Notes on certain p -valent functions ¹

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Abstract

Let A_p be the class of functions $f(z)$ which are analytic in the open unit disk U . Applying some subordinations for functions, some interesting properties for $f(z) \in A_p$ are showed. Our results are generalizations for results given before.

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1 Introduction

Let A_p be the class of functions $f(z)$ of the form:

$$(1) \quad f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \quad (p \in \mathbb{N} = \{1, 2, 3, \dots\})$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

For functions $f(z)$ and $g(z)$ which are analytic in U , we say that $f(z)$ is subordinate to $g(z)$ if there exists a function $w(z)$ which is analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, and such that $f(z) = g(w(z))$. We write this subordination by

$$(2) \quad f(z) \prec g(z), \quad z \in U.$$

If $g(z)$ is analytic and univalent in U , then the subordination (2) is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$. For the subordinations, Miller and Mocanu [2] gave the following lemma.

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Lemma 1.1 *If a function $p(z)$ is analytic in U with $p(0) = 1$ and satisfies the following subordination*

$$(3) \quad zp'(z) \prec z, \quad z \in U,$$

then $p(z)$ satisfies

$$(4) \quad p(z) \prec 1 + z, \quad z \in U.$$

The function $1 + z$ is the best dominant for (3).

Applying the above lemma, Ozaki and Nunokawa [5] have shown

Theorem 1.1 *If a function $f(z) \in A_1$ satisfies*

$$(5) \quad \operatorname{Re} \left(\frac{f^2(z)}{z^2 f'(z)} \right) \geq \frac{1}{2}, \quad z \in U,$$

then $f(z)$ is univalent in U .

Remark 1.1 *It is easy to see that the inequality (5) is equivalent to*

$$(6) \quad \left| \frac{z^2 f'(z)}{f^2(z)} - 1 \right| \leq 1, \quad z \in U.$$

Furthermore, Nunokawa, Obradović and Owa [4] have given the following result.

Theorem 1.2 *If a function $f(z) \in A_1$ satisfies $\frac{f(z)}{z} \neq 0$, $0 < |z| < 1$ and*

$$(7) \quad \left| \left(\frac{z}{f(z)} \right)'' \right| \leq 1, \quad z \in U,$$

then $f(z)$ is univalent in U .

In this present paper, we try to consider some generalizations for Theorem 1.1 and Theorem 1.2.

2 Applications of subordinations

Our first application of subordinations for $f(z) \in A_p$ is the following.

Theorem 2.1 *If a function $f(z) \in A_p$ satisfies*

$$(8) \quad \left| \left(\frac{z^{p-j}}{f^{(j)}(z)} \right)'' \right| \leq \frac{(p-j)!(p-j)(1-\alpha)}{p!}, \quad z \in U$$

for some real α ($0 \leq \alpha < 1$) and $j = 0, 1, 2, \dots, p-1$, then

$$(9) \quad \left| \frac{p!z^{p+1-j}f^{(j+1)}(z)}{(p-j)!(p-j)(f^{(j)}(z))^2} - 1 \right| < 1 - \alpha, \quad z \in U,$$

and that

$$(10) \quad \operatorname{Re} \left(\frac{(f^{(j)}(z))^2}{z^{p+1-j}f^{(j+1)}(z)} \right) > \frac{p!}{(p-j)!(p-j)(2-\alpha)}, \quad z \in U.$$

Proof. For $f(z) \in A_p$ satisfying (8), we define a function $p(z)$ by

$$(11) \quad p(z) = \frac{\frac{p!z^{p+1-j}f^{(j+1)}(z)}{(p-j)!(p-j)(f^{(j)}(z))^2} - \alpha}{1 - \alpha}$$

$$= \frac{1}{1 - \alpha} \left\{ \frac{p!z^{p-j}}{(p-j)!f^{(j)}(z)} - z \left(\frac{p!z^{p-j}}{(p-j)!(p-j)f^{(j)}(z)} \right)' - \alpha \right\}.$$

Then $p(z)$ is analytic in U with $p(0) = 1$ and

$$(12) \quad zp'(z) = \frac{-z^2}{1 - \alpha} \left(\frac{p!z^{p-j}}{(p-j)!(p-j)f^{(j)}(z)} \right)''.$$

