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Univalence Criteria for a General Integral Operator [1]

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Abstract

In this paper we introduce a new general integral operator for analytic functions in the open unit disk $\mathbb{U}$ and we obtain sufficient conditions for univalence of this integral operator.

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1 Introduction

Let $\mathcal{A}$ denote the class of the functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} :\mid z \mid< 1\}$$

and satisfy the following usual normalization conditions:

$$f(0) = f'(0) - 1 = 0,$$

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$\mathbb{C}$ being the set of complex numbers.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{R}(\lambda)$, $0 \leq \lambda < 1$, if

$$\mathrm{Re}\left[f^{'}(z)\right] > \lambda, \quad z \in \mathbb{U}.$$

We denote by $\mathcal{P}$ the class of the functions $p(z)$ which are analytic in $\mathbb{U}$ and satisfy the following conditions:

$$p(0) = 1, \quad \mathrm{Re}(p(z)) > 0, \quad z \in \mathbb{U}.$$

We consider the integral operator:

$$\mathcal{C}_n(z) = \left\{\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} e^{g_i(t)}\right)^{\alpha_i - 1} \left(h_i{}'(t)\right)^{\beta_i} \left(\frac{h_i(t))}{t}\right)^{\gamma_i}\right] \mathrm{dt}\right\}^{\frac{1}{\delta}}, \tag{2}$$

where f_i, g_i, h_i are analytic in $\mathbb{U}$ and $\delta, \alpha_i, \beta_i, \gamma_i$ are complex numbers for all $i = \overline{1,n}$, $n \in \mathbb{N} \setminus \{0\}$, $\delta \in \mathbb{C}$, with $\mathrm{Re}\delta > 0$..

This is a general integral operator of Pfaltzgraff, Kim-Merkes and Oversea types which extends also the other operators as follows:

Remark 1.1. *i) For $n = 1$, $\delta = 1$ and $\alpha_1 - 1 = \beta_1 = 0$ we obtain the integral operator which was studied by Kim-Merkes [7]*

$$\mathcal{F}_\alpha(z) = \int_0^z \left(\frac{f(t)}{t}\right)^\alpha dt.$$

ii) For $n = 1$, $\delta = 1$ and $\alpha_1 - 1 = \gamma_1 = 0$ we obtain the integral operator which was studied by Pfaltzgraff [18]

$$\mathcal{G}_\alpha(z) = \int_0^z \left(f'(t)\right)^\alpha dt.$$

iii) For $\alpha_i - 1 = \beta_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [2]

$$\mathcal{D}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha_i} dt\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pascu and Pescar [12].

iv) For $\alpha_i - 1 = \gamma_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz, Owa and N. Breaz [4]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left[f_i'(t)\right]^{\alpha_i} dt\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pescar and Owa in [17] .

v) For $\alpha_i - 1 = 0$ we obtain the integral operator which was defined and studied by Frasin [5]

$$\mathcal{F}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \left(f_i{}'(t)\right)^{\beta_i} dt\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Oversea in [9].

vi) For $n = 1$, $\delta = \beta$ and $\alpha_i - 1 = \alpha_i$ and $\beta_i = \gamma_i = 0$ we obtain the integral operator which was defined and studied by Stanciu in [19]

$$\mathcal{H}_1(z) = \left[\beta \int_0^z t^{\beta-1} \left(\frac{f(t)}{t} e^{g(t)}\right)^{\alpha} dt\right]^{\frac{1}{\beta}}.$$

Thus, the integral operator $\mathcal{C}_n$, introduced here by the formula (2), can be considered as an extension and a generalization of these operators above mentioned.

We need in our present investigation the following lemmas:

Lemma 1.2. *(Pascu [11]) Let γ, δ be complex numbers, $Re\gamma > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1 - |z|^{2Re\gamma}}{Re\gamma} \left|\frac{zf''(z))}{f'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, then for any complex number δ, $Re\delta \geq Re\gamma$, the function F_δ defined by

$$F_\delta(z) = \left(\delta \int_0^z t^{\delta-1} f'(t)\, dt\right)^{\frac{1}{\delta}},$$

is regular and univalent in $\mathcal{U}$.

