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Entropies related to integral operators [1]

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Abstract

We consider classical entropies associated with several continuous distributions of probabilities. Explicit expressions and properties of them are presented.

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1 Introduction

Let $p = (p_0(x), p_1(x), p_2(x), ...)$ be a probability distribution, where each p_k is a continuous function on an interval $\mathbf{I}$, $p_k(x) \geq 0$ and

$$\sum_k p_k(x) = 1, \ x \in \mathbf{I}.$$

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Given some points $x_k \in \mathbf{I}$, $k = 0, 1, 2, ...$, we can construct a positive linear operator

$$Lf(x) = \sum_k f(x_k) p_k(x), \ x \in \mathbf{I}, \tag{1.1}$$

for those functions $f \in \mathrm{C}(\mathbf{I})$, for which the right-hand side is defined.

On the other hand the index of coincidence associated with p is defined by

$$S(x) = \sum_k p_k^2(x), \ x \in \mathbf{I}, \tag{1.2}$$

and the corresponding Rényi entropy and Tsallis entropy of order 2 can be expressed as

$$R(x) = -\log S(x), \ x \in \mathbf{I}, \tag{1.3}$$

respectively

$$T(x) = 1 - S(x), \ x \in \mathbf{I}. \tag{1.4}$$

For details see, e.g., [10], [14] and the references therein.

For example, the binomial distribution

$$p_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k}, \ k = 0, 1, 2, ..., n; \ x \in [0,1]$$

is related to the classical *Bernstein* operators

$$B_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}, \ x \in [0,1].$$

The Poisson distribution

$$p_{n,k}(x) := e^{-nx} \frac{(nx)^k}{k!}, \ k = 0, 1, ...; \ x \geq 0$$

corresponds to the $Szász - Mirakjan$ operators

$$M_n f(x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) e^{-nx}\frac{(nx)^k}{k!},\ x \geq 0,$$

and the negative binomial distribution

$$p_{n,k}(x) := \begin{pmatrix} n+k-1 \\ k \end{pmatrix} x^k(1+x)^{-n-k},\ x \geq 0,\ k = 0, 1, 2, ... \ .$$

corresponds to the $Baskakov$ operators

$$V_n f(n) := \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) \begin{pmatrix} n+k-1 \\ k \end{pmatrix} x^k(1+x)^{-n-k},\ x \geq 0.$$

In these three classical cases, the corresponding indices of coincidence were denoted in [14] by

$$F_n(x) = \sum_{k=0}^{n} \left(\begin{pmatrix} n \\ k \end{pmatrix} x^k(1-x)^{n-k} \right)^2,\ x \in [0,1],$$

$$K_n(x) = \sum_{k=0}^{\infty} \left(e^{-nx}\frac{(nx)^k}{k!} \right)^2,\ x \in [0,\infty),$$

$$G_n(x) = \sum_{k=0}^{\infty} \left(\begin{pmatrix} n+k-1 \\ k \end{pmatrix} x^k(1+x)^{-n-k} \right)^2,\ x \in [0,\infty).$$

The $Bleimann - Butzer - Hahn$ operators

$$H_n f(x) := \sum_{k=0}^{n} f\left(\frac{k}{n-k+1}\right) \begin{pmatrix} n \\ k \end{pmatrix} x^k(1+x)^{-n},\ x \geq 0,$$

are associated with the distribution

$$p_{n,k}(x) := \begin{pmatrix} n \\ k \end{pmatrix} x^k(1+x)^{-n},\ k = 0, 1, ..., n;\ x \geq 0$$

and with index of concidence

$$U_n(x) := \sum_{k=0}^{n} \left(\begin{pmatrix} n \\ k \end{pmatrix} x^k(1+x)^{-n} \right)^2,\ x \geq 0.$$