With our condition (8), we see that

$$(13) \quad |zp'(z)| = \left| \frac{-z^2}{1 - \alpha} \left(\frac{p!z^{p-j}}{(p-j)!(p-j)f^{(j)}(z)} \right)'' \right|$$

$$\leq |z|^2 < 1, \quad z \in U.$$

This give us that

$$(14) \quad zp'(z) = \frac{-z^2}{1 - \alpha} \left(\frac{p!z^{p-j}}{(p-j)!(p-j)f^{(j)}(z)} \right)'' \prec z, \quad z \in U.$$

Therefore, applying Lemma 1.1 for $p(z)$, we obtain that

$$(15) \quad p(z) = \frac{\frac{p!z^{p+1-j}f^{(j+1)}(z)}{(p-j)!(p-j)(f^{(j)}(z))^2} - \alpha}{1 - \alpha} \prec 1 + z, \quad z \in U.$$

It follows from (15) that

$$(16) \quad \frac{p!z^{p+j-1}f^{(j+1)}(z)}{(p-j)!(p-j)(f^{(j)}(z))^2} - \alpha \prec (1 - \alpha)(1 + z), \quad z \in U,$$

that is, that

$$(17) \quad \frac{p!z^{p+j-1}f^{(j+1)}(z)}{(p-j)!(p-j)(f^{(j)}(z))^2} - 1 \prec (1-\alpha)z, z \in U.$$

This subordination (17) gives us the inequality (9). Further, it is easy to see that the inequality (9) is equivalent to (10).

Making $j = 0$ in Theorem 2.1, we have

Corollary 2.1 *If $f(z) \in A_p$ satisfies*

$$(18) \quad \left| \left(\frac{z^p}{f(z)} \right)'' \right| \leq p(1-\alpha), z \in U$$

for some real α , $0 \leq \alpha < 1$, then

$$(19) \quad \left| \frac{z^{p+1}f'(z)}{pf^2(z)} - 1 \right| < 1 - \alpha, z \in U,$$

and that

$$(20) \quad \operatorname{Re} \left(\frac{f^2(z)}{z^{p+1}f'(z)} \right) > \frac{1}{p(2-\alpha)}, z \in U.$$

Further, letting $p = 1$ in Corollary 2.1, we see

Corollary 2.2 *If $f(z) \in A_1$ satisfies*

$$(21) \quad \left| \left(\frac{z}{f(z)} \right)'' \right| \leq 1 - \alpha, z \in U$$

for some real α , $0 \leq \alpha < 1$, then

$$(22) \quad \left| \frac{z^2f'(z)}{f^2(z)} - 1 \right| < 1 - \alpha, z \in U,$$

and $f(z)$ is univalent in U .

Remark 2.1 *If we consider $\alpha = 0$ in Corollary 2.2, then we have Theorem 1.2 by Nunokawa, Obradović and Owa [4].*

Open Problem With Corollary 2.1, we have the following open problem.

For $f(z) \in A_p$ satisfying (18), can we say that $f(z)$ is univalent in U ?

By Corollary 2.2, if $p = 1$ in Corollary 2.1, we have $f(z)$ is univalent in U .

Next, we introduce Lemma 2 due to Hallenbeck and Ruscheweyh [1] (or Miller and Mocanu [3]).

Lemma 2.1 *Let a function $h(z)$ be convex in U with $h(0) = 1$ and $p(z)$ be analytic in U with $p(0) = 1$. If $p(z)$ satisfies the subordination*

$$(23) \quad p(z) + zp'(z) \prec h(z), \quad z \in U,$$

then

$$(24) \quad p(z) \prec q(z) \prec h(z), \quad z \in U,$$

where

$$(25) \quad q(z) = \frac{1}{z} \int_0^z h(t) dt.$$

The function $q(z)$ is convex in U and the best dominant of (24).

Applying the above lemma, we derive

Theorem 2.2 *If a function $f(z) \in A_p$ satisfies*

$$(26) \quad \frac{p!}{(p-j)!(p-j)} \left\{ \frac{z^{p+1-j} f^{(j+1)}(z)}{(f^{(j)}(z))^2} - z^2 \left(\frac{z^{p-j}}{f^{(j)}(z)} \right)'' \right\} \prec \frac{1 + (1-2\alpha)z}{1-z}, \quad z \in U$$

for some real α ($0 \leq \alpha < 1$), then

$$(27) \quad \frac{z^{p+1-j} f^{(j+1)}(z)}{(f^{(j)}(z))^2} \prec q(z), \quad z \in U,$$

where

$$(28) \quad q(z) = -\frac{(p-j)!(p-j)}{p!} \left\{ \frac{2(1-\alpha)}{z} \log(1-z) + 1 - 2\alpha \right\}$$

and $q(z)$ is the best dominant.