Lemma 1.3. *(Pescar [14]) Let δ be complex number, $Re\delta > 0$ and c a complex number, $|c| \leq 1$, $c \neq -1$, and $f \in \mathcal{A}$, $f(z) = z + a_2 z^2 + \ldots.$ If*

$$\left| c\,|z|^{2\delta} + \left(1 - |z|^{2\delta}\right) \frac{zf''(z))}{\delta f'(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, then the function F_δ defined by

$$F_\delta(z) = \left(\delta \int_0^z t^{\delta-1} f'(t)\, dt\right)^{\frac{1}{\delta}},$$

is regular and univalent in $\mathcal{U}$.

Lemma 1.4. *(General Schwarz Lemma [8]) Let f be the function regular in the disk $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R, R > 0\}$ with $|f(z)| < M$ for a fixed number $M > 0$ fixed. If $f(z)$ has one zero with multiplicity order bigger than a positive integer m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} z^m, \quad z \in \mathbb{U}_R.$$

The equality for $z \neq 0$ can hold only if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ is constant.

2 Main results

Our main results give sufficient conditions for the general integral operator $\mathcal{C}_n$ defined by (2) to be univalent in the open disk $\mathbb{U}$.

Theorem 2.1. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $c = Re\gamma > 0$, M_i, N_i, P_i, Q_i are real positive numbers, $i = \overline{1, n}$, and $f_i, g_i, h_i \in \mathcal{A}$, $f_i(z) = z + a_{2i} z^2 + a_{3i} z^3 + ...$, $g_i(z) = z + b_{2i} z^2 + b_{3i} z^3 + ...$, $h_i(z) = z + c_{2i} z^2 + c_{3i} z^3 + ...$, $i = \overline{1, n}$. If*

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| \leq M_i, \quad |g_i(z)| \leq N_i, \quad \left| \frac{z h_i''(z)}{h_i'(z)} \right| \leq P_i,$$

$$\left| \frac{z h_i'(z)}{h_i(z)} - 1 \right| \leq Q_i, \quad \left| \frac{z g_i'(z)}{g_i(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| (M_i + N_i) + |\beta_i| P_i + |\gamma_i| Q_i \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2} \tag{3}$$

then, for all δ complex numbers, $Re\delta \geq Re\gamma$, the integral operator $\mathcal{C}_n$, given by (2) is in the class $\mathcal{S}$.

Proof. Let us define the function

$$H_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} e^{g_i(t)} \right)^{\alpha_i - 1} \left(h_i'(t) \right)^{\beta_i} \left(\frac{h_i(t)}{t} \right)^{\gamma_i} \right] \mathrm{dt},$$

for $f_i, g_i, h_i \in \mathcal{A}$, $i = \overline{1, n}$.

The function H_n is regular in $\mathbb{U}$ and satisfy the following usual normalization conditions $H_n(0) = H_n'(0) - 1 = 0$.

After we calculate the first-order and second-order derivatives, we obtain

$$\frac{z H_n''(z)}{H_n'(z)} = \sum_{i=1}^{n} \left[(\alpha_i - 1) \left(\frac{z f_i'(z)}{f_i(z)} - 1 + z g_i'(z) \right) + \beta_i \frac{z h_i''(z)}{h_i'(z)} + \gamma_i \left(\frac{z h_i'(z)}{h_i(z)} - 1 \right) \right],$$

for all $z \in \mathbb{U}$.

Therefore

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq \frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\alpha_i - 1|\left(\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + \left|zg_i{}'(z)\right|\right)\right] +$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\beta_i|\left|\frac{zh_i''(z)}{h_i'(z)}\right| + |\gamma_i|\left|\frac{zh_i'(z)}{h_i(z)} - 1\right|\right],$$

for all $z \in \mathbb{U}$.

