General Mathematics Vol. 27, No. 2 (2019), 59–69

DOI:10.2478/gm-2019-0015

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q–analogue of generalized Ruschweyh operator related to a new subfamily of multivalent functions [1]

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Abstract

A new subfamily of p–valent analytic functions with negative coefficients in terms of q–analogue of generalized Ruschweyh operator is considered. Several properties concerning coefficient bounds, weighted and arithmetic mean, radii of starlikeness, convexity and close-to-convexity are obtained. A family of class preserving integral operators and integral representation are also indicated.

2010 Mathematics Subject Classification: 30C45, 30C50.

Key words and phrases: p–valent function, Fractional q–calculus operator, Ruschweyh operator, Coefficient inequality, Weighted mean, Arithmetic mean, Radii structures.

1 Introduction

Let $\mathcal{A}_p$ be the class of p–valent analytic functions defined in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and are of the type:

$$(1) \qquad f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k, \qquad (a_k \geqslant 0, \quad n, p \in \mathbb{N} = \{1, 2, \ldots\}).$$

[1] *Received 2 June, 2019*

Accepted for publication (in revised form) 21 October, 2019

In the theory of q–calculus, the q–shifted factorial $(w,q)_k$ for $w,q \in \mathbb{C}$ and $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ is defined by:

$$(w,q)_k = \begin{cases} 1 & , \quad k=0, \\ (1-w)(1-wq)\cdots(1-wq^{k-1}) & , \quad k \in \mathbb{N}, \end{cases} \tag{2}$$

and according to the gamma function:

$$(q^w,q)_k = \frac{\Gamma_q(w+k)(1-q)^k}{\Gamma_q(w)}, \qquad (k>0), \tag{3}$$

where the q–gamma function is given by:

$$\Gamma_q(y) = \frac{(q,q)_\infty(1-q)^{1-y}}{(q^y,q)_\infty}, \qquad (0<q<1), \tag{4}$$

where

$$(w,q)_\infty = \prod_{t=0}^{\infty}(1-wq^t), \qquad (|q|<1). \tag{5}$$

If $\Gamma_q(y)$ be the ordinary Eular gamma function, then it is easy to see that:

$$\lim_{q\to 1}\Gamma_q(y) = \Gamma(y), \tag{6}$$

and

$$\lim_{q\to 1^-}\frac{(q^w,q)_k}{(1-q)^k} = (w)_k, \tag{7}$$

where $(w)_k = w(w-1)\cdots(w+k-1)$ is the familiar Pochhammer symbol. For more details see [3] and [4].

The Jackson's q–derivative and q–integral are given by Gasper and Rahman as follow:

$$\mathcal{D}_{q,z}f(z) = \frac{f(z)-f(qz)}{z(1-q)}, \qquad (z\neq 0, \quad q\neq 0), \tag{8}$$

$$\int_0^z f(t)d_q(t) = z(1-q)\sum_{k=0}^{\infty}q^k f(zq^k). \tag{9}$$

See [3].

From (8), we conclude:

$$D_q z^k = \frac{z^k-(zq)^k}{z(1-q)} = [k]_q z^{k-1}, \tag{10}$$

where

$$[k]_q = \frac{1-q^k}{1-q}. \tag{11}$$

$[k]_q$ is called q–analogue of k, and we have:

$$\lim_{q\to 1}[k]_q = k. \tag{12}$$

Now, we recall some important definitions of fractional q–calculus operators of a complex-valued function $f(z)$, see [8].

Definition 1 *Let $f(z)$ is analytic in a simple-connected domain containing $z=0$. The fractional q–integral operator $\mathcal{I}^{\delta}_{q,z}$ of order $\delta>0$ is given by:*

$$\mathcal{I}^{\delta}_{q,z}f(z) = \mathcal{D}^{-\delta}_{q,z}f(z) = \frac{1}{\Gamma_q(\delta)}\int_0^z (z-qt)_{\delta-1}f(t)d_q(t), \tag{13}$$

where $(z-qt)_{\delta-1}$ is single-valued when:

$$\left|\arg\left(\frac{-tq^w}{z}\right)\right| < \pi, \qquad \left|\frac{tq^w}{z}\right| \qquad \textit{and} \qquad |\arg z| < \pi.$$

