Some strong and weak form of z -continuity via Cl^* ¹

A. R. Prasannan, J. Biswas

Abstract

This paper mainly dedicated on overview of zero sets in Ideal topological spaces. We also introduce a new class of functions which generalizes the class of continuous functions and investigate its position in the hierarchy of continuous functions on Ideal topological spaces. Moreover, these new sets ($zero^*$ - $\mathcal{I}$ -set) which is a pragmatic approach to characterize completely *Hausdorff* spaces.

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1 Introduction

In 1933, Kuratowski [13] applied the notion of topological ideals in topological spaces and later on Vidyanathswamy [18] formally applied the word topological ideal in the study of spaces which have properties “locally” and which in several interesting cases implies that space has the property “globally”. A systematic study of the general topological space via ideals was initiated by Hamlett and Janković in [8, 9, 10, 11, 12] and gave different open sets using “ cl^* ”-closure operator. Subsequently different authors generalized and extended the concept of open sets, separation axioms, compactness, connectedness, resolvability, decomposition of continuity, etc. to ideal topological spaces [4, 5, 6, 7, 14, 15]. Newcomb [14] applied the concept of ideal in defining $\mathcal{I}$ -compactness. Singal et.al. [16] introduced a new characterization of completely *Hausdorff* spaces by means of zero sets. Using this concept, we characterize completely *Hausdorff* space using topological ideal. In the process, we have to define many new terms and applied them to get desired results.

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Al-Omari [2, 3] also introduced zero sets in ideal topological spaces using z -local functions and studied its properties. Moreover, he obtained characterizations of z -compatibility of a topology τ with an ideal $\mathcal{I}$ via *cozero* sets. In this paper, we introduce the notion of $\text{zero}^*\text{-}\mathcal{I}$ set in ideal topological spaces and explore its relationship with existing zero sets and achieve our goal to characterize completely *Hausdorff* using topological ideal. Section 2 is devoted to some basic definition and preliminaries, in Section 3 we introduce a new form of continuous function and its interrelationship between other weak forms of continuities in topological space and ideal topological spaces. In Section 4 and 5, we introduce $\text{zero}^*\text{-}\mathcal{I}$ set and $z^*\text{-}\mathcal{I}$ continuity and study its basic properties. Also, a pictorial representation of the relationship between various forms of z -continuity has been presented which shows weak and strong forms of z -continuity. In Section 6, we studied topological properties of $z^*\text{-}\mathcal{I}$ continuity.

2 Preliminaries

Let X be a non-empty set. A non empty collection $\mathcal{I} \subseteq P(X)$ is said to be an ideal on the set X if it satisfies the following conditions:

1. If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
2. If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

Let (X, τ) be a topological space and $\mathcal{I}$ be an ideal on X , then $(X, \tau, \mathcal{I})$ is called an ideal topological space on X with an ideal $\mathcal{I}$. If for every $x \in X$, $\tau(x) = \{U \in \tau : x \in U\}$ is a neighbourhood system at x . Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$, then $A^*(\tau, \mathcal{I})$ (also denoted by A^*) = $\{x \in X : U \cap A \notin \mathcal{I} \text{ for each } U \in \tau(x)\}$ is called the local function of A with respect to $\mathcal{I}$ and τ . Now $cl^*(A) = A \cup A^*$ defines a Kuratowski closure operator for a topology τ^* on X , called the $*$ -topology which is finer than τ .

Throughout this paper, the members of τ^* are called $*$ -open sets.

Definition 1 [1] A mapping $f : (X, \tau) \rightarrow (Y, \sigma, \mathcal{I})$ is said to be weakly $\mathcal{I}$ -continuous if for each $x \in X$ and open set V in Y containing $f(x)$, there exists an open set U containing x such that $f(U) \subset cl^*(V)$.

Definition 2 [10] A mapping $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ is said to be $(\mathcal{I}, \mathcal{J})$ -continuous if for each $x \in X$ and $\mathcal{J}$ -open set V in Y containing $f(x)$, there exists a $*$ -open set U containing x such that $f(U) \subset V$.

Definition 3 [11] Let (X, τ) be a topological space, $\mathcal{I}$ an ideal on X . Then the collection

$$\mathcal{B} = \{O \setminus I : O \in \tau, I \in \mathcal{I}\}$$

is a base for τ^* .

Definition 4 A topological space (X, τ) is called *completely Hausdorff* if for each pair of distinct points x_1 and x_2 in X , there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(x_1)=0$ and $f(x_2)=1$.

The collection of all *cozero* subsets of (X, τ) which form a basis for a topology τ_s which is coarser than τ , called the complete-regularization of τ . The collection of all zero-sets of (X, τ) is denoted by $Z(X, \tau)$.

Definition 5 [16] A mapping $f : X \rightarrow Y$, where X and Y are topological spaces, is said to be *z -continuous* at $x \in X$, if for each cozero-set V in Y containing $f(x)$, there exists an open set U in X containing x such that $f(U) \subseteq V$.

Throughout this paper, we consider a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ is z -continuous if and only if $f : (X, \tau) \rightarrow (Y, \sigma)$ is z -continuous and $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ is continuous if and only if $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous.

3 w^* - $\mathcal{I}$ -continuous function

In this section, we define a class of functions which generalizes the class of continuous functions and the class of $*(\mathcal{I}, \mathcal{J})$ -continuous functions on ideal topological spaces. We also study its position in the hierarchy of continuous functions on ideal topological spaces already exists in the literature.

Definition 6 A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is said to be *w^* - $\mathcal{I}$ -continuous*, if for each $x \in X$ and each open neighborhood V containing $f(x)$ there exists a $*$ -open set U in X containing x such that $f(U) \subseteq V$.

It is evident from the above definition that a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is said to be w^* - $\mathcal{I}$ -continuous if and only if $f : (X, \tau^*) \rightarrow (Y, \sigma)$ is continuous.

Theorem 1 Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be a function, then the following are equivalent:

1. f is w^* - $\mathcal{I}$ -continuous.
2. $f(cl^*(A)) \subseteq cl(f(A)) \forall A \subseteq X$.
3. $cl^*(f^{-1}(B)) \subseteq f^{-1}(cl(B)) \forall B \subseteq Y$.
4. Inverse image of closed set is cl^* -closed.
5. Inverse image of open set is $*$ -open.

