$

$$= c_1 \sum_{i=1}^{k-1} e^{\frac{-(k-i)}{\tau 1}} u_i - \sum_{i=1}^{k-1} c_2(i) \cdot e^{\frac{-(k-i)}{\tau 2}} u_i \quad (14)$$

which can be easily seen to be exactly equivalent to $p(k) - p^*$, the deviation of the performance from its set point p^*, as given in Eq. (1).

Hence, the well-known fitness-fatigue model can be understood as a linear state-space model, linearized around its set point given by the initial performance p^*, and with the parameters $\boldsymbol{A}$, $\boldsymbol{B}_k$ and $\boldsymbol{C}$ as given above. This understanding opens a number of possibilities:

First, we can understand all performance measurements as inherently noisy, based on our model of state noise and measurement noise:

- The system model contains two components, x_1, the fitness, and x_2, the fatigue, which can both now be described as subject to some random fluctuations, e.g. caused by unmodeled exertions of the athlete or by possibly hidden health issues. These random (or at least unobserved) effects on fitness and fatigue can now be modeled by an explicit term, $\boldsymbol{v}_k$, the *state noise*.
- In addition, we are also explicitly modeling the *observation noise*, n_k, which can stem from a number of sources, such as actual measurement errors, or, in all-out performance estimation, possible differences caused by varying motivation or by varying measurement conditions.

Secondly, and interestingly, we can derive an improved performance modeling approach, which explicitly considers these random fluctuations of the states and the measurements, and which aims to compensate for them by optimal use of the performance measurements with the goal of continuously updating the model of the athlete. This approach, a Kalman-filtering-based fitness-fatigue-model, will be described in the following section.

Kalman Filtering for Optimal Tracking of Fitness and Fatigue

The Kalman filter uses measurements of a quantity containing statistical noise over time to produce a more reliable estimate in a recursive manner. It relies only on the last calculated state, a model-predicted subsequent state and a correction based on the last measured value to generate the present state estimate.

In addition to the linear system model parameters defined above, it uses noise covariance terms to describe random fluctuations in the state vector $\boldsymbol{x}_k$ and the observation y_k. $\boldsymbol{v}_k$ is the state noise with covariance $\boldsymbol{Q}$ and $\boldsymbol{R}$ is the noise variance of the observation noise n_k. The state reconstruction by the Kalman filter is done in two steps, cf. Ludyk, G. (1995):

First, the *a posteriori* state estimate $\hat{\boldsymbol{x}}_k$ is updated by the feedback

$$\hat{\mathbf{x}}_k = \mathbf{z}_k + \mathbf{K}_k(y_k - \mathbf{C}\mathbf{z}_k) \tag{15}$$

where y_k is the measurement value and the optimal Kalman gain, $\boldsymbol{K}_k$, is

$$\mathbf{K_k} = \mathbf{M_k}\mathbf{C^T}(\mathbf{R} + \mathbf{C}\mathbf{M_k}\mathbf{C^T})^{-1}. \tag{16}$$

The matrix $\boldsymbol{M}_k$ is iteratively computed as

$$\mathbf{M_{k+1}} = \mathbf{Q} + \mathbf{A}\mathbf{M_k}\mathbf{A^T} - \mathbf{A}\mathbf{M_k}\mathbf{C^T}(R + \mathbf{C}\mathbf{M_k}\mathbf{C^T})^{-1}\mathbf{C}\mathbf{M_k}\mathbf{A^T}. \tag{17}$$

Then, at the start of the next cycle, a predicted *a priori* state estimate $\boldsymbol{z}_k$ is formed using the system dynamics

$$\mathbf{z}_k = \mathbf{A}\hat{\mathbf{x}}_{k-1} + \mathbf{B}_{k-1}u_{k-1}. \tag{18}$$

To implement this approach, these equations, (15)-(18), are used together with the state-space model derived in subsection Linear System Model of Fitness and Fatigue. The implementation is done in Simulink and shown in Figure 1.