Then, we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq \frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\alpha_i - 1|\left(\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + \left|\frac{zg_i{}'(z)}{g_i(z)}\right||g_i(z)|\right)\right] +$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\beta_i|\left|\frac{zh_i''(z)}{h_i'(z)}\right| + |\gamma_i|\left|\frac{zh_i'(z)}{h_i(z)} - 1\right|\right], \tag{4}$$

for all $z \in \mathbb{U}$.

By applying the General Schwarz Lemma to the functions f_i, g_i, $i = \overline{1,n}$ we obtain

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i\,|z|, \quad |g_i(z)| \leq N_i\,|z|,$$

$$\left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq P_i\,|z|, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq Q_i\,|z|,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$.

Using these inequalities from (4) we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq \frac{1-|z|^{2c}}{c}\,|z|\sum_{i=1}^{n}\left[|\alpha_i - 1|\,(M_i + N_i) + |\beta_i|\,P_i + |\gamma_i|\,Q_i\right], \tag{5}$$

for all $z \in \mathbb{U}$.

Since

$$\max_{|z|\leq 1}\frac{\left(1-|z|^{2c}\right)|z|}{c} = \frac{2}{(2c+1)^{\frac{2c+1}{2c}}},$$

from (5) we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq \frac{2}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}\left[|\alpha_i - 1|\,(M_i + N_i) + |\beta_i|\,P_i + |\gamma_i|\,Q_i\right],$$

and hence, by (3) we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq \frac{2}{(2c+1)^{\frac{2c+1}{2c}}}\,\frac{(2c+1)^{\frac{2c+1}{2c}}}{2} = 1,$$

for all $z \in \mathbb{U}$.

So,

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le 1. \tag{6}$$

and using (6), by Lemma 1.2, it results that the integral operator $\mathcal{C}_n$, given by (2) is in the class $\mathcal{S}$. □

Theorem 2.2. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $c = Re\gamma > 0$ and $f_i, h_i \in \mathcal{S}$, $h_i{}' \in \mathcal{P}$, $g_i \in \mathcal{R}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $i = \overline{1,n}$. If*

$$4\sum_{i=1}^{n}|\alpha_i - 1| + \sum_{i=1}^{n}|\beta_i| + 2\sum_{i=1}^{n}|\gamma_i| \le \frac{c}{2}, \quad for \quad 0 < c < 1 \tag{7}$$

or

$$4\sum_{i=1}^{n}|\alpha_i - 1| + \sum_{i=1}^{n}|\beta_i| + 2\sum_{i=1}^{n}|\gamma_i| \le \frac{1}{2}, \quad for \quad c \ge 1 \tag{8}$$

then, for any complex numbers δ, $Re\delta \ge c$, the integral operator $\mathcal{C}_n$ defined by (2) is in the class $\mathcal{S}$.

Proof. From the proof of Theorem 2.1, we have:

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le \frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\alpha_i - 1|\left(\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| + \left|zg_i{}'(z)\right|\right)\right] +$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\beta_i|\left|\frac{zh_i''(z)}{h_i'(z)}\right| + |\gamma_i|\left|\frac{zh_i'(z)}{h_i(z)} - 1\right|\right],$$

for all $z \in \mathbb{U}$.

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le \frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\alpha_i - 1|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1 + \left|zg_i{}'(z)\right|\right)\right] +$$

$$+\frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\beta_i|\left|\frac{zh_i''(z)}{h_i'(z)}\right| + |\gamma_i|\left(\left|\frac{zh_i'(z)}{h_i(z)}\right| + 1\right)\right],$$

for all $z \in \mathbb{U}$.