Proof. Defining the function $p(z)$ by (11), we have $zp'(z)$ which is given by (12). Considering a function $h(z)$ given by

$$(29) \quad h(z) = \frac{1+z}{1-z}, \quad z \in U,$$

we know that $h(z)$ is analytic and convex in U .

We apply Lemma 2.1 for such $p(z)$ and $h(z)$. Then Lemma 2.1 gives us that

$$(30) \quad p(z) + zp'(z) \prec h(z), \quad z \in U$$

implies the subordination

$$(31) \quad p(z) \prec q(z) \prec h(z), \quad z \in U$$

with $q(z)$ given by (25).

We see that the subordination (30) is equivalent to (26). Further, it follows that

$$(32) \quad \frac{1}{z} \int_0^z \frac{1+t}{1-t} dt = \frac{1}{z} \int_0^z \left(\frac{2}{1-t} - 1 \right) dt = \frac{2}{z} \log \frac{1}{1-z} - 1.$$

Therefore, the subordination(31) becomes

$$(33) \quad p(z) \prec \frac{2}{z} \log \left(\frac{1}{1-z} \right) - 1 \prec \frac{1+z}{1-z}, \quad z \in U.$$

From (33), we have that

$$(34) \quad \frac{p! z^{p+1-j} f^{(j+1)}(z)}{(p-j)!(p-j) (f^{(j)}(z))^2} \prec \alpha + (1-\alpha) \left(\frac{2}{z} \log \left(\frac{1}{1-z} \right) - 1 \right), \quad z \in U,$$

that is, that

$$(35) \quad \frac{z^{p+1-j} f^{(j+1)}(z)}{(f^{(j)}(z))^2} \prec -\frac{(p-j)!(p-j)}{p!} \left\{ \frac{2(1-\alpha)}{z} \log \left(\frac{1}{1-z} \right) + 1 - 2\alpha \right\}, \quad z \in U.$$

This completes the proof of the theorem.

Letting $j = 0$ in Theorem 2.2, we have

Corollary 2.3 *If $f(z) \in A_p$ satisfies*

$$(36) \quad \frac{z^{p+1} f'(z)}{f^2(z)} - z^2 \left(\frac{z^p}{f(z)} \right)'' \prec p \frac{1 + (1-2\alpha)z}{1-z}, \quad z \in U$$

for some real α ($0 \leq \alpha < 1$), then

$$(37) \quad \frac{z^{p+1} f'(z)}{f^2(z)} \prec p \left(\frac{2(1-\alpha)}{z} \log \left(\frac{1}{1-z} \right) + 2\alpha - 1 \right), \quad z \in U.$$

Making $\alpha = \frac{1}{2}$ in Corollary 2.3, we obtain

Corollary 2.4 *If $f(z) \in A_p$ satisfies*

$$(38) \quad \frac{z^{p+1} f'(z)}{f^2(z)} - z^2 \left(\frac{z^p}{f(z)} \right)'' \prec \frac{p}{1-z}, \quad z \in U,$$

then

$$(39) \quad \frac{z^{p+1} f'(z)}{f^2(z)} \prec \frac{p}{z} \log \left(\frac{1}{1-z} \right), \quad z \in U.$$

Finally, putting $p = 1$ in Corollary 2.3, we see

Corollary 2.5 *If $f(z) \in A_1$ satisfies*

$$(40) \quad \frac{z^2 f'(z)}{f^2(z)} - z^2 \left(\frac{z}{f(z)} \right)'' \prec \frac{1 + (1 - 2\alpha)z}{1 - z}, \quad z \in U$$

for some real α ($0 \leq \alpha < 1$), then

$$(41) \quad \frac{z^2 f'(z)}{f^2(z)} \prec \frac{2(1 - \alpha)}{z} \log \left(\frac{1}{1 - z} \right) + 2\alpha - 1, \quad z \in U.$$

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