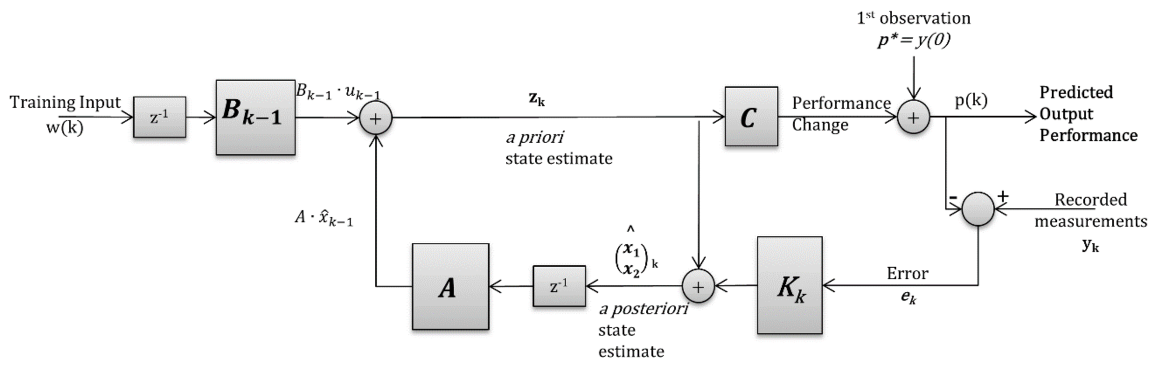

Figure 1. Block diagram of Kalman Filter System.

As it can be seen, this implies that the following process takes place on every time instant - i.e. on every day k - of our simulation:

1. The last estimated values of the states x_1 & x_2 (fitness & fatigue) produce the expected performance output through the output matrix $\boldsymbol{C}$.
2. The output is compared with the current performance measurement/observation (if

available) and a measurement residual (error) is calculated. If a measurement at that instant is not available, the error e_k is set to zero.

3. The Kalman gain $\boldsymbol{K}_k$ is calculated depending on the provided state and observation noise variances, using Eq. (16) and (17).
4. The Kalman filter uses the error feedback to correct the state estimates according to Eq. (15), which will affect the performance prediction in the next time step.
5. The new training impulse is added to the state variables after multiplication with the time-varying input matrix $\boldsymbol{B}_k$.

Using this method, we expect a more reliable performance estimate, especially in cases where the original model curve differs from measured values due to measurement errors or unmodeled effects on fitness and fatigue.

Experimental Setup

All experiments are based on the performance and training load data of five athletes, collected during a phase of 160 days. Training performed in water and on dryland was quantified daily according to Mujika et al. (1996). Swimming kilometers were divided into five intensity levels based on the swimming speed, multiplied by weighting coefficients (1, 2, 3, 5, 8) and cumulated at last. Dryland training was converted to water training equivalents as follows: 1-h dryland training corresponds to 2km of swimming and was weighted by its content (endurance*2, conditioning*5 and strength*8) followed by an accumulation. Finally, the training load is a single value for each day, which is essentially the intensity-weighted training volume. Regarding performance, a semi-tethered swimming test (20m without start) consisting of three repetitions (resistance increased trial by trial) was conducted weekly to determine the swimming-specific performance expressed in mean velocity reached for 60m (3x20m).

The effect on the fitness x_1 is an accumulation of the training input u_k weighted by c_1, whereas the contribution of the input u_k to fatigue x_2 is multiplied by a weighting factor of each individual input $c_2(k)$ calculated using τ_3 in (2). Absent an input, both states have an exponentially decreasing behavior according to their respective decay time constants (τ_1, τ_2). The magnitude factors c_1 and c_3 absorb the unit of measurement and have no direct physiological basis (Pfeiffer, 2008). The output generated by the model simply starts off from the first experimental observation ($p^* = y_0$).

The following parameters can be controlled in the model:

i. τ_1 Time constant for fitness decay

ii. τ_2 Time constant for fatigue decay

iii. τ_3 Time constant for decay of negative influence (weighting factor) of training

iv. c_1 Magnitude factor for fitness

v. c_3 Magnitude factor for fatigue

vi. $\boldsymbol{Q} = \begin{pmatrix} \sigma_{x_1}^2 & \sigma_{x_1,x_2} \\ \sigma_{x_1,x_2} & \sigma_{x_2}^2 \end{pmatrix}$ State noise covariance matrix, consisting of variance of fitness $\sigma_{x_1}^2$ and fatigue $\sigma_{x_2}^2$ and their covariance σ_{x_1,x_2}.

The state noise covariance $\boldsymbol{Q}$ influences the estimation error covariance matrix $\boldsymbol{M}_k$ used in the Kalman gain calculations. It governs how the Kalman gain evolves over time, denoting the strength of the filtering effect on the model. The matrix is defined so as to treat fitness and fatigue having some interdependence defined by σ_{x_1,x_2}. Therefore, an error will cause both states to change in the direction causing the performance output to move towards the measured value. This effect is illustrated in the Appendix.