Proof 1 (1. $\Rightarrow$ 2.) Let if possible, $f(cl^*(A)) \not\subseteq cl(f(A))$ for some $A \subseteq X$. This implies there exist $f(x) \in f(cl^*(A))$ but $f(x) \notin cl(f(A))$. This implies there exist a $V \in \sigma$ such that $f(x) \in V$ and $V \cap f(A) = \emptyset$. Since f is w^* - $\mathcal{I}$ -continuous at x , then there exist a $*$ -open set U containing x such that $f(U) \subseteq V$ and $f(U) \cap f(A) \subseteq$

$V \cap f(A) = \emptyset$ which in turn implies that $U \cap A = \emptyset$. Since $\mathcal{B} = \{O \setminus I : O \in \tau, I \in \mathcal{I}\}$ is a base for τ^* and U is $*$ -open, we have $U = \cup\{(U_i \setminus I_i) : I_i \in \mathcal{I} \text{ and } U_i \in \tau\}$. Hence, $\cup(U_i \setminus I_i) \cap A = (U_i \cap A) \cap I_i^c = \emptyset$, which shows that $U \cap A \in \mathcal{I}$. This shows $x \notin cl^*(A)$ and hence $f(x) \notin f(cl^*(A))$. This is a contradiction. Hence $f(cl^*(A)) \subseteq cl(f(A))$, $\forall A \subseteq X$.

(2. $\Rightarrow$ 3.) Let $B \subseteq Y$ and assume that $A = f^{-1}(B) \subseteq X$. Then by (2.) we get $f(cl^*(f^{-1}(B))) \subseteq cl(f(f^{-1}(B)))$ which implies that $f(cl^*(f^{-1}(B))) \subseteq cl(B)$. Hence $cl^*(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ for all $B \subseteq Y$.

(3. $\Rightarrow$ 4.) Let B be a closed set in Y , then $B = cl(B)$. We want to prove $cl^*(f^{-1}(B)) = f^{-1}(B)$ and by (3.) we get $cl^*(f^{-1}(B)) \subseteq f^{-1}(cl(B))$ for all $B \subseteq Y$. Hence $cl^*(f^{-1}(B)) \subseteq f^{-1}(B)$. Hence $cl^*(f^{-1}(B)) = f^{-1}(B)$.

(4. $\Rightarrow$ 5.) It can be proved very easily.

(5. $\Rightarrow$ 1.) Let B be any open set in Y containing $f(x)$. By (5.) we get $f^{-1}(B)$ is $*$ -open in X . Choose U to be $f^{-1}(B)$, we get f is w^* - $\mathcal{I}$ -continuous function.

3.1 Relationship with other forms of Continuity

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Theorem 2 Every continuous function is w^* - $\mathcal{I}$ -continuous.

Proof 2 Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be a continuous function and $x \in X$. Let V be any open neighbourhood of $f(x)$. Since f is continuous, there exist an open set U in X such that $f(U) \subseteq V$. Also, since $\tau \subseteq \tau^*$, f is w^* - $\mathcal{I}$ -continuous.

Converse of above theorem may not be true as shown in Example 1.

Example 1 Let $X = \mathbb{R}$, $\tau = \{\emptyset, X\}$, the indiscrete topology, σ be the cofinite topology and $\mathcal{I}_f$ be the ideal of finite subset of (X, τ) . Then τ^* is the cofinite topology on X with the ideal $\mathcal{I}_f$ (See Example 2.5 in [11]). Therefore, $\tau^* = \sigma$, the cofinite topology. Let $f : (X, \tau, \mathcal{I}_f) \rightarrow (X, \sigma)$ be the function defined by $f(x) = x + 1$ for all $x \in \mathbb{R}$. Let $x \in X$ and V be an open set containing $f(x)$ such that $V \neq X$. Then, there exist a $U \in \tau^*$ such that $f(U) \subset V$. Hence f is w^* - $\mathcal{I}$ -continuous. Now, since τ is indiscrete topology, we cannot find a $U \in \tau$ such that $f(U) \subseteq V$. So, f is not continuous.

Remark 1 It is evident from the following examples that there is no relationship between w^* - $\mathcal{I}$ -continuity and weakly- $\mathcal{I}$ -continuity.

²Throughout this paper, we mean a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ is continuous if and only if $f : (X, \tau) \rightarrow (Y, \sigma)$ is continuous.

Example 2 Let $X = Y = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{a, d\}\}$, $\sigma = \{\emptyset, Y, \{b\}, \{b, d\}\}$ and $\mathcal{I} = \mathcal{J} = \{\emptyset\}$. Let $i_X : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ be the identity function. Then i_X is weakly- $\mathcal{I}$ continuous because for each $V \in \sigma$, $cl^*(V) = Y$. However, i_X is not w^* - $\mathcal{I}$ -continuous because there does not exist a $*$ -open set in X containing the point b and is contained in $i_X^{-1}(b)$.

Example 3 Let $(X, \tau, \mathcal{I})$ and $(X, \sigma, \mathcal{I})$ be two ideal topological space defined by $X = \{a, b\}$, $\tau = \{\emptyset, X, \{a\}\}$, $\sigma = \{\emptyset, X, \{b\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Let $i_X : (X, \tau, \mathcal{I}) \rightarrow (X, \sigma, \mathcal{I})$ be the identity function. Then i_X is w^* - $\mathcal{I}$ -continuous but not weakly- $\mathcal{I}$ continuous as there doesn't exist any U in X containing b such that $i_X(U) \subseteq cl^*(\{b\})$.

Definition 7 Let $g : (X, \tau) \rightarrow (Y, \sigma, \mathcal{I})$ be a function from topological space (X, τ) to the ideal topological space $(Y, \sigma, \mathcal{I})$. Then g is said to be a $\star$ -continuous function on X , if for each $x \in X$ and for every $*$ -open set V containing $g(x)$ there exist an open set U containing x such that $g(U) \subseteq V$.