Since $f_i, h_i \in \mathcal{S}$ we have

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| \le \frac{1+|z|}{1-|z|}, \quad \left|\frac{zh_i'(z)}{h_i(z)}\right| \le \frac{1+|z|}{1-|z|},$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$.

For $h_i{}' \in \mathcal{P}$ we have

$$\left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq \frac{2\,|z|}{1-|z|^2},$$

for all $z \in \mathcal{U}$, $i = \overline{1,n}$.

For $g_i \in \mathcal{R}$ we have $g_i{}' \in \mathcal{P}$ and

$$\left|g_i{}'(z)\right| \leq \frac{1+|z|}{1-|z|}.$$

Using these relations we get

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq \frac{1-|z|^{2c}}{c}\left[\left(\frac{1+|z|}{1-|z|}+1+\frac{|z|\,(1+|z|)}{1-|z|}\right)\sum_{i=1}^{n}|\alpha_i-1|\right]+$$

$$+\frac{1-|z|^{2c}}{c}\frac{2\,|z|}{1-|z|^2}\sum_{i=1}^{n}|\beta_i|+\frac{1-|z|^{2c}}{c}\left[\left(\frac{1+|z|}{1-|z|}+1\right)\sum_{i=1}^{n}|\gamma_i|\right] \leq$$

$$\leq \frac{1-|z|^{2c}}{c}\left(\frac{2}{1-|z|}+\frac{|z|\,(1+|z|)}{1-|z|}\right)\sum_{i=1}^{n}|\alpha_i-1|+$$

$$+\frac{1-|z|^{2c}}{c}\frac{2\,|z|}{1-|z|^2}\sum_{i=1}^{n}|\beta_i|+\frac{1-|z|^{2c}}{c}\frac{2}{1-|z|}\sum_{i=1}^{n}|\gamma_i| \tag{9}$$

for all $z \in \mathbb{U}$.

For $0 < c < 1$, we have $1-|z|^{2c} \leq 1-|z|^2$, $z \in \mathbb{U}$ and by (9) we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq \left(\frac{4}{c}+\frac{4}{c}\right)\sum_{i=1}^{n}|\alpha_i-1|+\frac{2}{c}\sum_{i=1}^{n}|\beta_i|+\frac{4}{c}\sum_{i=1}^{n}|\gamma_i|\,, \tag{10}$$

for all $z \in \mathbb{U}$.

From (7) and (10) we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq 1, \tag{11}$$

for all $z \in \mathbb{U}$ and $0 < c < 1$.

For $c \geq 1$ we have $\frac{1-|z|^{2c}}{c} \leq 1-|z|^2$, for all $z \in \mathcal{U}$ and by (9) we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq (4+4)\sum_{i=1}^{n}|\alpha_i-1|+2\sum_{i=1}^{n}|\beta_i|+4\sum_{i=1}^{n}|\gamma_i|\,, \tag{12}$$

for all $z \in \mathbb{U}$ and $c \geq 1$.

From (8) and (12) we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq 1. \tag{13}$$

for all $z \in \mathbb{U}$ and $c \geq 1$.

And by (11), (13) and Lemma 1.2 it results that the integral operator $\mathcal{C}_n$, defined by (2) is in the class $\mathcal{S}$. □

Theorem 2.3. *Let α be complex number, $Re\alpha > 0$, $i = \overline{1,n}$, M_i, N_i, P_i are real positive numbers and $f_i, g_i, h_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $i = \overline{1,n}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| \leq M_i, \quad |g_i(z)| \leq N_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq 1,$$

$$\left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq P_i, \quad \left|\frac{z^2 g_i'(z)}{[g_i(z)]^2} - 1\right| < 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$(14) \qquad |c| \leq 1 - \left|\frac{\alpha - 1}{\alpha}\right| \left(M_i + 2N_i^2 + P_i + 3\right), \qquad c \in \mathbb{C}, \qquad c \neq -1,$$

then, the integral operator $\mathcal{C}_n^*$, given by

$$(15) \qquad \mathcal{C}_n^*(z) = \left[\alpha \int_0^z t^{\alpha-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t} e^{g_i(t)} h_i{}'(t) \frac{h_i(t))}{t}\right)^{\alpha-1} dt\right]^{\frac{1}{\alpha}}$$

is in the class $\mathcal{S}$.