The dynamics of the model are implemented in a Simulink model, called by a MATLAB script supplying input vectors and collecting outputs of interest from the simulation run.

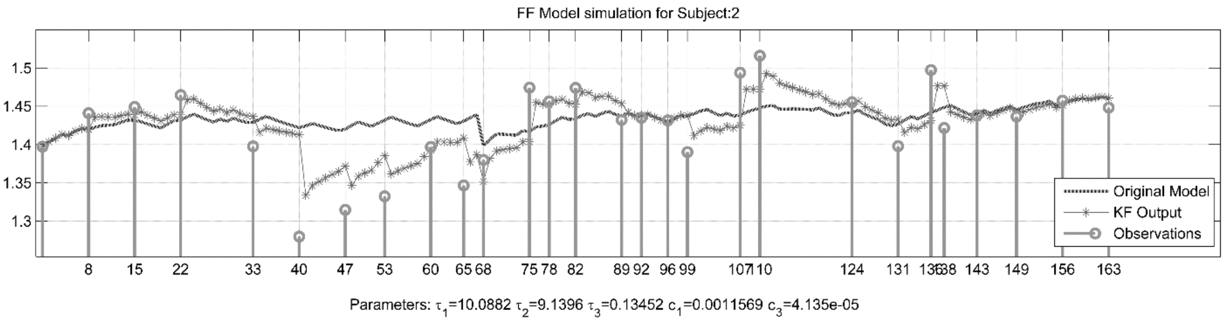

Figure 2. Simulation results for subject 2 with full range calibration. The original $MAPE$ value is 2.69% and $MAPE_{Kalman} = 2.31\%$.

A comparison between an FF-model with and without Kalman feedback is possible in Figure 2. The original model performs well in this case but fails to explain large/faster changes in performance. The Kalman filter model however, has the disadvantage of occasionally relying heavily on the experimental observations. The parameters here show a case where almost equal time constants of 9 days for fitness and 8 days for fatigue were estimated. The method used to estimate these values will now be discussed.

Optimization / Model Fitting

The model parameters are optimized so as to minimize the sum of squared errors between predicted and measured data. The mean absolute percentage error ($MAPE$) is used to assess the quality of Kalman filtered response of the model.

First, an array of the weighting factors c_2 for the complete range of days was calculated by Eq. (2). The remaining set of model parameters (τ_1, τ_2, τ_3, c_1, c_3 and all 3 components of $\boldsymbol{Q}$) mentioned in section Experimental Setup was then determined using the multi-start interior point search algorithm minimizing the RSS (Residual Sum of Squares) between the model output and measured data.

The standard deviation of the observation noise n_k is fixed to a value of 0.0126 calculated as 1% of the average measured output performance over all test subjects. The model is designed to start from a baseline performance, p^*, which in our case is the first available experimental performance measurement y_0. Fitness and fatigue start from zero as initial values and may be increased or decreased by the Kalman filter, however, negative values are not allowed.

MATLAB's multi-start optimization method is used, which finds multiple local minima using the interior point algorithm. The lower and upper bounds for model parameters are based on ranges from previous studies. Some constraints are introduced to keep model parameters within reasonable physiological ranges e.g. the fitness time constant (τ_1) must be at least 3 times and the fatigue time constant (τ_2) at least 2 times longer than the negative training effect time constant (τ_3). The fatigue time constant is also constrained to be 1.1 times smaller the fitness time constant. This is done to prevent the model from having a largely constant performance response, and also because it is understood that fatigue is short-lived compared to fitness in the physiological system.

The magnitude factors c_1 and c_3 depend on the output scale and are therefore also optimized. Finally, in the KF-parameters being optimized as components of the state noise covariance matrix, $\boldsymbol{Q}$, the variances of fitness and fatigue, $\sigma^2_{x_1}$ and $\sigma^2_{x_2}$, are restricted to less than 1000 and 1×10^8, respectively, while their covariance is limited to less than 1×10^6. To keep the covariance matrix positive semi-definite, a nonlinear constraint is applied.

To obtain individual models per athlete, two types of calibration are carried out separately for each individual. First, the entire range of available experimental data are used to obtain a best fit for the Kalman-filtered performance curve, the so-termed *full-range optimization*. The second type, being more important for the purpose of future performance prediction, optimizes the parameters based only on the first half of experimental observations and examines the model evolution for the latter half of the training season in comparison to the observations. Hence, it is referred to as *half-range optimization* in the following. Results for both these sets of athlete-specific parameters are given in Tables 1 and 2.