Theorem 3 Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be a function, then the following are equivalent:

1. f is $\star$ -continuous.
2. $f(cl(A)) \subseteq cl^*(f(A))$, $\forall A \subseteq X$.
3. $cl(f^{-1}(B)) \subseteq cl^*(f^{-1}(B))$, $\forall B \subseteq Y$.
4. Inverse image of cl^* -closed set is closed.
5. Inverse image of $*$ -open set is open.

Theorem 4 Every $\star$ -continuous function is continuous.

The converse of above theorem may not be true which is illustrated in the following example.

Example 4 Let $X = \mathbb{R}$, τ be the indiscrete topology and $\mathcal{I}_c$ be the collection of countable subsets of X [11]. Take $i_X : (X, \tau) \rightarrow (X, \tau, \mathcal{I}_c)$ is continuous. Now we will prove i_X is not $\star$ -continuous. Let $x \in V = X \setminus \{x + n : n \in \mathbb{N}\}$. Then there does not exist any proper subset U in X such that $f(U) \subset V$. Therefore f is not $\star$ -continuous.

Theorem 5 Every $\star$ -continuous function is weakly $\mathcal{I}$ -continuous.

Proof 3 Let $f : (X, \tau) \rightarrow (Y, \tau_1, \mathcal{I})$ be a $\star$ -continuous. Then for each $x \in X$ and for every $*$ -open set V containing $f(x)$ there exist an open set U in X such that $f(U) \subseteq V$. Let $V_1 = O \setminus I$ where $O \in \tau_1$ containing $f(x)$ and $I \in \mathcal{I}$ be a basis element of τ^* containing $f(x)$. Then we can find a U_1 in τ such that $f(U_1) \subseteq O \setminus I = V_1$ this implies $f(U_1) \subseteq O \subseteq cl^*(O)$. Hence f is weakly $\mathcal{I}$ -continuous.

The converse of above theorem fails which is illustrated in the following example.

Example 5 Suppose $X = \{a, b\}$, $\tau = \{\emptyset, X, \{a\}\}$ and $I = \{\emptyset, \{b\}\}$. Take $i_X : (X, \tau) \rightarrow (X, \tau, I)$ be an identity function. Then i_X is weakly $\mathcal{I}$ -continuous at b but is not $\star$ -continuous.

Lemma 1 A function $f : (X, \tau) \rightarrow (Y, \sigma^*)$ is continuous if and only if f is $\star$ -continuous from (X, τ) to $(Y, \sigma, \mathcal{I})$.

Proof 4 Proof is very simple.

Theorem 6 Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ and $g : (Y, \sigma, \mathcal{J}) \rightarrow (Z, \eta, K)$ be any two functions such that f is onto, $\star$ -continuous and g be w^* - $\mathcal{I}$ -continuous. Then $g \circ f$ is continuous.

Proof 5 Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ and $g : (Y, \sigma, \mathcal{J}) \rightarrow (Z, \eta, K)$ be any two functions such that f be a $\star$ -continuous onto function and g be w^* - $\mathcal{I}$ -continuous. Let P be any open set containing $g(y)$, $y \in Y$. Since g is w^* - $\mathcal{I}$ -continuous function, there exist a $\star$ -open set V such that $g(V) \subseteq P$. Since f is onto, then there exist $x \in X$ such that $f(x) = y$. Also, f is $\star$ -continuous there exist an open set U in X such that $f(x) \in f(U) \subseteq V$. This further implies that there exist an open set U in X such that $g \circ f(U) \subseteq g(V) \subseteq P$. Hence $g \circ f$ is continuous.

Theorem 7 Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \eta)$ be any two functions such that f be w^* - $\mathcal{I}$ -continuous, onto function and g be a continuous. Then $g \circ f$ is w^* - $\mathcal{I}$ -continuous.

Proof 6 It can be proved easily.

The following figure summarizes the relationship between the different form of continuities in $\mathcal{I}$ -topological spaces. It is already shown that the reverse implications are not true.

4 z^* - $\mathcal{I}$ open set

Definition 8 Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is said to be a zero * - $\mathcal{I}$ set (denoted by z^* - $\mathcal{I}$ set) in X , if there exist a real valued w^* - $\mathcal{I}$ -continuous function $f : (X, \tau, \mathcal{I}) \rightarrow (\mathbb{R}, T)$ such that $f^{-1}(0) = A$ where T is the usual topology on $\mathbb{R}$.

The complement of a z^* - $\mathcal{I}$ set is called cozero * - $\mathcal{I}$ -set (denoted by cz^* - $\mathcal{I}$ set).

Theorem 8 Every z^* - $\mathcal{I}$ set is cl^* -closed.

Proof 7 By theorem 3 (4), inverse image of closed set is cl^* -closed and $\{0\}$ is closed in $(\mathbb{R}, T)$. Hence $f^{-1}(\{0\})$ is cl^* -closed.

Theorem 9 Every zero set is z^* - $\mathcal{I}$ set.

Proof 8 By theorem 2, every continuous function is w^* - $\mathcal{I}$ -continuous and hence every zero-set is a z^* - $\mathcal{I}$ set.

Corollary 1 Every cozero set is an cz^* - $\mathcal{I}$ set.

Remark 2 Following example shows that every z^* - $\mathcal{I}$ set may not be zero-set.

Example 6 Let $X = \mathbb{R}$, τ be indiscrete topology and $\mathcal{I} = \mathcal{P}(X)$. Then every subset of $\mathbb{R}$ is z^* - $\mathcal{I}$ set but it is not even closed set. Hence it is not a zero set.

Theorem 10 Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be a w^* - $\mathcal{I}$ -continuous function then inverse image of cozero set is cz^* - $\mathcal{I}$ set.

Proof 9 Let V be any zero-set. Then there exist a continuous real valued function g such that $V = g^{-1}(0)$. Then $f^{-1}(V) = f^{-1}(g^{-1}(0)) = (g \circ f)^{-1}(0)$. By theorem 7 $g \circ f$ is w^* - $\mathcal{I}$ -continuous function and hence $(g \circ f)^{-1}(0)$ is z^* - $\mathcal{I}$ -set.

