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Dilation and Modelling of Sands in the Light of Experimental Data

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Abstract

The problem of dilation is discussed in the context of classical Cam-Clay model, which was developed on the basis of a specific assumption regarding the plastic work. This assumption leads to a special form of the dilation function, from which a shape of yield function is derived. The above mentioned assumption is compared with the results of the triaxial tests, performed on the model "Skarpa" sand. It is shown that the Cam-Clay approach is not realistic, as it is based on the assumption which is not consistent with experimental data. Some general considerations and discussion of this problem are also presented.

Key words: sands, dilation, constitutive equations

1. Introduction

Dilation denotes a change in volume of granular materials due to shearing. This phenomenon was first discovered and defined by Reynolds in 1885, and still plays an important role in soil mechanics. First investigations were concentrated on explanation of the shear strain-stress curves obtained from the simple shear tests. These curves show that the shear stress reaches its maximum value for moderate strain, and then drops to its residual value for larger deformation at constant volume. Different values of the angle of internal friction correspond to these maximum and residual shear stresses at failure. It is a common practice in soil mechanics that the observed angle of friction is the sum of the angle of friction at constant volume and the angle of dilation, see Bolton (1986), Houlsby (1991). The phenomenon of dilation is associated with the dissipation of work in a frictional soil. On this basis, the famous Cam-Clay model was developed, Schofield & Wroth (1968), which then established a kind of standard for soil mechanics investigations.

In this paper, the problem of dilation is discussed in the context of classical Cam-Clay model, which was developed on the basis of a specific assumption regarding the plastic work. This assumption leads to a special form of the dilation function, from which a shape of yield function is derived. The above mentioned assumption is compared with the results of the triaxial tests, performed on the model “Skarpa” sand. It is shown that the Cam-Clay approach is not realistic, as it is based on the assumption which is not consistent with experimental data. Some general considerations and discussion of this problem are also presented.

2. Basic Definitions

The soil mechanics sign convention is applied, where compression is defined as positive. Considerations presented in this paper are restricted to the configuration of the triaxial apparatus, where σ_1 = vertical stress, σ_3 = radial stress, ε_1 = vertical strain, ε_3 = radial strain. The effective stresses are defined as follows: $\sigma_1' = \sigma_1 - u$ and $\sigma_3' = \sigma_3 - u$, where u denotes the pore pressure. The following stress and strain invariants are defined for the triaxial configuration:

$$p = \frac{\sigma_1 + 2\sigma_3}{3} = \text{ mean stress,} \tag{1}$$

$$q = \sigma_1 - \sigma_3 = \text{ deviatoric stress,} \tag{2}$$

$$\varepsilon_v = \varepsilon_1 + 2\varepsilon_3 = \text{ volumetric strain,} \tag{3}$$

$$\varepsilon_q = \frac{2}{3}(\varepsilon_1 - \varepsilon_3) = \text{ deviatoric strain.} \tag{4}$$

In Eqs. (3) and (4), there appear the total strains. In elasto-plasticity, the total strain tensor $\boldsymbol{\varepsilon}$ is decomposed onto the elastic $\boldsymbol{\varepsilon}^{el}$ and plastic $\boldsymbol{\varepsilon}^{pl}$ parts respectively, i.e.

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{el} + \boldsymbol{\varepsilon}^{pl}. \tag{5}$$

The fundamental problem is how to determine the elastic and plastic parts of the strain tensor. The plastic strains are usually determined from the flow rule, which has to be defined for a concrete plasticity theory (model). Eqs. (1)–(3) are also valid for the stress and strain increments, denoted as $d\boldsymbol{\sigma}$ and $d\boldsymbol{\varepsilon}$.

The following definition of dilation function is adapted:

$$D = \frac{d\varepsilon_v}{d\varepsilon_q} = \tan\psi, \tag{6}$$

where ψ denotes the angle of dilation.