Definition 2 *The fractional q–derivative of order δ $(0\leqslant\delta<1)$, is given by:*

$$\mathcal{D}^{\delta}_{q,z}f(z) = \mathcal{D}_{q,z}\mathcal{I}^{1-\delta}_{q,z}f(z) = \frac{1}{\Gamma_q(1-\delta)}\mathcal{D}_{q,z}\int_0^z (z-qt)_{-\delta}f(t)d_q(t). \tag{14}$$

Definition 3 *Under the same assumption of Definition 2, the extended fractional q–derivative of order δ $(0\leqslant\delta<1)$ is defined by:*

$$\mathcal{D}^{\delta}_{q,z}f(z) = \mathcal{D}^{m}_{q,z}\mathcal{I}^{m-\delta}_{q,z}f(z), \qquad (m-1\leqslant\delta<m, \quad m\in\mathbb{N}_0). \tag{15}$$

By applying known extensions of q–differintegral, we consider the linear operator:

$$\begin{aligned}\Omega^{\delta}_{q,p}f(z) &= \frac{\Gamma_q(p+1-\delta)}{\Gamma_q(p+1)}z^{\delta}\mathcal{D}^{\delta}_{q,z}f(z)\\ &= z^p + \sum_{k=n+p}^{\infty}\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}a_k z^k.\end{aligned} \tag{16}$$

We can easily check that $\Omega^{\delta}_{q,z}$ is fractional q–integral of order δ when $-\infty<\delta<p+1$. Also, if $\delta=0$, then $\Omega^{0}_{q,p}f(z)=f(z)$.

Now, we consider the generalize Al-Oboudi differential operator which was introduced by Selvakumaran et al. [11], as follows:

$$\begin{aligned}&\mathcal{D}^{\delta,m}_{q,p,\lambda}:\mathcal{A}_p\to\mathcal{A}_p,\\ &\mathcal{D}^{\delta,0}_{q,p,\lambda}f(z) = f(z),\\ &\mathcal{D}^{\delta,1}_{q,p,\lambda}f(z) = (1-\lambda)\Omega^{\delta}_{q,p}f(z) + \frac{\lambda z}{[p]_q}\mathcal{D}_q\big(\Omega^{\delta}_{q,p}f(z)\big),\\ &\mathcal{D}^{\delta,m}_{q,p,\lambda}f(z) = \mathcal{D}^{\delta,1}_{q,p,\lambda}\big(\mathcal{D}^{\delta,m-1}_{q,p,\lambda}f(z)\big),\end{aligned} \tag{17}$$

where $\lambda \geqslant 0$ and $m \in \mathbb{N}$.

We note that, by specializing the parameters, the above operator reduces to many well-known operators. For example see [2, 1] and [10].

Analogously, we consider the generalized Ruschweyh operator $\mathcal{R}^{\delta,m}_{q,p,\lambda} : \mathcal{A}_p \to \mathcal{A}_p$ as follows:

$$\begin{aligned}
&\mathcal{R}^{\delta,0}_{q,p,\lambda} f(z) = f(z),\\
&\mathcal{R}^{\delta,1}_{q,p,\lambda} f(z) = \frac{z}{[p]_q}\mathcal{D}_{q,p}\big(\Omega^{\delta}_{q,p} f(z)\big),\\
(18)\quad &(m+1)\mathcal{R}^{\delta,m+1}_{q,p,\lambda} f(z) = \frac{z}{[p]_q}\mathcal{D}_q\big(\mathcal{R}^{\delta,m}_{q,p,\lambda} f(z)\big) + m\mathcal{R}^{\delta,m}_{q,p,\lambda} f(z), \qquad (m \in \mathbb{N}).
\end{aligned}$$

For $f(z) \in \mathcal{A}_p$ given by (1), by using (18), we get:

$$\mathcal{R}^{\delta,m}_{q,p,\lambda} f(z) = z^p - \sum_{k=p+1}^{\infty} \left(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}\right)^m C_m\left(\frac{[k]_q}{[p]_q}\right) a_k z^k, \tag{19}$$

where

$$C_m(x) = \frac{x(x+1)\cdots(x+m)}{m!}. \tag{20}$$

See [5].