Theorem 11 Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $U \subseteq X$. If U is both * -open and cl^* -closed then U is cz^* - $\mathcal{I}$ set.

Proof 10 Let U be both * -open and cl^* -closed in X . It is enough to show that $X \setminus U$ is a z^* - $\mathcal{I}$ set i.e. there exist a w^* - $\mathcal{I}$ -continuous function f such that $X \setminus U = f^{-1}(0)$. Consider the function χ_U defined by

$$\chi_U(x) = \begin{cases} 0, & \text{if } x \notin U \\ 1, & \text{if } x \in U \end{cases}$$

It is obvious that χ_U is continuous. Then χ_U is w^* - $\mathcal{I}$ -continuous and $X \setminus U = \chi_U^{-1}(0)$. Hence U is cz^* - $\mathcal{I}$ set.

Definition 9 A set G is said to be z^* - $\mathcal{I}$ open if for each $x \in G$ there exist a cz^* - $\mathcal{I}$ set H_x such that $x \in H_x \subseteq G$. i.e. $G = \bigcup_{x \in G} H_x$.

The complement of a z^* - $\mathcal{I}$ open set is called z^* - $\mathcal{I}$ closed set.

Lemma 2 Every cz^* - $\mathcal{I}$ set is z^* - $\mathcal{I}$ open.

Proof 11 Let H be a cz^* - $\mathcal{I}$ set, for each $x \in H$, $x \in H_x \subseteq H$. Hence H is z^* - $\mathcal{I}$ open.

Theorem 12 If (X, τ^*) is completely regular then every * -open set is z^* - $\mathcal{I}$ open.

Proof 12 Suppose that P is * -open then $X \setminus P$ is $\mathcal{I}$ -closed and $x \in P$. Since (X, τ^*) is completely regular, then there exist a continuous function f such that $f(x) = 0$ and $X \setminus P \subseteq f^{-1}(\{1\})$. Then $H_x = \{y : f(y) < \frac{1}{3}\}$ is a cozero-set containing the point x . Since every cozero-set is cz^* - $\mathcal{I}$ set and $x \in H_x \subseteq P$. Hence P is z^* - $\mathcal{I}$ open.

Lemma 3 Every z -open set is z^* - $\mathcal{I}$ open set.

Proof 13 Let U be any z -open set, then for each x in U there exist a cozero-set H_x such that $U = \bigcup_{x \in U} H_x$. Since every cozero-set is cz^* - $\mathcal{I}$ set, then H_x is cz^* - $\mathcal{I}$ set for each $x \in U$ and hence U is z^* - $\mathcal{I}$ open.

Converse of the above lemma may not be true. See Example 6

Definition 10 A set $U \subseteq X$ is said to be a cz^* - $\mathcal{I}$ neighborhood of a point $x \in X$ if there exist a cz^* - $\mathcal{I}$ set V such that $x \in V \subseteq U$.

Definition 11 A point $x \in X$ is said to be z^* - $\mathcal{I}$ -adherent point of a set $A \subseteq X$, if every cz^* - $\mathcal{I}$ set containing x intersects with A . The set of z^* - $\mathcal{I}$ -adherent point of A is denoted by $A_{z^*-\mathcal{I}}$.

i.e, $A_{z^*-\mathcal{I}} = \{x \in X : N_x \cap A \neq \emptyset, \text{ for each } cz^*- \mathcal{I} \text{ neighbourhood } N_x \text{ of } x\}$.

Lemma 4 Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A be any subset of X , then $A \subseteq A_{z^*-\mathcal{I}}$.

Proof 14 Let if possible $A \not\subseteq A_{z^*-\mathcal{I}}$. Then there exist $x \in A$ such that $x \notin A_{z^*-\mathcal{I}}$. Then there exist a cz^* - $\mathcal{I}$ set N_x containing x such that $N_x \cap A = \emptyset$, then $x \notin A$. This is a contradiction, thus $x \in A_{z^*-\mathcal{I}}$. Hence $A \subseteq A_{z^*-\mathcal{I}}$.

Theorem 13 A is z^* - $\mathcal{I}$ closed if and only if $A_{z^*-\mathcal{I}} = A$.

Proof 15 Let A be z^* - $\mathcal{I}$ closed. By Lemma 4, $A \subseteq A_{z^*-\mathcal{I}}$. Now, it is enough to show that $A_{z^*-\mathcal{I}} \subseteq A$. Let if possible $A_{z^*-\mathcal{I}} \not\subseteq A$, then there exist a point $x \in A_{z^*-\mathcal{I}}$ and $x \notin A$. This implies that every cz^* - $\mathcal{I}$ set containing x intersects A but $x \notin A$. Now $X \setminus A$ is z^* - $\mathcal{I}$ open and $x \in X \setminus A$, then there exist a cz^* - $\mathcal{I}$ set H_x such that $x \in H_x \subseteq X \setminus A$. This implies $H_x \cap A = \emptyset$ and hence $x \notin A_{z^*-\mathcal{I}}$, which is a contradiction. Thus $A_{z^*-\mathcal{I}} \subseteq A$.

Conversely, let $A_{z^*-\mathcal{I}} = A$, then we have to prove A is z^* - $\mathcal{I}$ -closed. Let if possible A is not z^* - $\mathcal{I}$ -closed which implies $X \setminus A$ is not z^* - $\mathcal{I}$ open. This shows there exist a point $x \in X \setminus A$ such that every cz^* - $\mathcal{I}$ set H_x which is not contained in $X \setminus A$. Hence $H_x \cap A \neq \emptyset$, which implies $x \in A_{z^*-\mathcal{I}} = A$ which is a contradiction. Hence A is z^* - $\mathcal{I}$ -closed.

Theorem 14 Let $(X, \tau, \mathcal{I})$ be an Ideal topological space and A, B, C be any three subsets of X . Then the following properties hold:

1. $\emptyset_{z^*-I} = \emptyset$
2. if $A \subseteq B$ then $A_{z^*-I} \subseteq B_{z^*-I}$.
3. if $A_{z^*-I} \subseteq C_{z^*-I}$ and $B_{z^*-I} \subseteq C_{z^*-I}$ then $A_{z^*-I} \cup B_{z^*-I} \subseteq C_{z^*-I}$
4. $A \subseteq A_{z^*-I}$
5. $(A_{z^*-I})_{z^*-I} = A_{z^*-I}$
6. $A_{z^*-I} \cup B_{z^*-I} = (A \cup B)_{z^*-I}$

Proof 16 (1), (2), (3) and (4) can be proved easily.