Proof. Let us consider the function

$$H_n(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t} e^{g_i(t)} h_i{}'(t) \frac{h_i(t))}{t}\right)^{\alpha-1} \mathrm{dt},$$

for $f_i, g_i, h_i \in \mathcal{A}$, $i = \overline{1,n}$. The function H_n is regular in $\mathbb{U}$.

Also, a simple computation yields

$$\frac{zH_n''(z)}{H_n'(z)} = \sum_{i=1}^n \left[(\alpha - 1)\left(\frac{zf_i'(z)}{f_i(z)} - 1 + zg_i{}'(z) + \frac{zh_i''(z)}{h_i'(z)} + \frac{zh_i'(z)}{h_i(z)} - 1\right)\right],$$

for all $z \in \mathbb{U}$.

Therefore

$$\left|c\,|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right)\frac{zH_n''(z)}{\alpha H_n'(z)}\right| =$$

$$= \left|c\,|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right)\frac{1}{\alpha}\sum_{i=1}^n \left[(\alpha - 1)\left(\frac{zf_i'(z)}{f_i(z)} - 1 + zg_i{}'(z)\right)\right]\right|$$

$$+ \left|c\,|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right)\frac{1}{\alpha}\sum_{i=1}^n \left[(\alpha - 1)\left(\frac{zh_i''(z)}{h_i'(z)} + \frac{zh_i'(z)}{h_i(z)} - 1\right)\right]\right|,$$

for all $z \in \mathbb{U}$.

Then, we obtain

$$\left| c\,|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zH_n''(z)}{\alpha H_n'(z)} \right| \le$$

(16)
$$\le |c| + \left| \frac{\alpha - 1}{\alpha} \right| \left[\left(\left| \frac{zf_i'(z)}{f_i(z)} \right| + 1 \right) + \left| \frac{z^2 g_i{}'(z)}{[g_i(z)]^2} \right| \left| \frac{[g_i(z)]^2}{z} \right| + \left| \frac{zh_i''(z)}{h_i'(z)} \right| + \left(\left| \frac{zh_i'(z)}{h_i(z)} \right| + 1 \right) \right],$$

for all $z \in \mathbb{U}$.

By applying the General Schwarz Lemma to the functions g_i, $i = \overline{1, n}$ we obtain

$$|g_i(z)| \le N_i\,|z|\,,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$.

Using these inequalities from (16) we have

$$\left| c\,|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zH_n''(z)}{\alpha H_n'(z)} \right| \le |c| + \left| \frac{\alpha - 1}{\alpha} \right| \left[M_i + 1 + 2N_i^2 + 1 + P_i + 1 \right],$$

for all $z \in \mathbb{U}$.

So

$$\left| c\,|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zH_n''(z)}{\alpha H_n'(z)} \right| \le |c| + \left| \frac{\alpha - 1}{\alpha} \right| \left[M_i + 2N_i^2 + P_i + 3 \right], \tag{17}$$

and hence, by inequality (14) we have

$$\left| c\,|z|^{2\alpha} + \left(1 - |z|^{2\alpha}\right) \frac{zH_n''(z)}{H_n'(z)} \right| \le 1,$$

for all $z \in \mathbb{U}$.