In special case, $\delta = 0$, $p = 1$ and $q \to 0$, then above operator reduces to the operator introduced by Ruschweyh [9].

Definition 4 *For the functions f and F, analytic in $\mathbb{U}$, we say f is subordinate to F denoted by $f \prec F$ if for some analytic function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$,*

$$f(z) = F(w(z)), \qquad (z \in \mathbb{U}).$$

Definition 5 *A function $f(z) \in \mathcal{A}_p$ is said to be a member of the class $\mathcal{XR}(\alpha, \beta, \gamma)$ if and only if:*

$$\frac{z\big(\mathcal{R}^{\delta,m}_{q,p,\lambda} f(z)\big)'}{f_t(z)} \prec \frac{p + \big(\alpha p + (\beta-\alpha)(p-\gamma)\big)z}{1+\alpha z}, \tag{21}$$

where $-1 \leqslant \alpha < \beta \leqslant 1$, $0 \leqslant t < 1$, $0 < \gamma < p$,

$$f_t(z) = (1-t)z^p + t f(z), \qquad \big(f(z) \in \mathcal{A}_p\big), \tag{22}$$

and $\mathcal{R}^{\delta,m}_{q,p,\lambda} f(z)$ is defined in (19).

2 Main results

In this section, we obtain coefficient inequality and conclude weighted and arithmetic mean properties.

Theorem 1 *Let $f(z) = z^p \sum_{k=n+p}^{\infty} a_k z^k \in \mathcal{A}_p$ be analytic in $\mathbb{U}$. Then $f(z) \in \mathcal{XR}(\alpha,\beta,\gamma)$ if and only if:*

(23)
$$\sum_{k=n+p}^{\infty}\left[\left(k\Big(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}\Big)^m C_m\Big(\frac{[k]_q}{[p]_q}\Big)-pt\right)(1-\alpha)+t(\beta-\alpha)(p-\gamma)\right]a_k \leqslant (\beta-\alpha)(p-\gamma)$$

where $C_m(x)$ is given in (20).

Proof. The relation (21) is equivalent to the condition given by:

$$\left|\frac{\dfrac{z\big(\mathcal{R}_{q,p,\lambda}^{\delta,m}f(z)\big)'}{f_t(z)}-p}{\big(\alpha p+(\beta-\alpha)(p-\gamma)\big)-\alpha z\dfrac{\big(\mathcal{R}_{q,p,\lambda}^{\delta,m}f(z)\big)'}{f_t(z)}}\right|<1. \tag{24}$$

Let $|z|=1$ and (23) holds true. So we have:

$$\begin{aligned}
W &= \left|z\big(\mathcal{R}_{q,p,\lambda}^{\delta,m}f(z)\big)'-pf_t(z)\right|-\left|\big(\alpha p+(\beta-\alpha)(p-\gamma)\big)+\alpha z\big(\mathcal{R}_{q,p,\lambda}^{\delta,m}f(z)\big)'\right| \\
&= \Bigg|z\left(pz^{p-1}-\sum_{k=n+p}^{\infty}ka_k\Big(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}\Big)^m C_m\Big(\frac{[k]_q}{[p]_q}\Big)z^{k-1}\right) \\
&\quad -p\Big((1-t)z^p+tf(z)\Big)\Bigg| = \Bigg|\big(\alpha p+(\beta-\alpha)(p-\gamma)\big)\big((1-t)z^p+tf(z)\big) \\
&\quad -\alpha z\left(pz^{p-1}-\sum_{k=n+p}^{\infty}ka_k\Big(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}\Big)^m C_m\Big(\frac{[k]_q}{[p]_q}\Big)z^{k-1}\right)\Bigg| \\
&= \left|-\sum_{k=n+p}^{\infty}\left[k\Big(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}\Big)^m C_m\Big(\frac{[k]_q}{[p]_q}\Big)-pt\right]a_kz^k\right| \\
&\quad -\Bigg|(\beta-\alpha)(p-\gamma)z^p-\sum_{k=n+p}^{\infty}\Big[t\big(\alpha p+(\beta-\alpha)(p-\gamma)\big) \\
&\quad -\alpha k\Big(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}\Big)^m C_m\Big(\frac{[k]_q}{[p]_q}\Big)\Big]a_kz^k\Bigg|.
\end{aligned}$$