Secondly a generalized set of parameters is computed across all individuals. This method is expected to give a more reliable estimate of the parameters, as there is effectively more experimental data involved in the calibration, which can thus augment the sparse inital data of a new subject for the computation of parameter estimates during their initial period of training. The aim is also to find parameters that can be used for performance prediction on any subject without any prior data about the individual. The optimizer hence searches for a parameter set that minimizes the residual sum of squares over all five test subjects for full- and half-range calibration (also shown in Table 3).

Results

Calibration per subject

In the full-range-calibrated model, the MAPE average of the subjects improves from 3.35% to 2.31% by using Kalman filtering, which is shown in detail in Table 1. However, a beneficial effect is immediately visible in the performance curve comparison graphically where the filtered version improves considerably at times when the original model might have drifted to a wrong direction, see Figure 2.

Table 1. Full-range-calibrated individual parameters.

Parameter / Subject	τ_1	τ_2	τ_3	c_1	c_3	$\sigma^2_{x_1}$	$\sigma^2_{x_2}$	σ_{x_1,x_2}	$MAPE$ Original (%)	$Kal\ gain$ converged	$MAPE$ Kalman (%)
P1	37.3	33.8	0.86	0.0012	0.000028	5	8831	200	2.54	$\begin{pmatrix} -2.9 \\ -191.2 \end{pmatrix}$	2.54
P2	10.1	9.1	0.13	0.0012	0.000041	69	99051	22	2.69	$\begin{pmatrix} 216 \\ -10576 \end{pmatrix}$	2.31
P3	15.3	9.6	0.15	0.0008	0.000044	37	30022	1048	1.84	$\begin{pmatrix} -257.5 \\ -735.8 \end{pmatrix}$	1.82
P4	16.1	3.1	0.11	0.0003	0.000060	2663	99991	723	1.94	$\begin{pmatrix} 1509 \\ -5906 \end{pmatrix}$	1.81
P5	15.0	13.6	5.00	0.0009	0.000008	367	10000	1	7.75	$\begin{pmatrix} 771 \\ -177 \end{pmatrix}$	3.06
Average	18.7	13.9	1.3	0.0009	0.000036	628	49579	399	3.35	$\begin{pmatrix} 448 \\ -3517 \end{pmatrix}$	2.31

In the half-range-calibrated model, it is evident that the goodness of fit is largely dependent on the number of data points and their fluctuations in the calibration range. The state noise variance values in $\boldsymbol{Q}$ usually have to assume relatively higher values in this case to account for the fluctuations. The estimated FF-model parameters are different from the full-range optimized ones and may therefore not be the best representation of each individual athlete's response. However, it is the only realistic scenario for judging prediction quality, as the full-range-calibrated model is actually, unrealistically, using training data for the testing phase. Hence, the half-range calibration, tested on the second half of the data points, offers a practically applicable test scenario. In this case, the Kalman filter again proves to be very helpful, often even leading to greater improvements than for the full-range-calibration scenario, since the Kalman filter reduces the error through its feedback mechanism, as seen for example for Subject 1 in Figure 3. Improvement of the average MAPE between the original FF-model (4.12%) and the Kalman filter model (3.56%), as seen in Table 2 indicates the usefulness of using a feedback-based prediction method, especially when data available for calibration is relatively scarce.

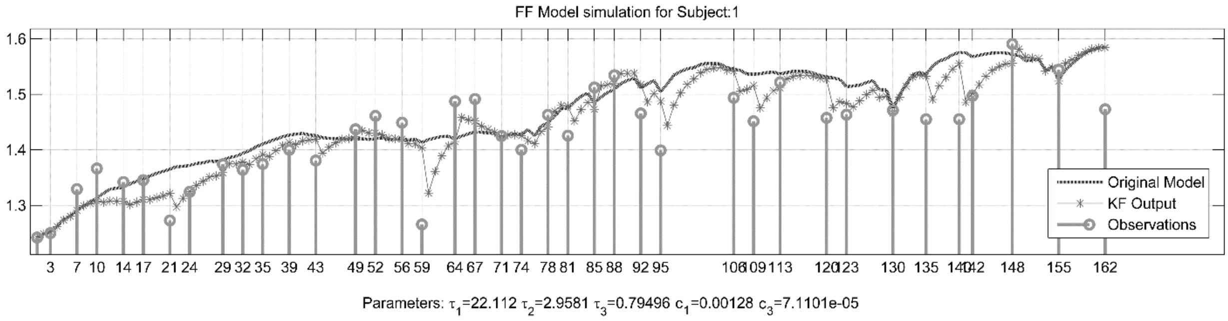

Figure 3. Simulation results for subject 1 with half range calibration. The original $MAPE$ value is 3.17% and $MAPE_{Kalman} = 2.72\%$.