An alternative definition deals with the plastic parts of strain increments:

$$D^{pl} = \frac{d\varepsilon_v^{pl}}{d\varepsilon_q^{pl}}. \tag{7}$$

The increment of plastic work (dissipation) is defined as follows:

$$dW^{pl} = pd\varepsilon_v^{pl} + qd\varepsilon_q^{pl}. \tag{8}$$

Comment: Note, that during the triaxial tests, only the total strains are measured. The plastic part of the strain tensor is deduced/calculated from the flow rule which, in turn, should be assumed or deduced from independent assumptions. This means that such a decomposition of the strain tensor is not objective, as it depends on theory applied. Objectively, we only know the total strains, as they can be measured directly. Sometimes, for simple loading paths, the soil sample can be unloaded, and the plastic and elastic strains determined on this basis. However, such a procedure can be misleading in many cases as, for example, during the reverse shearing where progressive compaction takes place.

3. Dilation and Cam-Clay Plasticity

In soil mechanics, there is a single milestone defining the strength of granular materials, which was formulated already in the 18-th century by Coulomb. His strength criterion is still applied in contemporary soil mechanics, as better alternative has not been found yet, although some authors have tried to modify his strength criterion by adding the effect of dilation, Rowe (1962), Bolton (1986), Houlsby (1991), Schanz and Vermeer (1996), Vaid and Sasitharan (1992). Still unsolved problem in geomechanics is determination of the pre-failure behaviour of these materials, see Sawicki (2012) who provides extensive references. The most common approach to this task is based on the theory of plasticity. There exists a large number of such models, see Gryczmański (1995), Saada and Bianchini (1989), etc. Many of these models, even those applied to describe the behaviour of granular soils, have their origin in the Cam-Clay methodology, proposed in Cambridge, cf. Schofield and Wroth (1968), Wood (1990), Atkinson (1993), Ortigao (1995).

The Cam-Clay approach is based on interesting assumption regarding the shape of plastic work function, which leads to a specific form of dilation equation. Subsequently, having known this function, the yield surface is determined and respective theory developed. An advantage of such approach is that a single material parameter is introduced, which is well seen in soil mechanics, as it is required that the number of material parameters should be as minimal as possible. The other tendency in soil mechanics is to develop constitutive models from very basic assumptions ("first principles"), and the Cam-Clay approach satisfies these both requirements. However, such simplicity does not always lead to precise description of the real soils behaviour, see Sawicki (2003). Let us see consequences which follow from the basic assumption proposed by Roscoe et al (1963), and then applied by their followers.

The above authors have proposed the following shape of dissipation function:

$$dW^{pl} = \mathrm{M}pd\varepsilon_q^{pl}, \tag{9}$$

where M is a certain material parameter, Jarzębowski (1990).

Eqs. (8) and (9) lead to the following formula for the dilation function:

$$D^{pl} = \frac{d\varepsilon_v^{pl}}{d\varepsilon_q^{pl}} = \mathrm{M} - \eta, \tag{10}$$

where $\eta = q/p$.

The parameter M is related to the critical state of soil, and is given by the following equation, in the case of triaxial compression of sand:

$$\mathrm{M} = \frac{6\sin\varphi}{3 - \sin\varphi}, \tag{11}$$

where φ denotes the angle of internal friction.

Subsequent steps lead to determination of the shape of yield function, assuming the associated flow rule, see Jarzębowski (1990). The other authors as, for example, Burland (1965) have followed this methodology, proposing alternative shapes of the dissipation function. A fundamental question is how the assumption (9), and subsequently a specific shape of the dilation function (10), is related to experimental data?

4. Experimental Data

The experimental results reported in this paper were performed in the triaxial apparatus GDS Instruments, equipped with special gauges enabling the local measurement of both the vertical and radial strains. The experiments were performed on the quartz sand "Skarpa", characterized by the following parameters: median size of grains $D_{50} = 420\,\mu$m, uniformity coefficient $C_u = 2.5$, maximum and minimum void ratios $e_{\max} = 0.677$ and $e_{\min} = 0.432$, angles of internal friction $\varphi = 34^\circ$ and 41° for loose and dense sand respectively, Sawicki and Świdziński (2010).