Applying Lemma 1.3, we conclude that the integral operator $\mathcal{C}_n^*$, given by (15) is in the class $\mathcal{S}$. □

3 Corollaries and consequences

First of all, upon setting $\delta = 1$ in Theorem 2.1, we immediately arrive at the following corollary:

Corollary 3.1. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \le 1$, M_i, N_i, P_i, Q_i are real positive numbers and $f_i, g_i, h_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \ldots.$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + \ldots.$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + \ldots.$, $i = \overline{1, n}$. If*

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \le M_i, \quad |g_i(z)| \le N_i, \quad \left| \frac{zh_i''(z)}{h_i'(z)} \right| \le P_i,$$

$$\left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| \le Q_i, \quad \left| \frac{zg_i'(z)}{g_i(z)} \right| \le 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *and*

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \left(M_i + N_i \right) + |\beta_i| \, P_i + |\gamma_i| \, Q_i \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then, the integral operator $\mathcal{K}_n$ *defined by*

$$\mathcal{K}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} e^{g_i(t)} \right)^{\alpha_i - 1} \left(h_i{}'(t) \right)^{\beta_i} \left(\frac{h_i(t))}{t} \right)^{\gamma_i} \right] dt, \tag{18}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\gamma_i = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 3.2. *Let* $\gamma, \alpha_i, \beta_i$ *be complex numbers,* $0 < Re\gamma \leq 1$, $c = Re\gamma$, M_i, N_i, P_i *are real positive numbers and* $f_i, g_i, h_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $i = \overline{1,n}$. *If*

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| \leq M_i, \quad |g_i(z)| \leq N_i, \quad \left| \frac{z h_i''(z)}{h_i'(z)} \right| \leq P_i, \quad \left| \frac{z g_i'(z)}{g_i(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *and*

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \left(M_i + N_i \right) + |\beta_i| \, P_i \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then, the integral operator $\mathcal{T}_n$ *defined by*

$$\mathcal{T}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} e^{g_i(t)} \right)^{\alpha_i - 1} \left(h_i{}'(t) \right)^{\beta_i} \right] dt, \tag{19}$$

is in the class $\mathcal{S}$.

Remark 3.3. *In the integral operator given by (19) if we take* $\gamma_i = 0$, *we obtain known result proven in [19].*

If we consider $\delta = 1$ and $\beta_i = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 3.4. *Let* $\gamma, \alpha_i, \gamma_i$ *be complex numbers,* $0 < Re\gamma \leq 1$, $c = Re\gamma$, M_i, N_i, Q_i *are real positive numbers and* $f_i, g_i, h_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + ...$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + ...$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $i = \overline{1,n}$. *If*

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| \leq M_i, \quad |g_i(z)| \leq N_i, \quad \left| \frac{z h_i'(z)}{h_i(z)} - 1 \right| \leq Q_i, \quad \left| \frac{z g_i'(z)}{g_i(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *and*

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \left(M_i + N_i \right) + |\gamma_i| \, Q_i \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then, the integral operator $\mathcal{R}_n$ *defined by*

$$\mathcal{R}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} e^{g_i(t)} \right)^{\alpha_i - 1} \left(\frac{h_i(t))}{t} \right)^{\gamma_i} \right] dt, \tag{20}$$

is in the class $\mathcal{S}$.

Remark 3.5. *Putting* $\beta_i = 0$ *in (20) we obtain the integral operator introduced in [19].*

If we consider $\delta = 1$ and $\alpha_i = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 3.6. *Let* $\gamma, \beta_i, \gamma_i$ *be complex numbers,* $0 < Re\gamma \leq 1$, $c = Re\gamma$, P_i, Q_i *are real positive numbers and* $h_i \in \mathcal{A}$, $h_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + \ldots.$, $i = \overline{1, n}$. *If*

$$\left| \frac{z h_i''(z)}{h_i'(z)} \right| \leq P_i, \quad \left| \frac{z h_i'(z)}{h_i(z)} - 1 \right| \leq Q_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ *and*

$$\sum_{i=1}^{n} \left[|\beta_i| \, P_i + |\gamma_i| \, Q_i \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then, the integral operator $\mathcal{I}_n$ *defined by*

$$\mathcal{I}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left({h_i}'(t) \right)^{\beta_i} \left(\frac{h_i(t))}{t} \right)^{\gamma_i} \right] dt, \tag{21}$$

is in the class $\mathcal{S}$.