For the $Meyer - König$ *and* $Zeller$ operators

$$R_nf(x) := \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right)\left(\begin{array}{c} n+k \\ k \end{array}\right) x^k(1-x)^{n+1},\ x \in [0,1],$$

the corresponding probability distribution is

$$p_{n,k}(x) := \left(\begin{array}{c} n+k \\ k \end{array}\right) x^k(1-x)^{n+1},\ k = 0,1,2,...;\ x \in [0,1],$$

and the index of coincidence is

$$J_n(x) := \sum_{k=o}^{\infty}\left(\left(\begin{array}{c} n+k \\ k \end{array}\right) x^k(1-x)^{n+1}\right)^2,\ x \geq 0.$$

The above indices of coincidence were studied in [1], [3], [4], [7], [9], [11], [14], [15], [16].

The logarithmic convexity of F_n has been established recently in [2], [4], [12].

Consider now an integral operator of the form

$$Lf(x) = \int_{\mathbf{I}} K(x,t)f(t)dt,\ x \in \mathbf{I}, \tag{1.5}$$

where the kernel K is positive and continuous on **IxI**. Suppose that

$$\int_{\mathbf{I}} K(x,t)dt = 1,\ x \in \mathbf{I}, \tag{1.6}$$

so that $K(x,\bullet)$ can be considered as a probability density function on **I**, for each $x \in \mathbf{I}$.

Let us define (see also [10])

$$S(x) := \int_{\mathbf{I}} (K(x,t))^2\, dt,\ x \in \mathbf{I}. \tag{1.7}$$

The function $S(x)$ is called the *information potential* associated with $K(x, \bullet)$.

The associated Rényi and Tsallis entropies will be $R(x) := -\log S(x)$, $T(x) := 1 - S(x)$, $x \in \mathbf{I}$. These definitions, related to the integral operators (1.5), correspond to the definitions (1.2), (1.3), (1.4), related to the discrete operator (1.1).

In this paper we shall compute $S(x)$ from (1.7), for some integral operators of the form (1.5), and establish bounds for $S(x)$.

2 The Post-Widder operators

For the definition of these operators see, e.g., [5, p.114].

$$L_n f(x) = \frac{\left(\frac{n}{x}\right)^n}{(n-1)!} \int_0^\infty e^{-\frac{nt}{x}} t^{n-1} f(t) dt, \ x > 0$$

Theorem 1 *The corresponding information potential is*

$$S_n = \binom{2n-2}{n-1} \frac{n}{2^{2n-1}} \frac{1}{x}, \ x > 0. \tag{2.1}$$

Proof. The corresponding kernel is

$$K(x, t) := \frac{\left(\frac{n}{x}\right)^n}{(n-1)!} e^{-\frac{nt}{x}} t^{n-1}, \ t, x > 0,$$

so that

$$S_n(x) = \int_0^\infty \frac{\left(\frac{n}{x}\right)^{2n}}{(n-1)!^2} e^{-\frac{2nt}{x}} t^{2n-2} dt.$$

Setting $\frac{2n}{x} t = s$, we get

$$S_n(x) = \frac{n^{2n}}{x^{2n}} \frac{1}{(n-1)!^2} \left(\frac{x}{2n}\right)^{2n-1} \Gamma(2n-1)$$

$$= \frac{n^{2n}}{x^{2n}} \frac{(2n-2)!}{(n-1)!^2} \frac{x^{2n-1}}{(2n)^{2n-1}} = \binom{2n-2}{n-1} \frac{n}{2^{2n-1}} \frac{1}{x}.$$

Corollary 1 *The function $S_n(x)$ given by (2.1) satisfies the inequalities*

$$\frac{n}{2\sqrt{\pi(n+2)}}\frac{1}{x} < S_n(x) < \frac{n}{2\sqrt{\pi(n-2)}}\frac{1}{x},\ x>0,\ x\geq 3. \tag{2.2}$$