Figs 1–3 illustrate the stress-strain curves during pure shearing. These curves were obtained from various experiments, each of them performed for different value of $p =$ const. It was shown that such experimental results can be represented by single plots as those shown in Figs 1–3, see Sawicki (2012), Sawicki and Świdziński (2010).

Note that character of the volumetric strains depends on the initial state of sand, defined either as contractive or dilative. Such a distinction is not taken into account in most of soil mechanics publications, including pioneering works of Roscoe et al (1963), Burland (1965) or Schofield and Wroth (1968) and even in more recent works as, for example, Boukpeti et al (2002), Mróz et al (2003). This is not the aim of this paper to provide extensive treatise on this problem, for details see Sawicki (2012), Sawicki and Świdziński (2010). However, it should be mentioned that, in order to determine the initial state of sand, the steady state line (SSL) must be determined first. This line divides the plane $\log p' - e$, where e is the void ratio, onto two regions. The region lying above SSL corresponds to initially contractive states, whilst the region below this line defines initially dilative states, see Sawicki and Świdziński (2010).

The volumetric strains in the initially dilative sand first increase (small compaction) and after reaching some threshold they start to decrease (dilation), see Fig. 1.

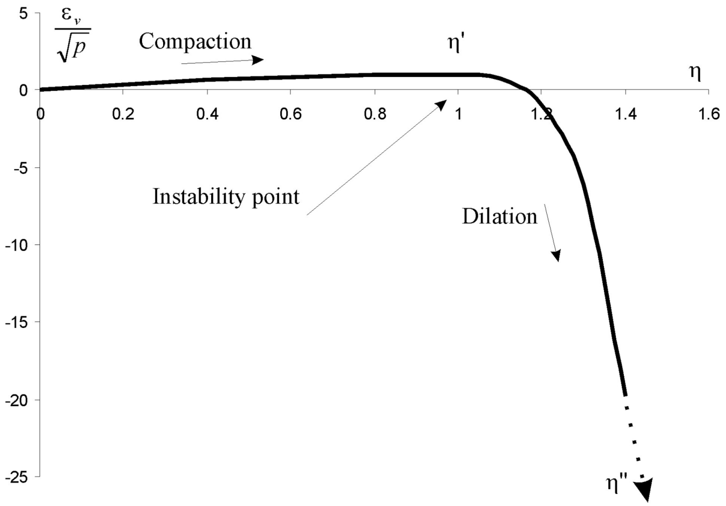

Fig. 1. Volumetric strains developed during pure shearing of initially dilative sand

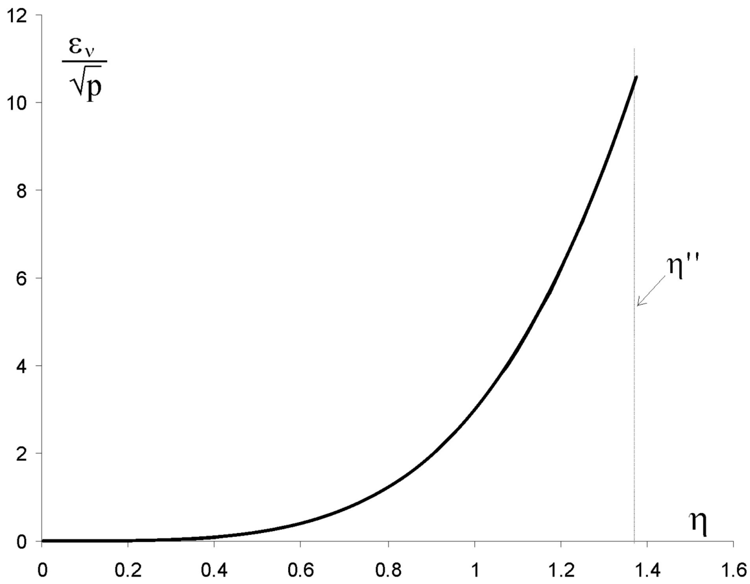

Fig. 2. Volumetric strains developed during pure shearing of initially contractive sand

This threshold corresponds to instability line (IL), the other important object in soil mechanics. The volumetric behaviour of initially contractive sand is different, as during pure shearing such sand compacts, see Fig. 2. Character of deviatoric strains is similar for both, initially contractive and dilative sands, see Fig. 3.