By putting:

$$t\big(\alpha p+(\beta-\alpha)(p-\gamma)\big)-\alpha k\Big(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}\Big)^m C_m\Big(\frac{[k]_q}{[p]_q}\Big)$$
$$=t(\beta-\alpha)(p-\gamma)-\Bigg[k\Big(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}\Big)^m C_m\Big(\frac{[k]_q}{[p]_q}\Big)-pt\Bigg]\alpha,$$

the above expression reduces to:

(25)
$$W\leqslant \left|\sum_{k=n+p}^{\infty}\Bigg[\Bigg(k\Big(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}\Big)^m C_m\Big(\frac{[k]_q}{[p]_q}\Big)-pt\Bigg)(1-\alpha)+t(\beta-\alpha)(p-\gamma)\Bigg]a_k - (\beta-\alpha)(p-\gamma)\right|.$$

By (23), we have $W\leqslant 0$, so $f(z)\in\mathcal{XR}(\alpha,\beta,\gamma)$.

To prove the converse, let $f(z)\in\mathcal{XR}(\alpha,\beta,\gamma)$, thus:

$$Y=\left|\frac{\dfrac{z\big(\mathcal{R}^{\delta,m}_{q,p,\lambda}f(z)\big)'}{f_t(z)}-p}{\big(\alpha p+(\beta-\alpha)(p-\gamma)\big)-\alpha z\dfrac{\big(\mathcal{R}^{\delta,m}_{q,p,\lambda}f(z)\big)'}{f_t(z)}}\right|$$
$$=\frac{\left|z\Big(pz^{p-1}-\sum_{k=n+p}^{\infty}ka_kQ^*z^{k-1}\Big)-p\Big[(1-t)z^p+tf(z)\Big]\right|}{\left|\big(\alpha p+(\beta-\alpha)(p-\gamma)\big)\Big[(1-t)z^p+tf(z)\Big]-\alpha z\Big(pz^{p-1}-\sum_{k=n+p}^{\infty}ka_kQ^*z^{k-1}\Big)\right|}$$
$$<1,$$

where

$$Q^*=\Big(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}\Big)^m C_m\Big(\frac{[k]_q}{[p]_q}\Big). \tag{26}$$

Since for all z, $\mathrm{Re}\{z\}\leqslant|z|$, so we have:

$$\mathrm{Re}\{Y\}=\mathrm{Re}\left\{\frac{\sum_{k=n+p}^{\infty}\big(kQ^*-pt\big)a_kz^k}{(\beta-\alpha)(p-\gamma)z^p-\sum_{k=n+p}^{\infty}\Big[t\big(\alpha p+(\beta-\alpha)(p-\gamma)\big)-\alpha kQ^*\Big]a_kz^k}\right\}$$
$$<1,$$

where Q^* is given (26).

By letting $z \to 1$ through positive real values and choose the values of z such that $\frac{z(\mathcal{R}^{\delta,m}_{q,p,\lambda} f(z))'}{f_t(z)}$ is real, we have:

$$\sum_{k=n+p}^{\infty} \big(kQ^* - pt\big) a_k \leqslant (\beta-\alpha)(p-\gamma) - \sum_{k=n+p}^{\infty} \Big[t\big(\alpha p + (\beta-\alpha)(p-\gamma)\big) - \alpha k Q^*\Big] a_k,$$

where Q^* is defined by (26).

By (25), we get:

$$\sum_{k=n+p}^{\infty} \left[\left(k\Big(\frac{\Gamma_q(p+1-\delta)\Gamma_q(k+1)}{\Gamma_q(p+1)\Gamma_q(k+1-\delta)}\Big)^m C_m\Big(\frac{[k]_q}{[p]_q}\Big) - pt\right)(1-\alpha) + t(\beta-\alpha)(p-\gamma)\right] a_k$$
$$\leqslant (\beta-\alpha)(p-\gamma),$$

and this completes the proof.