(5) From (4) we have $(A_{z^*-I}) = (A_{z^*-I})_{z^*-I}$. Let if possible there exist a $x \in (A_{z^*-I})_{z^*-I}$ but $x \notin A_{z^*-I}$ then by theorem 13. $x \in X \setminus A_{z^*-I}$ which is z^* - I open. Therefore there exist a cz^* - I set H_x such that $x \in H_x \subseteq X \setminus A_{z^*-I}$ which implies $H_x \cap A_{z^*-I} = \emptyset$ which means $x \notin (A_{z^*-I})_{z^*-I}$, which is a contradiction. Hence $(A_{z^*-I})_{z^*-I} = A_{z^*-I}$.

(6) from (2) and (3) we have $A_{z^*-I} \cup B_{z^*-I} \subseteq (A \cup B)_{z^*-I}$. Now take $x \in (A \cup B)_{z^*-I}$ then every cz^* - I set, U_x such that $U_x \cap (A \cup B) \neq \emptyset$, this implies $U_x \cap A \neq \emptyset$ or $U_x \cap B \neq \emptyset$. Hence $x \in A_{z^*-I}$ or $x \in B_{z^*-I}$, which means $A_{z^*-I} \cup B_{z^*-I} = (A \cup B)_{z^*-I}$.

Remark 3 The following example shows that the notion of z^* - I open is different from the notion of zero sets defined by Al-Omari in [2].

Example 7 Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a, b\}, \{c, d\}, \{b\}, \{b, c, d\}\}$ and $\mathcal{I} = \{\emptyset, \{b\}, \{c\}, \{b, c\}\}$. $Cl_z^*(\{c\}) = \{c\}$ therefore $\{c\}$ is a zero closed set in the sense of Al-Omari in [2]. Now $\{c\}_{z^*-I} = \{c, d\} \neq \{c\}$. Therefore $\{c\}$ is not z^* - I closed. Consider the set $\{a\}$, then $Cl_z^*(\{a\}) = \{a, b\}$ which shows that $\{a\}$ is not a zero closed set in the sense of Al-Omari, however, $\{a\}$ is z^* - I closed.

Following is the diagrammatic representation of the relationship between $z^* - I$ -open sets and other form of open sets in ideal topological spaces.

5 z^* - I -continuous function

Definition 12 A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ is said to be z^* - I continuous, if for each $x \in X$ and each cz^* - $\mathcal{J}$ set V containing $f(x)$ there exists a z^* -open set U containing x such that $f(U) \subseteq V$.

Definition 13 A function $g : (X, \tau) \rightarrow (Z, \eta, \mathcal{I})$ is said to be strongly z^* - I continuous if for each $x \in X$ and each cz^* - $\mathcal{I}$ set V containing $g(x)$ there exists an open set U containing x such that $g(U) \subseteq V$.

Theorem 15 For a function $f : X \rightarrow Y$, the following statements are equivalent

1. f is strongly z^* - $\mathcal{I}$ -continuous.
2. $f(\text{cl}(A)) \subseteq (f(A))_{z^*-\mathcal{I}}, \forall A \subseteq X$.
3. $\text{cl}(f^{-1}(B)) \subseteq f^{-1}(B_{z^*-\mathcal{I}}), \forall B \subseteq Y$.
4. Inverse image of z^* - $\mathcal{I}$ closed set is closed.
5. Inverse image of z^* - $\mathcal{I}$ open set is open.

Proof 17 (1. $\Rightarrow$ 2.) Let if possible, $f(\text{cl}(A)) \not\subseteq (f(A))_{z^*-\mathcal{I}}$ for some $A \subseteq X$. This implies there exist some $x \in X$ such that $f(x) \in f(\text{cl}(A))$ and $f(x) \notin (f(A))_{z^*-\mathcal{I}}$. This implies there exist a cz^* - $\mathcal{I}$ set, $N_{f(x)}$ such that $N_{f(x)} \cap f(A) = \emptyset$. Since f is strongly z^* - $\mathcal{I}$ -continuous therefore $f^{-1}(N_{f(x)})$ is open in X and $f^{-1}(N_{f(x)}) \cap A = \emptyset$. Hence $x \notin \text{cl}(A)$ which is not possible. Hence $f(\text{cl}(A)) \subseteq (f(A))_{z^*-\mathcal{I}}$ for all $A \subseteq X$.

(2. $\Rightarrow$ 3.) Let $B \subseteq Y$ and now assume $A = f^{-1}(B) \subseteq X$, therefore by (2.) we get $f(\text{cl}(A)) \subseteq (f(A))_{z^*-\mathcal{I}}$. Hence $f(\text{cl}(f^{-1}(B))) \subseteq (f(f^{-1}(B)))_{z^*-\mathcal{I}}$, which implies $f(\text{cl}(f^{-1}(B))) \subseteq B_{z^*-\mathcal{I}}$. Hence $\text{cl}(f^{-1}(B)) \subseteq f^{-1}(B_{z^*-\mathcal{I}})$ for all $B \subseteq Y$.

(3. $\Rightarrow$ 4.) Let B be a z^* - $\mathcal{I}$ -closed set in Y , then $B = B_{z^*-\mathcal{I}}$. We want to prove $\text{cl}(f^{-1}(B)) = f^{-1}(B)$ and by (3.) we get $\text{cl}(f^{-1}(B)) \subseteq f^{-1}(B_{z^*-\mathcal{I}})$ for all $B \subseteq Y$. Hence $\text{cl}(f^{-1}(B)) \subseteq f^{-1}(B)$. Hence $\text{cl}(f^{-1}(B)) = f^{-1}(B)$.

(4. $\Rightarrow$ 5.) It can be proved very easily.