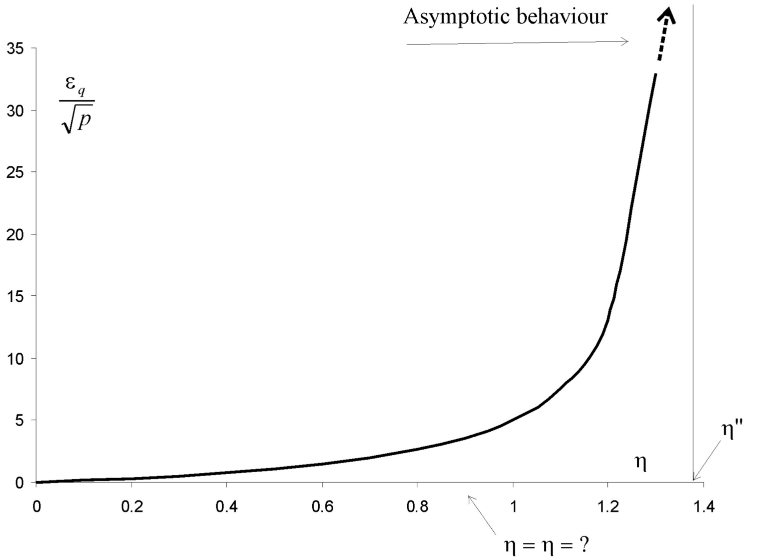

Fig. 3. Deviatoric strains developed during pure shearing of sand. A character of the stress-strain curve is similar for initially dilative and contractive specimens

It should be added that the behaviour of sand shown in Fig. 3 is very different from that presented in Mróz et al (2003). They show, in their Fig. 7, some diagrams, which essentially differ from the stress-strain diagram shown in Fig. 3. We have not obtained such diagrams from our empirical investigations. For example, they show a kind of S – character of the $\varepsilon_q - q$ curves, which we have never obtained in the laboratory, and than they perform dubious energetic considerations. The behaviour presented in Figs 1–3 can be approximated, with a good accuracy, by the equations listed in Table 1, see Sawicki (2010).

Table 1. Analytical approximations of the stress-strain curves from Figs 1–3

Stress-strain curve	**Analytical approximation**
Volumetric strains during pure shearing of initially dilative sand (Fig. 1).	$\frac{\varepsilon_v}{\sqrt{p}} = a_1\eta^2 + a_2\eta, \quad 0 \le \eta \le \eta'$ $\frac{\varepsilon_v}{\sqrt{p}} = (a_3\eta + a_4)\exp(a_5\eta), \quad \eta' \le \eta \le \eta''$
Volumetric strains during pure shearing of initially contractive sand (Fig. 2).	$\frac{\varepsilon_v}{\sqrt{p}} = c_1\eta^4$
Deviatoric strains during pure shearing of both initially dilative and contractive sand (Fig. 3).	$\frac{\varepsilon_q}{\sqrt{p}} = \frac{c\eta}{\eta'' - \eta}$

Explanation: The stress-strain curves shown in Figs 1–3 illustrate the real behaviour of sand, tested in triaxial apparatus. Analytical approximations of these results, presented in Table 1, describe sufficiently well these empirical data, and can also be used for testing various models of sand. The parameters appearing in Table 1 were determined for **stress unit** 10^5 N/m^2 and **strain unit** 10^{-3}. This means that $\varepsilon_v = 3$ corresponds to the volumetric strain of $3 \times 10^{-3} = 0.003$, or $p = 1$ corresponds to the mean stress of 1×10^5 N/m^2 = 100 kPa, etc. Values of respective coefficients for "Skarpa" sand are the following: $a_1 = -1$; $a_2 = 2$; $a_3 = -0.0106$; $a_4 = 0.0123$; $a_5 = 6.4$; $\eta' = 1$; $c_1 = 2.97$; $c = 1.9$. The parametr η' corresponds to the instability line, and parameter η'' corresponds to the steady state of sand. The steady state of sand is identified with the critical state, as both designations deal with the continuous plastic flow of sand under constant volume and stresses. We also identify these states with the Coulomb-Mohr yield condition. Therefore $\eta'' = \mathrm{M}$, cf. Eq. (11). There is $\eta'' = 1.68$ for initially dense sand and $\eta'' = 1.375$ for initially loose sand.