Remark 1 *We note that:*

(i) The function:

$$G(z) = z^p - \sum_{k=n+p}^{\infty} \frac{(\beta-\alpha)(p-\gamma)}{\big(kQ^* - pt\big)(1-\alpha) + t(\beta-\alpha)(p-\gamma)} z^k,$$

shows that the inequality (23) *is sharp.*

(ii) If $f(z) \in \mathcal{XR}(\alpha,\beta,\gamma)$, then:

$$a_k \leqslant \frac{(\beta-\alpha)(p-\gamma)}{\big(kQ^* - pt\big)(1-\alpha) + t(\beta-\alpha)(p-\gamma)}, \qquad (k \geqslant n+p),$$

where Q^* is given in (26).

By applying Theorem 1, we can easily prove that the class $\mathcal{XR}(\alpha,\beta,\gamma)$ is closed under weighted and arithmetic mean. Also it is easy to obtain radii of starlikeness, convexity and close-to-convexity. So we state the following three theorems without proof.

Theorem 2 *If $f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k$ and $g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k$, be in the class $\mathcal{XR}(\alpha,\beta,\gamma)$, then the weighted mean of f and g given by:*

$$h_j(z) = \frac{1}{2}\Big[(1-j)f(z) + (1+j)g(z)\Big],$$

is also in $\mathcal{XR}(\alpha,\beta,\gamma)$.

Theorem 3 *If $f_j(z) = z^p - \sum_{k=n+p}^{\infty} a_{k,j} z^k$, $j = 1, 2, \ldots, \ell$, be in the class $\mathcal{XR}(\alpha, \beta, \gamma)$, then the arithmetic mean of $f_j(z)$ given by:*

$$h(z) = \frac{1}{\ell} \sum_{j=1}^{\ell} f_j(z),$$

is also in the same class.

Theorem 4 *Let the function $f(z)$ defined by* (1) *be in the class $\mathcal{XR}(\alpha, \beta, \gamma)$, then:*

(i) $f(z)$ is starlike of order θ $(0 \leqslant \theta < p)$ in $|z| < R_1$, where:

$$R_1 = \inf_{k \geqslant n+p} \left\{ \frac{p-\theta}{k-\theta} \left[\frac{\big(kQ^* - pt\big)(1-\alpha)}{(\beta-\alpha)(p-\gamma)} + t \right] \right\}^{\frac{1}{k-p}}.$$

(ii) $f(z)$ is convex of order θ $(0 \leqslant \theta < p)$ in $|z| < R_2$, where:

$$R_2 = \inf_{k \geqslant n+p} \left\{ \frac{p(p-\theta)}{k(k-\theta)} \left[\frac{\big(kQ^* - pt\big)(1-\alpha)}{(\beta-\alpha)(p-\gamma)} + t \right] \right\}^{\frac{1}{k-p}}.$$

(iii) $f(z)$ is close-to-convex of order θ $(0 \leqslant \theta < p)$ in $|z| < R_3$, where:

$$R_3 = \inf_{k \geqslant n+p} \left\{ \frac{p-\theta}{k} \left[\frac{\big(kQ^* - pt\big)(1-\alpha)}{(\beta-\alpha)(p-\gamma)} + t \right] \right\}^{\frac{1}{k-p}}.$$

In relations R_1, R_2 and R_3, Q^* and $C_m(x)$ are defined by (26) *and* (20) *respectively.*

3 Preserving properties and integral representation

In this section, we investigate some class preserving integral operators. We recall the Komatu [7] and generalized Jung-Kim-Srivastava [6] operators defined by:

$$\mathcal{K}_p^{c,d} f(z) = \frac{(c+p)^d}{\Gamma(d) z^c} \int_0^z t^{c-1} \big(\log \frac{z}{t}\big)^{d-1} f(t) dt,$$
$$\mathcal{JKS}_p^{c,d} f(z) = \frac{\Gamma(d+c+p)}{\Gamma(c+p)\Gamma(d) z^c} \int_0^z t^{c-1} \big(1 - \frac{t}{z}\big)^{d-1} f(t),$$

where $d > 0$, $c > -p$, $f \in \mathcal{A}_p$ and $z \in \mathbb{U}$.

Finally, in the end of this section, we introduce integral representation for $\mathcal{R}_{q,p,\lambda}^{\delta,m} f(z)$, where $f(z) \in \mathcal{XR}(\alpha, \beta, \gamma)$.