(5. $\Rightarrow$ 1.) Let A be any cz^* - $\mathcal{I}$ set in Y . Since every cz^* - $\mathcal{I}$ set is z^* - $\mathcal{I}$ open and using (5.) we get $f^{-1}(A)$ is open in X . Hence there exist an open set $U = f^{-1}(A)$ such that $f(U) \subseteq A$. Hence f is strongly z^* - $\mathcal{I}$ -continuous.

Theorem 16 For a function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$, the following statements are equivalent

1. f is z^* - $\mathcal{I}$ -continuous.
2. $f(\text{cl}_{\mathcal{I}}^*(A)) \subseteq (f(A))_{z^*-\mathcal{J}}, \forall A \subseteq X$.
3. $\text{cl}_{\mathcal{I}}^*(f^{-1}(B)) \subseteq f^{-1}(B_{z^*-\mathcal{J}}), \forall B \subseteq Y$.
4. Inverse image of z^* - $\mathcal{J}$ closed set is $\text{cl}_{\mathcal{I}}^*$ -closed.
5. Inverse image of z^* - $\mathcal{J}$ open set is * -open.

Proof 18 Proof is very easy.

Theorem 17 Every strongly z^* - $\mathcal{I}$ -continuous function is z -continuous.

Proof 19 Theorem can be proved easily.

The converse may not be true as illustrated in the following example.

Example 8 Let $X = Y = \mathbb{R}$, τ be Smirnov Topology [17] and $\mathcal{I}_K$ be the collection $\{U_0 \setminus K : U_0 \in \tau(0) \text{ and } K = \{\frac{1}{n} : n \in \mathbb{N}\}\}$. Let $i : (X, \tau) \rightarrow (Y, \tau, \mathcal{I}_K)$ be the identity function. Let V be an z -open set in $(Y, \tau, \mathcal{I}_K)$. Since every z -open set is open, V is open in (Y, τ) . Therefore $i^{-1}(V)$ is open in X . Hence i is z -continuous. Now we will prove i is not strongly z^* - $\mathcal{I}$ -continuous. Let $V = K$. Since $\mathbb{R} \setminus K \in \mathcal{I}_K$, $cl^*(\mathbb{R} \setminus K) = \mathbb{R} \setminus K$ which implies $\mathbb{R} \setminus K$ is cl^* -closed thus $K \in (Y, \tau^*)$. Since every closed set is cl^* -closed this implies K is cl^* -closed in $(Y, \tau, \mathcal{I}_K)$. Therefore K is z^* - $\mathcal{I}$ open in Y . Now $i^{-1}(K)$ is closed in (X, τ) . Hence by Theorem 15 (iv), i is not strongly z^* - $\mathcal{I}$ -continuous.

Theorem 18 Every strongly z^* - $\mathcal{I}$ continuous function is z^* - $\mathcal{I}$ continuous.

Proof 20 Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma, \mathcal{J})$ be a strongly z^* - $\mathcal{I}$ continuous. Then for each $x \in X$ and each cz^* - $\mathcal{J}$ set V containing $f(x)$ there exists an open set, U containing x such that $f(U) \subseteq V$. Since every open set is * -open, f is z^* - $\mathcal{I}$ -continuous.

Remark 4 Converse of Theorem 18 may not be true.

Example 9 Let the space (X, τ) and the ideal $\mathcal{I}_K$ be defined as in Example 8. Let $i_X : (X, \tau, \mathcal{I}_K) \rightarrow (X, \tau, \mathcal{I}_K)$ be the identity mapping. Let V be any z^* - $\mathcal{I}$ open in X . Since every z^* - $\mathcal{I}$ open is * -open in X . Then $i_X^{-1}(V)$ is V which is open in X . Then i_X is z^* - $\mathcal{I}$ -continuous. However i_X is not strongly z^* - $\mathcal{I}$ -continuous as shown in Example 8.

Theorem 19 Every $\star$ -continuous function is strongly z^* - $\mathcal{I}$ -continuous.

Proof 21 Let $f : (X, \tau) \rightarrow (Y, \sigma, \mathcal{I})$ be a $\star$ -continuous function and let V be any z^* - $\mathcal{I}$ open subset of Y . Since every z^* - $\mathcal{I}$ open set is * -open and f is $\star$ -continuous. Then the inverse image of * -open set is open in X . Hence f is strongly z^* - $\mathcal{I}$ -continuous by Theorem 15.

Corollary 2 Every $\star$ -continuous function is z^* - $\mathcal{I}$ -continuous.

Following example shows converse of theorem 19 may not be true.

Example 10 Let $X = \{a, b\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$ and τ be the indiscrete topology on X . Suppose $f : (X, \tau) \rightarrow (X, \tau, \mathcal{I})$ be the identity mapping then f is z^* - $\mathcal{I}$ -continuous but not $\star$ -continuous at $x = a$.

5.1 $\mathcal{I}^*$ -z-continuous

Definition 14 A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is said to be $\mathcal{I}^*$ -z-continuous at $x \in X$, if for each cozero-set V in Y containing $f(x)$. Then there exists a $*$ -open set U in X containing x such that $f(U) \subseteq V$.

Theorem 20 Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be an onto function, then the following are equivalent:

1. f is $\mathcal{I}^*$ -z-continuous.
2. Inverse image of every cozero set is $*$ -open.
3. Inverse image of every zero set is cl^* -closed.

Proof 22 (1. $\Rightarrow$ 2.) Let f be a $\mathcal{I}^*$ -z-continuous and V be any cozero-set in Y . Then for each $x \in f^{-1}(V)$, there exists a $*$ -open set U_x containing x such that $f(U_x) \subseteq V$. Then $f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$ is $*$ -open because arbitrary union of $*$ -open is $*$ -open set.

Hence inverse image of every cozero set is $*$ -open.

(2. $\Rightarrow$ 3.) Let C be any zero set then $Y \setminus C$ is a cozero set in Y . Then inverse image of $Y \setminus C$ is $*$ -open in X by (2). Since f is onto $f^{-1}(Y \setminus C) = X \setminus f^{-1}(C)$ is $*$ -open which in turn implies that $f^{-1}(C)$ is cl^* -closed.