Table 2. Half-range-calibrated individual parameters.

Parameter / Subject	τ_1	τ_2	τ_3	c_1	c_3	$\sigma^2_{x_1}$	$\sigma^2_{x_2}$	σ_{x_1,x_2}	$MAPE$ Original (%)	$Kal\ gain$ converged	$MAPE$ Kalman (%)
P1	22.1	3.0	0.79	0.0012	0.000071	9998	35246353	593633	3.17	$\begin{pmatrix} 44 \\ -13255 \end{pmatrix}$	2.72
P2	13.6	2.0	0.13	0.0002	0.000046	9972	99922	31540	3.07	$\begin{pmatrix} 4773 \\ -11728 \end{pmatrix}$	2.54
P3	5.6	5.1	0.48	0.0020	0.000070	9671	30240	225	2.07	$\begin{pmatrix} 507 \\ -43 \end{pmatrix}$	1.95
P4	6.7	6.1	0.12	0.0016	0.000141	1151	30009	4528	5.23	$\begin{pmatrix} 751 \\ 1908 \end{pmatrix}$	3.88
P5	3.2	2.3	0.09	0.0071	0.000376	5688	100700	23914	7.05	$\begin{pmatrix} 180 \\ 756 \end{pmatrix}$	6.69
Average	10.2	3.7	0.32	0.0024	0.000141	7296	7101445	130768	4.12	$\begin{pmatrix} 1251 \\ -4472 \end{pmatrix}$	3.56

Calibration across subjects

In the second set of experiments, we have tested performance of the parameter set obtained by optimization across individuals (referred to as *generalized parameters*). The parameter set obtained over full-range calibration behaves as an averaged response of all athletes and tends to expect minimal change in performance from training inputs. However, variances assume higher values, thus allowing larger corrections based on experimental observations. Generalized parameters obtained via half-range calibration perform better in predicting latter half performance on average. Therefore, an individual calibration may not always be best suited for prediction, most definitely not when only little training data is available for that individual. The result of simulation using the generalized parameter set for subject 4 is shown in Figure 4 and all results are collected in Table 3.

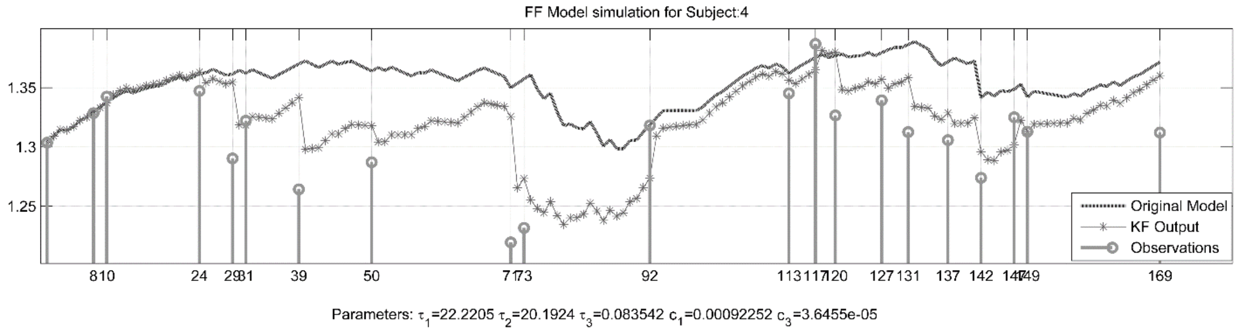

Figure 4. Simulation results for subject 4 with generalized parameters from half-range calibration across subjects. The original $MAPE$ value is 3.75% and $MAPE_{Kalman} = 2.43\%$.

Table 3. Generalized parameters calibrated across subjects.