Remark 3.7. *The integral operator from Corollary 3.6, given by (21) is a known result proven in [5].*

If we consider $\delta = 1$, $\beta_i = 0$ and $\gamma_i = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 3.8. *Let* γ, α_i *be complex numbers,* $0 < Re\gamma \leq 1$, $c = Re\gamma$, M_i, N_i *are real positive numbers and* $f_i, g_i \in \mathcal{A}$, $f_i(z) = z + a_{2i}z^2 + a_{3i}z^3 + \ldots.$, $g_i(z) = z + b_{2i}z^2 + b_{3i}z^3 + \ldots.$, $i = \overline{1, n}$. *If*

$$\left| \frac{z f_i'(z)}{f_i(z)} - 1 \right| \leq M_i, \quad |g_i(z)| \leq N_i, \quad \left| \frac{z g_i'(z)}{g_i(z)} \right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ *and*

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| \left(M_i + N_i \right) \right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then, the integral operator $\mathcal{I}_n$ *defined by*

$$\mathcal{I}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} e^{g_i(t)} \right)^{\alpha_i - 1} \right] dt, \tag{22}$$

is in the class $\mathcal{S}$.

Remark 3.9. *This integral operator given by (22) was defined in [19].*

If we consider $\delta = 1$, $\alpha_i = 0$ and $\gamma_i = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 3.10. *Let γ, β_i be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, P_i are real positive numbers and $h_i \in \mathcal{A}$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $i = \overline{1,n}$. If*

$$\left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq P_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\sum_{i=1}^{n} \left[|\beta_i| P_i\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then, the integral operator $\mathcal{I}_n$ defined by

$$\mathcal{I}_n(z) = \int_0^z \prod_{i=1}^{n} \left(h_i{}'(t)\right)^{\beta_i} dt, \tag{23}$$

is in the class $\mathcal{S}$.

Remark 3.11. *The integral operator from Corollary 3.10, given by (23) is another known result proven in [4].*

If we consider $\delta = 1$, $\alpha_i = 0$ and $\beta_i = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 3.12. *Let γ, γ_i be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, Q_i are real positive numbers and $h_i \in \mathcal{A}$, $h_i(z) = z + c_{2i}z^2 + c_{3i}z^3 + ...$, $i = \overline{1,n}$. If*

$$\left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq Q_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\sum_{i=1}^{n} \left[|\gamma_i| Q_i\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then, the integral operator $\mathcal{I}_n$ defined by

$$\mathcal{I}_n(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{h_i(t))}{t}\right)^{\gamma_i} dt, \tag{24}$$

is in the class $\mathcal{S}$.

Remark 3.13. *This integral operator given by (24) is a well know result proven in [2].*

If we consider $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.1, obtain the next corollary:

Corollary 3.14. *Let α be complex number, $Re\alpha > 0$, M, N, P, Q are real positive numbers and $f, g, h \in \mathcal{A}$, $f(z) = z + a_2z^2 + a_3z^3 + ...$, $g(z) = z + b_2z^2 + b_3z^3 + ...$, $h(z) = z + c_2z^2 + c_3z^3 + ...$ If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M, \quad |g(z)| \leq N, \quad \left|\frac{zh''(z)}{h'(z)}\right| \leq P, \quad \left|\frac{zh'(z)}{h(z)} - 1\right| \leq Q, \quad \left|\frac{zg'(z)}{g(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$ and

$$|\alpha - 1| (M + N + P + Q) \leq \frac{(2Re\alpha + 1)^{\frac{2Re\alpha+1}{2Re\alpha}}}{2},$$

then, the integral operator $\mathcal{C}$ defined by

$$\mathcal{C}(z) = \left\{\alpha \int_0^z \left[f(t)e^{g(t)}h'(t)\frac{h(t))}{t}\right]^{\alpha-1} dt\right\}^{\frac{1}{\alpha}}, \tag{25}$$

is in the class $\mathcal{S}$.