Consequently, for a fixed $n \geq 1$,

$$\lim_{x\to\infty} S_n(x) = 0, \qquad \lim_{x\to 0} S_n(x) = \infty, \tag{2.3}$$

and for a fixed $x > 0$,

$$\lim_{n\to\infty} S_n(x) = \infty, \tag{2.4}$$

Proof. We use the inequalities (see [6, (1.9)])

$$\frac{1}{\sqrt{\pi(n+3)}} < \frac{1}{4^n}\left(\begin{array}{c} 2n \\ n \end{array}\right) < \frac{1}{\sqrt{\pi(n-1)}},\ n\geq 2, \tag{2.5}$$

Now (2.2) follows from (2.1) and (2.5). Let us remark that

$$S_1(x) = S_2(x) = \frac{1}{2x},\ x>0. \tag{2.6}$$

Thus (2.3) and 2.4 are also valid, and the proof is finished.

3 The Gamma operators

These operators are defined, e.g., in [5, p.114]:

$$G_n f(x) = \frac{x^{n+1}}{n!}\int_0^\infty e^{-ux}u^n f\left(\frac{n}{u}\right)du,\ x>0.$$

Theorem 2 *The associated information potential is*

$$S_n(x) = \left(\begin{array}{c} 2n \\ n \end{array}\right)\frac{(n+1)(2n+1)}{n4^{n+1}}\frac{1}{x},\ x>0. \tag{3.1}$$

Proof.

Setting $\frac{n}{u} = y$, we get

$$G_n f(x) = \frac{x^{n+1}}{n!}\int_0^\infty e^{-\frac{nx}{y}}\left(\frac{n}{y}\right)^n f(y)\frac{n}{y^2}dy = \frac{(nx)^{n+1}}{n!}\int_0^\infty e^{-\frac{nx}{y}}\frac{1}{y^{n+2}}f(y)dy.$$

$$\text{Therefore } S_n(x) = \frac{(nx)^{2n+2}}{n!^2} \int_0^\infty e^{-\frac{2nx}{y}} \frac{1}{y^{2n+4}} dy.$$

Let $z = \frac{2nx}{y}$. Then

$$S_n(x) = \frac{(nx)^{2n+2}}{n!^2} \int_0^\infty e^{-z} \frac{z^{2n+4}}{(2nx)^{2n+4}} \frac{2nx}{z^2} dz =$$

$$= \frac{(nx)^{2n+2}}{n!^2} \frac{1}{2^{2n+3}(nx)^{2n+3}} \int_0^\infty e^{-z} z^{2n+3-1} dz =$$

$$= \frac{1}{nx} \frac{1}{n!^2 2^{2n+3}} (2n+2)! = \left(\begin{array}{c} 2n \\ n \end{array} \right) \frac{(n+1)(2n+1)}{n4^{n+1}} \frac{1}{x}.$$

Corollary 2 *The function* $S_n(x)$ *from* (3.1) *satisfies*

$$S_1(x) = \frac{3}{4x}, \ x > 0$$

and

$$\frac{(n+1)(2n+1)}{4n\sqrt{\pi(n+3)}} \frac{1}{x} < S_n(x) < \frac{(n+1)(2n+1)}{4n\sqrt{\pi(n-1)}} \frac{1}{x}, \ x > 0, \ n \geq 2.$$

It also satisfies (2.3) *and* (2.4).

The proof is similar to that of Corollary 1.

4 The Rayleigh operator

This operator is related to the probability density function (see [8, p.458]) $K(x,t) := 2xte^{-xt^2}, \ x, y > 0$, so that

$$Lf(x) = \int_0^\infty 2xte^{-xt^2} dt, \ x > 0.$$

Let $e_k(t) := t^k$, $k = 0, 1, \ldots$. We start by computing the moments $Le_k(x)$ of the operator L, for $k = 0, 1, 2, \ldots$.