(3. $\Rightarrow$ 1.) Let $x \in X$ and V be any cozero set containing $f(x)$. Then $x \notin f^{-1}(Y \setminus V)$ which is a cl^* -closed set in X by (3). This shows that x belongs to the $*$ -open set $f^{-1}(V) = U$. Hence x belongs to the $*$ -open set U such that $f(U) \subseteq V$. Thus f is $\mathcal{I}^*$ -z-continuous.

Theorem 21 Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be a function, then the following are equivalent:

1. $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is $\mathcal{I}^*$ -z-continuous.
2. $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma_s)$ is w^* - $\mathcal{I}$ -continuous.
3. $f : (X, \tau^*) \rightarrow (Y, \sigma_s)$ is continuous.

where σ_s is the complete regularisation of (Y, σ) .

Proof 23 The proof is very easy.

Theorem 22 A mapping $f : (X, \tau) \rightarrow (Y, \sigma, \mathcal{I})$ be $\star$ -continuous and $g : (Y, \sigma, \mathcal{I}) \rightarrow (Z, \eta)$ be $\mathcal{I}^*$ -z-continuous. Then $g \circ f$ is z-continuous.

Proof 24 Let f be a $\star$ -continuous function and $g : (Y, \sigma, \mathcal{I}) \rightarrow (Z, \eta)$ be $\mathcal{I}^*$ -z-continuous. Let V be any cozero set and by theorem 20, we get $g^{-1}(V)$ is $*$ -open set in Y . Now f is $\star$ -continuous then $f^{-1}(g^{-1}(V))$ is open in X . Hence $(g \circ f)^{-1}(V)$ is an open set in X . Hence $g \circ f$ is z-continuous.

Theorem 23 Let (X, τ) be a normal space. A mapping $f : (X, \tau) \rightarrow (Y, \sigma, \mathcal{J})$ be strongly z^* - $\mathcal{J}$ -continuous and $g : (Z, \eta, \mathcal{I}) \rightarrow (X, \tau)$ be $\mathcal{I}^*$ - z -continuous. Then $f \circ g$ is z^* - $\mathcal{I}$ -continuous.

Proof 25 Let g be a $\mathcal{I}^*$ - z -continuous function and $f : (X, \tau) \rightarrow (Y, \sigma, \mathcal{J})$ be a strongly z^* - $\mathcal{J}$ -continuous. Since (X, τ) is normal and V be any z^* - $\mathcal{J}$ -open set in Y then $f^{-1}(V)$ is open in X . Again since (X, τ) is normal then every open set in X is a cozero set. By theorem 10, we get $f^{-1}(V)$ is a cozero set in X . Now g is $\mathcal{I}^*$ - z -continuous then $g^{-1}(f^{-1}(V))$ is * -open in Z . Hence $(f \circ g)$ is z^* - $\mathcal{I}$ -continuous.

Theorem 24 Every z -continuous function is $\mathcal{I}^*$ - z -continuous.

Proof 26 Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be a z -continuous. Then for each $x \in X$ and each cozero-set V containing $f(x)$ there exists an open set U containing x such that $f(U) \subseteq V$. Since every open set is * -open, f is $\mathcal{I}^*$ - z -continuous.

Converse of the above result may not be true, which is illustrated in following example.

Example 11 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, X\}$. Suppose i_X be the identity map from $(X, \tau, \mathcal{I})$ to (X, σ) . Then i_X is $\mathcal{I}^*$ - z -continuous but not z -continuous at b .

Theorem 25 Every w^* - $\mathcal{I}$ -continuous function is $\mathcal{I}^*$ - z -continuous.

Proof 27 Let $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ be a w^* - $\mathcal{I}$ -continuous. Let $x \in X$ and V be a cozero-set containing $f(x)$. Since every cozero-set is open set and f is w^* - $\mathcal{I}$ -continuous then there exists a * -open set U containing x such that $f(U) \subseteq V$. Hence f is $\mathcal{I}^*$ - z -continuous.

Converse of the above theorem may not be true as shown in the following example.

Example 12 Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$ and $\sigma = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$. Suppose i_X be an identity map from $(X, \tau, \mathcal{I})$ to (X, σ) . Then i_X is $\mathcal{I}^*$ - z continuous but not w^* - $\mathcal{I}$ -continuous at c .

The relationship between the various form of z -continuities discussed above is summarised in the following figure.

Note: None of the implications are irreversible.

6 Topological properties of z^* - $\mathcal{I}$ -continuous function

Theorem 26 A mapping $f : (X, \tau) \rightarrow (Y, \sigma, \mathcal{I})$ be strongly z^* - $\mathcal{I}$ -continuous and $g : (Y, \sigma, \mathcal{I}) \rightarrow (Z, \eta)$ be w^* - $\mathcal{I}$ -continuous. Then $g \circ f$ is continuous if the space (Z, η) is normal space.

Proof 28 Let f be a strongly z^* - $\mathcal{I}$ -continuous function, $g : (Y, \sigma, \mathcal{I}) \rightarrow (Z, \eta)$ be w^* - $\mathcal{I}$ -continuous. Let (Z, η) be a normal space and V be any closed set in Z . Since (Z, η) is normal then every closed set in Z is a zero set. By theorem 10, we get $g^{-1}(V)$ is z^* - $\mathcal{I}$ set in Y . Now f is strongly z^* - $\mathcal{I}$ -continuous then $f^{-1}(g^{-1}(V))$ is closed in X . Hence $(g \circ f)^{-1}(V)$ is a closed set in X .

The converse of above theorem may not be true as shown in the following example.

Example 13 Let $X = \{a, b, c\}$, $\mathcal{I} = \{\emptyset, \{a\}\}$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$, $\sigma = \{\emptyset, X, \{a\}\}$ and $\eta = \{\emptyset, X, \{a\}\}$. Suppose $f : (X, \tau) \rightarrow (Y, \sigma, \mathcal{I})$ be strongly z^* - $\mathcal{I}$ -continuous and $g : (Y, \sigma, \mathcal{I}) \rightarrow (Z, \eta)$ be w^* - $\mathcal{I}$ -continuous, then $g \circ f$ is continuous but it is evident that (Z, η) is not a normal space.