Parameter / Subject	τ_1	τ_2	τ_3	c_1	c_3	$\sigma^2_{x_1}$	$\sigma^2_{x_2}$	σ_{x_1,x_2}	Average $MAPE$ Original (%)	$Kal\ gain$ converged	Average $MAPE$ Kalman (%)
Full	54.4	49.3	0.12	0.0005	0.000018	326	100013	911	5.07	$\begin{pmatrix} 802 \\ -7267 \end{pmatrix}$	3.66
Half	22.2	20.2	0.08	0.0009	0.000036	93	100024	285	5.00	$\begin{pmatrix} 261 \\ -11029 \end{pmatrix}$	3.57

Discussion and Conclusion

It has been shown that the fitness-fatigue model, widely used in athletic performance modeling, can be represented equivalently by a linear, time-variant state-space model. For such models, an optimal tracking algorithm exists in the form of the Kalman filter, which utilizes error feedback to incrementally update and improve its state estimate.

Based on this understanding, we have introduced a new method for efficient estimation of athletic performance. For this purpose, we start out with the three-time-constant fitness fatigue model, re-write it as a state-space model, introduce state and measurement noise, and utilize the update equations of the Kalman filter. This approach offers advantages over conventional performance prediction by optimally using available measurement data for correcting the state estimate online. This is helpful with respect to a number of issues. For example, the original fitness-fatigue model relies on the accuracy of the performance measurements, and on them always being carried out at the same level of intensity, whereas the suggested, stochastic version has a much higher tolerance for measurement errors.

In the second part of the paper, we evaluate different partitionings of training and test data, to contrast individual with generalized, and full-range-calibrated with half-range-calibrated models. Here, the Kalman filter has proven helpful in all conditions. This makes it applicable for many use cases of fitness tracking: for online training planning as well as for an analysis of past data, and also for new users with sparse data, where a generalized set of parameters across individuals can be advantageously combined with the Kalman filtering approach. In all these partitionings, the average $MAPE$ is improved significantly, with the most notable improvement, from 5.0% to 3.57% $MAPE$, achieved for the generalized parameter set.

The variability of the model parameters during the training period was not considered in our study, but its inclusion is quite promising to improve the estimation further. Again, the view of the fitness-fatigue model as a state-space model can help in this endeavor, as optimal adaptation approaches exist for this purpose, as well. In addition, a state-space analysis and subsequent design of a Kalman filter can be applied on other training-performance models, e.g. to those which quantify training inputs separately via time and intensity.

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Appendix: Interpretation of parameters

Table 4. Kalman filter parameters.

Parameter	Typical value	Deduction
$\sigma_{x_1}^2, \sigma_{x_2}^2$ Standard deviation of fitness and fatigue noise in v_k	Estimated: $\sigma_{x_1} = 33\sim100$, $\sigma_{x_1} = 173\sim6000$	By calibration
σ_{Px} Standard deviation of observation noise n k	Chosen $\sigma_{Px} = 0.0126$ w.r.t. output data value ranges.	Measurement tolerance was set to 1% of the average output performance which is 1.26 over all test subjects
$\boldsymbol{Q} = \begin{pmatrix} \sigma_{x_1}^2 & \sigma_{x_1,x_2} \\ \sigma_{x_1,x_2} & \sigma_{x_2}^2 \end{pmatrix}$ State noise covariance matrix $\boldsymbol{M}_0 = \boldsymbol{Q}$ which is updated to $\boldsymbol{M}_k$ in every iteration	$\boldsymbol{M}_0 = \begin{pmatrix} 628 & 399 \\ 399 & 49579 \end{pmatrix}$ For subject 5 evolves to $\boldsymbol{M}_{end} = \begin{pmatrix} 485 & 547 \\ 547 & 73185 \end{pmatrix}$	There exists some covariance between the states. The initial covariance $\boldsymbol{M}_0$ quickly evolves to a stable value via Eq. (17).
R Observation noise covariance matrix	Fixed to $(0.0126)^2$	Output measurement is scalar, hence a suitable value can be chosen based on the observed variability of the data..
$\boldsymbol{K}_k$ Kalman gain A higher gain has a higher effect on the change of the estimated state via $\boldsymbol{K}_k(y_k - \boldsymbol{C}\,\boldsymbol{z}_k)$ according to Eq. (15) and (18).	Within the range $\binom{-300}{-11000}$ to $\binom{1500}{-200}$ for full range, $\binom{40}{-4500}$ to $\binom{5000}{2000}$ for half range calibration. Correction from gain typically moves state away from direction of error.	Kalman gain is computed via Eq. (16) and (17).