It should be added that the functions approximating the volumetric strains of initially dilative sand (Fig. 1) are continuous at η', having there the same values, and also have the same first derivative.

5. Cam-Clay Dilation against Experimental Data

The Cam-Clay philosophy is based on interesting assumption (9) which leads to a specific form of the dilation function (10), and subsequently to the shape of yield surface, and other consequences of such assumption. The Cam-Clay approach has become almost as standard in soil mechanics, although some researchers as, for example Wood (1990), treat it as a kind of "students' model". From one side, it is proper approach as in engineering we deal with simple approximations of the real world, in the form of ideal bodies, as elastic, viscous, plastic, etc. On the other hand, we are looking for more and more better approximations of the real behaviour of materials and structures. The "student's models" are very good, as they allow for better understanding of the real phenomena, but they cannot be a basis for developing reliable engineering models.

In this paper, the basic assumptions of the Cam-Clay approach are compared with the real experimental data, presented in previous Section. It should be noted first that the volumetric strain – deviatoric stress curves are different for the initially dilative and contractive sands, cf. Figs. 1 and 2. Therefore, respective dilation functions will also be different. Consider initially contractive sand first. Differentiation of respective formulae from Table 1, with respect to η, leads to increments of the volumetric and deviatoric strains. Substitution of these increments into Eq. (6) leads to the following form of dilation function:

$$D = \frac{4c_1\eta^3(\eta'' - \eta)^2}{c\eta''}, \tag{12}$$

which significantly differs from Eq. (10), derived from the Cam-Clay assumptions. Note that Eq. (12) deals with total strains, whilst the plastic strain increments are taken into account in Eq. (10). By introducing the plastic strain increments, we obtain the following function, cf. Sawicki (2010):

$$D^{pl} = \frac{4c_1\eta^3(\eta'' - \eta)^2}{c\eta'' - (\eta'' - \eta)^2 b_q}, \tag{13}$$

where $b_h = 0.76$ denotes the coefficient of deviatoric unloading. Eq. (13) is certainly very much different from Eq. (10). Fig. 4 illustrates the dilation functions (12) and (13) for data presented in previous Section, after Sawicki (2010).

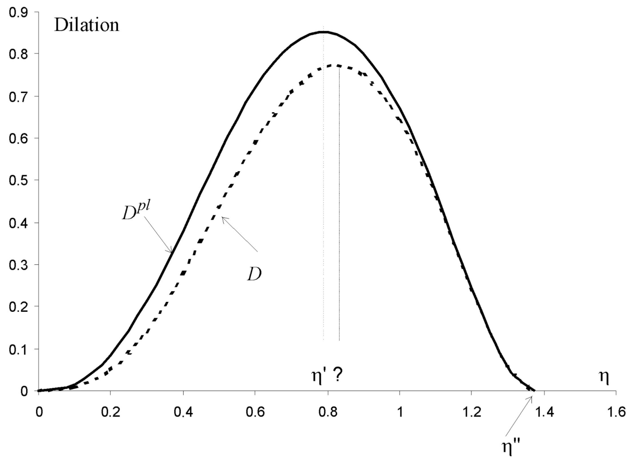

Fig. 4. Dilation function of contractive sand during shearing. The total and plastic dilations do not differ so much

Note that Eq. (10) represents a straight line with the slope of -1. That line has its maximum $D^{pl} = \mathrm{M}$ at $\eta = 0$ which is very different from the diagram presented in Fig. 4. Fig. 5 presents the plastic dilation function of initially dilative sand, after Sawicki (2010). The plastic dilation decreases down to η' (which approximately corresponds to instability line) and then remains zero. This function is also much different from Eq. (10).