Theorem 27 A mapping $f : (X, \tau) \rightarrow (Y, \sigma, \mathcal{I})$ is strongly z^* - $\mathcal{I}$ -continuous if and only if the inverse image of every z^* - $\mathcal{I}$ set in Y is a zero-set in X .

Proof 29 Let f be a strongly z^* - $\mathcal{I}$ -continuous function and Z be any z^* - $\mathcal{I}$ -set in $(Y, \sigma, \mathcal{I})$. Then there exist w^* - $\mathcal{I}$ -continuous function $g : (Y, \sigma, \mathcal{I}) \rightarrow (\mathbb{R}, T)$, where T is usual topology on $\mathbb{R}$, such that $Z = g^{-1}(0)$. Since, $(\mathbb{R}, T)$ is normal, then by theorem 26, $g \circ f$ is continuous. Therefore $(g \circ f)^{-1}(0) = f^{-1}(g^{-1}(0)) = f^{-1}(Z)$ is a zero-set.

Converse is obvious.

Corollary 3 A mapping $f : (X, \tau, \mathcal{J}) \rightarrow (Y, \sigma, \mathcal{I})$ is z^* - $\mathcal{I}$ -continuous if and only if the inverse image of every zero*- $\mathcal{I}$ -set in Y is zero*- $\mathcal{J}$ -set in X .

Proof 30 Every closed set in X is a $cl^*_\mathcal{J}$ -closed. Then there exist w^* - $\mathcal{I}$ -continuous function, $g : (Y, \sigma, \mathcal{I}) \rightarrow (\mathbb{R}, T)$ such that $(g \circ f)^{-1}(0)$ is $cl^*_\mathcal{J}$ -closed in X . Thus $g \circ f$ is w^* - $\mathcal{I}$ -continuous. Therefore the inverse image of z^* - $\mathcal{I}$ open set is z^* - $\mathcal{J}$ -open.

Corollary 4 Composition of two z^* - $\mathcal{I}$ -continuous functions is z^* - $\mathcal{I}$ -continuous.

Theorem 28 A mapping $f : (X, \tau) \rightarrow (Y, \sigma, \mathcal{I})$ and the space (Y, σ^*) is completely regular space. Then f is strongly z^* - $\mathcal{I}$ -continuous if and only if f is $\star$ -continuous.

Proof 31 By Theorem 19, every $\star$ -continuous function is strongly z^* - $\mathcal{I}$ -continuous. Conversely, let f be a strongly z^* - $\mathcal{I}$ -continuous. Since (Y, σ^*) is completely regular, every $\star$ -open set in Y is z^* - $\mathcal{I}$ open. Let $x \in X$ and let P be any $\star$ -open set containing $f(x)$. Then there exist a cz^* - $\mathcal{I}$ set, V such that $f(x) \in V \subseteq P$. Again since f is strongly z^* - $\mathcal{I}$ -continuous then there exist an open set U in X such that $f(U) \subseteq V$. Therefore f is $\star$ -continuous at x .

Theorem 29 If $f : (X, \tau) \rightarrow (Y, \sigma)$ be a continuous function and $g : (Y, \sigma) \rightarrow (Z, \eta, \mathcal{I})$ be strongly z^* - $\mathcal{I}$ -continuous, then $g \circ f$ is strongly z^* - $\mathcal{I}$ -continuous.

Proof 32 Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a continuous function and $g : (Y, \sigma) \rightarrow (Z, \eta, \mathcal{I})$ be a strongly z^* - $\mathcal{I}$ -continuous. Let V be any z^* - $\mathcal{I}$ open set in Z . **Claim** for any z^* - $\mathcal{I}$ open subset V in Z , $(g \circ f)^{-1}(V)$ is open set in X . Since g is z^* - $\mathcal{I}$ -continuous, then $g^{-1}(V)$ is open in Y . Again since f is continuous then $(g \circ f)^{-1}(V)$ is open in X . Therefore $g \circ f$ is strongly z^* - $\mathcal{I}$ -continuous.

Theorem 30 If there exists a bijective strongly z^* - $\mathcal{I}$ -continuous $f : (X, \tau) \rightarrow (Y, \sigma, \mathcal{I})$, where (Y, σ^*) is completely Hausdorff, then X is also completely Hausdorff.

Proof 33 Let x_1 and x_2 be a pair of distinct points and $f(x_1) \neq f(x_2)$. Since (Y, σ^*) is completely Hausdorff, then there exist disjoint z -open sets V_1 and V_2 in (Y, σ^*) such that $f(x_1) \in V_1$ and $f(x_2) \in V_2$. Since every z -open set is z^* - $\mathcal{I}$ -open in (Y, σ^*) and f is strongly z^* - $\mathcal{I}$ -continuous which implies $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are two disjoint cozero sets in X containing x_1 and x_2 . Therefore, X is completely Hausdorff.

Remark 5 Even if the topological space (Y, σ) is not completely Hausdorff but, we can assure that X is completely Hausdorff by considering a suitable $*$ -topology on Y which is completely Hausdorff.

Corollary 5 If for each pair of distinct points x_1 and x_2 in a topological space X , there exists a injective z^* - $\mathcal{I}$ -continuous f from $(X, \tau, \mathcal{J})$ onto $(Y, \sigma, \mathcal{I})$, where (Y, σ^*) is completely Hausdorff, then (X, τ^*) is also completely Hausdorff.

Proof 34 Let x_1 and x_2 be pair of distinct points and $f(x_1) \neq f(x_2)$. Since (Y, σ^*) is completely Hausdorff, there exist disjoint z^* - $\mathcal{I}$ open sets V_1 and V_2 such that $f(x_1) \in V_1$ and $f(x_2) \in V_2$. Since f is z^* - $\mathcal{I}$ -continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are two disjoint z^* - $\mathcal{J}$ -open sets in X containing x_1 and x_2 . Hence (X, τ^*) is completely Hausdorff.

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A.R. Prasannan

University of Delhi

Department of Mathematics

Maharaja Agrasen College, India

e-mail: arprasannan@gmail.com

Jayanta Biswas

University of Delhi

Department of Mathematics

New Delhi- 110 007

e-mail: mbiswas15@gmail.com