References

[1] C. L. Aldea, V. Pescar, *Univalence Criteria for a general integral operator*, Transilvania University of Brasov, vol. 10, no. 2, 2017, 19-30.

[2] D. Breaz, N. Breaz, *Two Integral Operators*, Studia Univ."Babes-Bolyai", Cluj-Napoca, Mathematica, vol. 47, no. 3, 2002, 13-21.

[3] D. Breaz, N. Breaz, H. M. Srivastava, *An extension of the univalent condition for a family of integral operators*, Appl. Math. Lett., vol. 22, no. 3, 2009, 41-44.

[4] D. Breaz, S. Owa, N. Breaz, *A new integral univalent operator*, Acta Universitatis Apulensis, vol. 16, 2008.

[5] B. A. Frasin, *Order of convexity and univalence of general integral operator*, Journal of the Franklin, vol. 348, 2011, 1012-1019.

[6] B. A. Frasin, *Sufficient conditions for the univalence of an integral operator*, Asia Pacific Journal of Mathematics, vol.5, no.1, 2018, 85-91.

[7] I. J. Kim, E. P. Merkes, *On an integral of powers of a spirallike function*, Kyungpook Math. J., vol. 12, no. 2, 1972, 249-253.

[8] O. Mayer, *The Functions Theory of the One Variable Complex* , Acad. Ed., Bucuresti, Romania, 1981, 101-117.

[9] H. Oversea, *Integral operators of Bazilvic type*, Bull. Math. Bucuresti, vol. 37, 1993, 115-125.

[10] S. Ozaki, M. Nunokawa, *The Schwarzian derivative and univalent functions*, Proceedings of the American Mathematical Society, Mathematics, vol. 33, 1972, 392-394.

[11] N. N. Pascu, *An improvement of Beker's univalence criterion*, Proceedings of the Commemorative Session Simion Stoilov, University of Braso, 1987, 43-48.

[12] N. N. Pascu, V. Pescar, *On the integral operators Kim-Merkes and Pfaltzgraff*, Mathematica, UBB, Cluj-Napoca, vol. 32, no. 2, 1990, 185-192.

[13] V. Pescar, *New univalence criteria for some integral operators*, Studia Univ."Babes-Bolyai", Cluj-Napoca, Mathematica, vol. 59, no. 2, 2014, 185-192.

[14] V. Pescar, *A new generalization of Ahlfors's and Becker's criterion of univalence*, Bulletin of Malaysian Mathematical Society, vol. 19, no.2, 1996, 53-54.

[15] V. Pescar, *Univalence criteria of certain integral operators*, Acta Ciencia Indica, Mathematics, vol. 29, no. 1, 2003, 135-138.

[16] V. Pescar, *On the univalence of some integral operators*, General Mathematics, vol. 14, no. 2, 2006, 77-84.

[17] V. Pescar, S. Owa, *Univalence of certain integral operators*, Int. J. Math. Math. Sci., vol. 23, 2000, 697-701.

[18] J. Pfaltzgraff, *Univalence of the integral of* $(f'(z))^\lambda$, Bull. London Math. Soc., vol. 7, no. 3, 1975, 254-256.

[19] L. Stanciu, *The Univalence conditions of some integral operators* , Abstract and Applied Analysis, ID 924645, 2012, 9 pages.

[20] L. F. Stanciu, D. Breaz, H. M. Srivastava, *Some criteria for univalence of a certain integral operator*, Novi Sad J. Math. vol. 43, no. 2, 2013, 51-57.

[21] P. T. Mocanu, T. Bulboaca, G. S. Salagean, *Teoria geometrica a functiilor univalente*, Casa Cartii de Stiinta, Cluj Napoca, 1999, 77-81.

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