Above considerations can be summarized as follows: (a) The Cam-Clay assumption (10) is not realistic. Real experimental data show that the shape of dilation function very much differs from that assumed in the Cam-Clay approach; (b) In the Cam-Clay approach, no distinction is taken into account between the initially contractive or dilative state of sand, which is a serious shortcoming; (c) There are serious doubts whether the Cam-Clay "philosophy" is correct from the view point of construction of constitutive equations of soils. A single assumption about the shape of dilation function, for particular case of pure shearing (even wrong in this simple case) cannot be a basis for derivation of general constitutive equations. Subsequent arguments against the Cam-Clay approach will be presented in the following Sections.

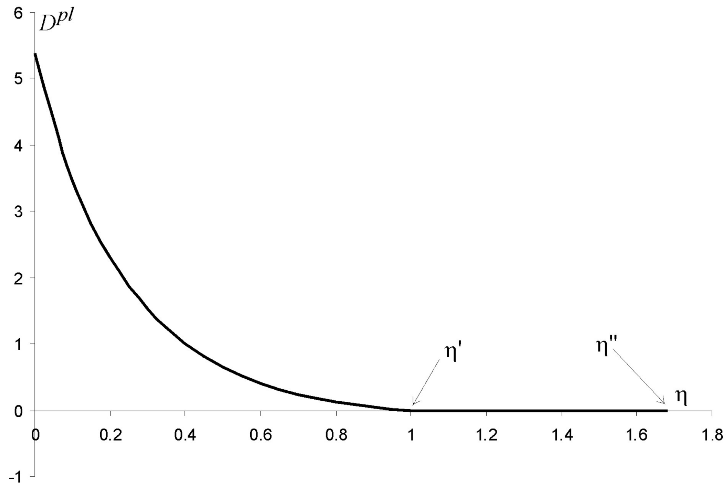

Fig. 5. The plastic dilation function of initially dilative sand

6. Dilation during Isotropic Compression

Consider the case of isotropic compression, i.e. the process of loading when $dp > 0$ and $dq = 0$. Experimental investigations show that during such a process both volumetric and deviatoric strains develop, see Sawicki and Świdziński (2010), Sawicki (2012). Their increments can be approximated with good accuracy by the following equations:

$$d\varepsilon_v = \frac{A_v}{2\sqrt{p}}dp, \tag{14}$$

$$d\varepsilon_q = \frac{A_q}{2\sqrt{p}}dp, \tag{15}$$

where A_v and A_q are coefficients. Their average values for initially loose sand are 6 and −0.95 respectively. For an initially dense sand, we have obtained the corresponding values of 3.47 and −0.53 (recall units applied!). Note that $d\varepsilon_q \neq 0$, which means that sand displays anisotropic properties, usually ignored in most soil mechanics models. During spherical unloading $dp < 0$, $dq = 0$, some (elastic) strains are recovered. They can be approximated by similar formulae:

$$d\varepsilon_v^{el} = \frac{A_v^{el}}{2\sqrt{p}}dp, \tag{16}$$

$$d\varepsilon_q^{el} = \frac{A_q^{el}}{2\sqrt{p}}dp, \tag{17}$$

where the average values of A_v^{el} and A_q^{el} are 4.4 and −0.45, respectively, for an initially loose sand, and 2.9 and −0.21, respectively, for an initially dense sand.

Substitution of the above relations into Eqs. (6) and (7) leads to the following expressions:

$$D = \frac{A_v}{A_q}, \tag{18}$$

$$D^{pl} = \frac{A_v - A_v^{el}}{A_q - A_q^{el}}. \tag{19}$$

Substitution of the above values of the corresponding coefficients leads to almost similar values of D for initially loose and dense sands, i.e. −6.316 and −6.55, respectively. These values correspond to the angle of dilation $\psi \approx -81° = \text{const}$. It is not clear whether this is a coincidence or not. Eq. (19) gives $D^{pl} = -3.2$ for an initially loose sand and −1.76 for an initially dense sand, both having constant values. Formal substitution of the above results into Eq. (10) leads to a contradiction, as it gives $\eta = \mathrm{M} - D^{pl} = \text{const}$, whereas $\eta = q/p$ is a stress variable that certainly is not constant. Also note that this constant depends on the initial state of sand, which is another strange result.