Theorem 5 *If $f(z) \in \mathcal{XR}(\alpha, \beta, \gamma)$, then $\mathcal{K}_p^{c,d} f(z) \in \mathcal{XR}(\alpha, \beta, \gamma)$.*

Proof. Let $f(z) \in \mathcal{XR}(\alpha,\beta,\gamma)$ be defined by (1). It is easy to show that:

$$\mathcal{K}_p^{c,d} f(z) = z^p - \sum_{k=n+p}^{\infty} \Big(\frac{c+p}{c+k+p}\Big)^d a_k z^k, \qquad (a_k \geqslant 0, \quad p \in \mathbb{N}).$$

But $\mathcal{K}_p^{c,d} f(z) \in \mathcal{XR}(\alpha,\beta,\gamma)$, if:

$$\begin{aligned} L = &\sum_{k=n+p}^{\infty} \big[\big(kQ^* - pt\big)(1-\alpha) + t(\beta-\alpha)(p-\gamma)\big] \Big(\frac{c+p}{c+k+p}\Big)^d a_k \\ &\leqslant (\beta-\alpha)(p-\gamma), \end{aligned}$$

where Q^* is given in (26).

Since for $k \in \mathbb{N}$, $\frac{c+p}{c+k+p} \leqslant 1$, so it clear that:

$$L \leqslant \sum_{k=n+p}^{\infty} \big[\big(kQ^* - pt\big)(1-\alpha) + t(\beta-\alpha)(p-\gamma)\big]\, a_k$$

then by (23), we have:

$$L \leqslant (\beta-\alpha)(p-\gamma),$$

Therefore $\mathcal{K}_p^{c,d} f(z) \in \mathcal{XR}(\alpha,\beta,\gamma)$.

Theorem 6 *If $f(z) \in \mathcal{XR}(\alpha,\beta,\gamma)$, then $\mathcal{JKS}_p^{c,d} f(z)$ is in the same class.*

Proof. Let $f(z) \in \mathcal{XR}(\alpha,\beta,\gamma)$ be defined by (1). It can be easily verified that:

$$\mathcal{JKS}_p^{c,d} f(z) = z^p - \sum_{k=n+p}^{\infty} \frac{\Gamma(d+c+p)\Gamma(c+k)}{\Gamma(c+p)\Gamma(d+c+k)} a_k z^k.$$

By the similar steps as in the proof of Theorem 5, we can state the proof concerning $\mathcal{JKS}_p^{c,d} f(z)$, so the details are omitted.

Theorem 7 *Let $f(z) \in \mathcal{XR}(\alpha,\beta,\gamma)$, then:*

$$\mathcal{R}_{q,p,\lambda}^{\delta,m} f(z) = \int_0^z \frac{f_t(z)\Big[\big(\alpha p + (\beta-\alpha)(p-\gamma)\big)H(s) + p\Big]}{s\big(1+\alpha H(s)\big)} ds,$$

where $|H(z)| < 1$ and f_t is given in (23).

Proof. For $f(z) \in \mathcal{XR}(\alpha,\beta,\gamma)$, we have the subordination relation (21), or equivalently the inequality (24). Thus:

$$\frac{z\big(\mathcal{R}_{q,p,\lambda}^{\delta,m} f(z)\big)' - pf_t(z)}{\big(\alpha p + (\beta-\alpha)(p-\gamma)\big)f_t(z) - \alpha z\big(\mathcal{R}_{q,p,\lambda}^{\delta,m} f(z)\big)'} = H(z)$$

where $|H(z)| < 1$. Therefore we get:

$$z\big(\mathcal{R}^{\delta,m}_{q,p,\lambda} f(z)\big)' - p f_t(z) = \big(\alpha p + (\beta-\alpha)(p-\gamma)\big) H(z) f_t(z) - \alpha z H(z)\big(\mathcal{R}^{\delta,m}_{q,p,\lambda} f(z)\big)',$$

or

$$\big(\mathcal{R}^{\delta,m}_{q,p,\lambda} f(z)\big)'\big(z + \alpha z H(z)\big) = f_t(z)\Big[\big(\alpha p + (\beta-\alpha)(p-\gamma)\big)H(z) + p\Big].$$

After integration, we get the required result. So the proof is complete.

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