$$Le_k(x) = \int_0^\infty 2xt^{k+1}e^{-xt^2}dt, \ x > 0.$$

Setting $xt^2 = y$, we get

$$Le_k(x) = \int_0^\infty 2x\left(\sqrt{\frac{y}{x}}\right)^{k+1} e^{-y}\frac{1}{2\sqrt{x}\sqrt{y}}dy =$$

$$= x^{-\frac{k}{2}}\int_0^\infty y^{\frac{k}{2}}e^{-y}dy = x^{-\frac{k}{2}}\Gamma\left(\frac{k}{2}+1\right), \ k \geq 0.$$

In particular, $Le_0(x) = 1$, $Le_1(x) = \frac{1}{2}\sqrt{\frac{\pi}{x}}$, $Le_2(x) = \frac{1}{x}$, $x > 0$.

Theorem 3 *For the Rayleigh probability density function we have*

$$S(x) = \sqrt{\frac{\pi x}{8}}, \qquad V(x) = \frac{4-\pi}{4x}, \ x > 0.$$

Proof. By definition,

$$S(x) = \int_0^\infty 4x^2t^2e^{-2xt^2}dt.$$

Let $2xt^2 = y$. Then

$$S(x) = \int_0^\infty 4x^2\frac{y}{2x}e^{-y}\frac{1}{2\sqrt{2x}\sqrt{y}}dy = \sqrt{\frac{x}{2}}\int_0^\infty y^{\frac{1}{2}}e^{-y}dy =$$

$$= \sqrt{\frac{x}{2}}\Gamma(\frac{3}{2}) = \sqrt{\frac{\pi x}{8}}, \ x > 0.$$

Furthermore,

$$V(x) = Le_2(x) - (Le_1(x))^2 = \frac{1}{x} - \frac{\pi}{4x} = \frac{4-\pi}{4x}, \ x > 0.$$

5 Other operators

Let $n \in \mathrm{N}$, $n \geq 2$. Consider the operators

$$L_n f(x) := (n-1)x^{n-1} \int_0^\infty \frac{f(t)}{(x+t)^n} dt, \ x > 0.$$

For $0 \leq k \leq n-2$, the moments $Le_k(x)$ are given by

$$L_n e_k(x) = (n-1)x^{n-1} \int_0^\infty \frac{t^k}{(x+t)^n} dt.$$

Setting $t = xy$, we get

$$L_n e_k(x) = (n-1)x^{n-1} \int_0^\infty \frac{x^k y^k}{x^n (1+y)^n} x dy =$$

$$= (n-1)x^k \int_0^\infty \frac{y^{(k+1)-1}}{(1+y)^{(k+1)+(n-k-1)}} dy =$$

$$= (n-1)x^k B(k+1, n-k-1) = (n-1)x^k \frac{\Gamma(k+1)\Gamma(n-k-1)}{\Gamma(n)},$$

and finally,

$$L_n e_k(x) = \begin{pmatrix} n-2 \\ k \end{pmatrix}^{-1} x^k, \ x > 0.$$

Theorem 4 *With the above notation, we have*

$$S(x) = \frac{(n-1)^2}{2n-1} \frac{1}{x}, \qquad V(x) = \frac{n-1}{(n-2)^2(n-3)} x^2, \ x > 0, \ n \geq 4.$$

Proof.

By definition,

$$S(x) = \int_0^\infty (n-1)^2 x^{2n-2} \frac{1}{(x+t)^{2n}} dt =$$

$$= (n-1)^2 x^{2n-2} \frac{1}{(2n-1)x^{2n-1}},$$

so that

$$S(x) = \frac{(n-1)^2}{2n-1} \frac{1}{x}, \ x > 0.$$

Moreover,

$$V(x) = Le_2(x) - (Le_1(x))^2 = \frac{(n-1)x^2}{(n-2)^2(n-3)},\ n \geq 4.$$

Remark 1 *All the properties of the information potential, mentioned above, can be used to derive properties of the Rényi entropy and Tsallis entropy, described by* (1.3) *and* (1.4)*. For the sake of brevity we omit the details.*

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