Summary: (a) The dilation function is constant during the spherical compression of sand, which follows from the initial anisotropy of samples investigated in the triaxial apparatus (transverse isotropy induced by gravity); (b) This effect is not taken into account in most soil mechanics models, which usually assume isotropy of sand; (c) Experimental results contradict again the theoretical assumption (10), which is a basis of the Cam-Clay philosophy.

7. General Considerations

The experimental results can be approximated by the following incremental formulae, see Sawicki (2012), Sawicki and Świdziński (2010):

$$d\varepsilon_v = Mdp + Ndq, \tag{20}$$

$$d\varepsilon_q = Pdp + Qdq, \tag{21}$$

where M, N, P, Q are functions of the stress state.

Incremental equations for plastic strains have a similar form:

$$d\varepsilon_v^{pl} = Adp + Bdq, \tag{22}$$

$$d\varepsilon_q^{pl} = Cdp + Ddq, \tag{23}$$

where A, B, C, D are again functions of the stress state. Recall that some of these functions have different forms for initially contractive and dilative sands, see the previous sections or original publications by Sawicki and Świdziński (2010), Sawicki (2012).

Substitution of Eqs. (22) and (23) into (7) gives:

$$D^{pl} = \frac{Adp + Bdq}{Cdp + Ddq}. \tag{24}$$

Particular forms of Eq. (24) are presented by relations (13) and (19). Experimentally obtained relations (20)–(23) show that the shape of the dilation function D^{pl} also depends on the stress path. This means that such hypotheses as (10) are artificial, as they are not consistent with empirical data. The yield function, determined from such an assumption, is also vague, as there could be an infinite number of such surfaces, each of them corresponding to a particular stress path.

8. Discussion and Conclusions

The elasto-plastic modelling of sand behaviour is almost a standard in contemporary soil mechanics. There exists a large number of various models, but none of them has become commonly accepted, perhaps except for the Cam-Clay approach, which however has serious shortcomings, as already explained. A common feature of the elasto-plastic approach is an assumption about the yield surface, which is located inside the statically admissible region, bounded by the limit surface. The limit surface usually corresponds to the Coulomb-Mohr yield condition. Contemporary attempts, such as, for example, the critical state line, are probably based on false interpretations of experimental data, cf. Sawicki (2012).

Yield surfaces introduced in numerous soil mechanics models, which evolve inside the limit surface, pose a considerable problem. Most of them are introduced artificially, like the Cam-Clay yield surface, which follows from a doubtful assumption about the dilation function. Consequences of such *ad hoc* assumptions are serious, as they lead to results inconsistent with experimental data. The role of the yield surface is to determine whether a subsequent load increment causes loading or unloading. During loading, both elastic and plastic strains develop, and during unloading elastic strains are recovered. We can imagine only two models, according to which a given load increment produces either elastic and plastic deformations or leads only to recoverable strain. The simplest example are the Tresca and Huber-Mises yield conditions, since in the corners of the Tresca limit surface such ambiguity does exist. The development of plastic strains is an objective physical phenomenon, and it cannot depend on the model assumed, as is often the case in soil mechanics.

The basic question is how to determine these yield surfaces objectively, and whether it is even possible. The Cam-Clay approach is interesting, as it is based on a single theoretical assumption regarding the form of the dilation function. It is always temptating to derive the simplest possible models of reality, possibly from the first principles, with a minimal set of parameters. However, such an ideal approach has not yet led to realistic results. The question is whether it is possible to determine these various yield surfaces from experimental investigations There have been some rare attempts to do so, for example, Tatsuoka (1972), see also Wood (1990), but such methods are not convincing. It is possible, however, to determine experimentally the Coulomb-Mohr limit surface, as it corresponds to maximum stresses that can be supported by soils. Such experiments can be performed in standard geotechnical laboratories to determine the basic strength characteristics of soils, such as the angle of internal friction φ and cohesion c.

The main conclusion from the present paper is that the Cam-Clay approach should be re-examined, since the main assumption, on which this model was developed, is inconsistent with experimental data.

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