General Mathematics Vol. 27, No. 1 (2019), 29–36
DOI:10.2478/gm-2019-0003

Properties of the intermediate point from a mean value theorem of the integral calculus - II [1]

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Abstract

In this paper we consider two continuous functions $f, g : [a, b] \to \mathbb{R}$ and we study for these ones, under which circumstances the intermediate point function is four order differentiable at the point $x = a$ and we calculate its derivative.

2010 Mathematics Subject Classification: 26A24, 11H60.
Key words and phrases: intermediate point, mean-value theorem.

1 Introduction and Preliminaries

We start by considering the discussions related to the second mean value theorem of the integral calculus (or Bonnet theorem) and also, the idea from the paper [7] and some results from the paper [2].

Let us consider two continuous functions $f, g : [a, b] \to \mathbb{R}$ with the properties:

(a) the function f is decreasing on $[a, b]$,

(b) $f(x) \geq 0$, for all $x \in [a, b]$,

then, for each $x \in (a, b]$, there exists a point $c_x \in [a, x]$ such that

$$\int_a^x f(t)\, g(t)\, \mathrm{d}t = f(a) \int_a^{c_x} g(t)\, \mathrm{d}t. \tag{1}$$

If for each $x \in (a, b]$ we choose one $c_x \in [a, x]$ such that (1) holds, we can define the function $c : (a, b] \to [a, b]$ by

$$c(x) = c_x, \text{ for all } x \in (a, b], \tag{2}$$

with the property

$$\int_a^x f(t)\, g(t)\, \mathrm{d}t = f(a) \int_a^{c(x)} g(t)\, \mathrm{d}t, \text{ for all } x \in (a, b]. \tag{3}$$

Moreover, from ([7]) we have that

[1] *Received 27 March, 2019*
Accepted for publication (in revised form) 8 May, 2019

Theorem 1 *Let $a, b \in \mathbb{R}$ with $a < b$.If the continuous functions $f, g : [a, b] \to \mathbb{R}$ satisfies properties:*
(a) the function f is decreasing on $[a, b]$,
(b) $f(x) \geq 0$, for all $x \in [a, b]$,
then there exists a function $c : (a, b] \to [a, b]$ such that (3) holds.

When $x \in (a, b]$ tends to a, because $|c(x) - a| \leq |x - a|$, we have

$$\lim_{x \to a} c(x) = a.$$

Then the function $\overline{c} : [a, b] \to [a, b]$ defined by

$$\overline{c}(x) = \begin{cases} c(x), & \text{if } x \in (a, b] \\ a, & \text{if } x = a. \end{cases}$$

is continuous at $x = a$.

For $x \in (a, b]$, we have

$$\frac{\overline{c}(x) - \overline{c}(a)}{x - a} = \frac{c(x) - a}{x - a}.$$

If we denote by

$$\theta(x) = \frac{c(x) - a}{x - a},$$

then

$$\theta(x) \in (0, 1)$$

and

$$c(x) = a + (x - a)\theta(x)$$

and hence

$$\int_a^x f(t) g(t) \, \mathrm{d}t = f(a) \int_a^{a+(x-a)\theta(x)} g(t) \, \mathrm{d}t, \text{ for all } x \in (a, b]. \tag{4}$$

Hence, the following result is true.

Theorem 2 (*see [7]*) *Let $a, b \in \mathbb{R}$ with $a < b$.If the continuous functions $f, g : [a, b] \to \mathbb{R}$ satisfies properties:*
(a) the function f is decreasing on $[a, b]$,
(b) $f(x) \geq 0$, for all $x \in [a, b]$,
then there exists a function $\theta : (a, b] \to [0, 1]$ such that (4) holds.

If F be is a primitive of the function fg and G is a primitive of the function g, then, from the Leibniz Newton Theorem, the equality (4) becomes

$$F(x) - F(a) = f(a)[G(a + (x - a)\theta(x)) - G(a)], \text{ for all } x \in (a, b]. \tag{5}$$

In the paper [7], we had proved the following theorems.

Theorem 3 (*see [7]*) *Let $a, b \in \mathbb{R}$ with $a < b$ and let f, g be two continuous functions on $[a, b]$. If*

(*a*) *the function f is decreasing on $[a, b]$,*
(*b*) *$f(x) \geq 0$, for all $x \in [a, b]$,*
(*c*) *$f(a)g(a) \neq 0$,*

then

1° *The function $\theta : (a, b] \to [0, 1]$ has limit at the point $x = a$ and*

$$\lim_{x \to a} \theta(x) = 1.$$

2° *The function $\bar{c}$ is derivable at a and*

$$\bar{c}'(a) = 1.$$

Let $\overline{\theta} : [a, b] \to [0, 1]$ the function defined by

$$\overline{\theta}(x) = \begin{cases} \theta(x), & \text{if } x \in (a, b] \\ 1, & \text{if } x = a. \end{cases}$$

Theorem 4 (*see [7]*) *Let $a, b \in \mathbb{R}$ with $a < b$ and let f, g be two differentiable functions on $[a, b]$. If*

(*a*) *the function f is decreasing on $[a, b]$,*
(*b*) *$f(x) \geq 0$, for all $x \in [a, b]$,*
(*c*) *f' and g' are continuous at $x = a$,*
(*d*) *$f(a)g(a) \neq 0$,*

then

1° *The function $\overline{\theta}$ is differentiable at $x = a$ and*

$$\overline{\theta}'(a) = \frac{f'(a)}{2f(a)}.$$

2° *The function $\bar{c}$ is twice differentiable at $x = a$ and*

$$\bar{c}''(a) = \frac{f'(a)}{f(a)}.$$

Theorem 5 (*see [7]*) *Let $a, b \in \mathbb{R}$ with $a < b$ and let f, g be two twice differentiable functions on $[a, b]$. If*

(*a*) *the function f is decreasing on $[a, b]$,*
(*b*) *$f(x) \geq 0$, for all $x \in [a, b]$,*
(*c*) *f'' and g'' are continuous at $x = a$,*
(*d*) *$f(a)g(a) \neq 0$,*

then

1° *The function $\overline{\theta}$ is twice differentiable at $x = a$ and*

$$\overline{\theta}''(a) = \frac{f''(a)g(a) - f'(a)g'(a)}{3f(a)g(a)}.$$

2° *The function* $\overline{c}$ *is third differentiable at* $x = a$ *and*

$$\overline{c}'''(a) = \frac{f''(a)g(a) - f'(a)g'(a)}{f(a)g(a)}.$$

The purpose of this paper is to establish under which circumstances the function $\overline{c}$ is the four order differentiable at the point $x = a$ and to compute its derivative $\overline{c}^{(4)}(a)$. Do the derivative $\overline{c}^{(4)}(a)$ depend upon the functions f and g? If there exist several functions $\overline{c}$ which satisfy (3), do the derivative of the function $\overline{c}$ at $x = a$ depend upon the function $\overline{c}$ we choose?

These types of discussions, related to the differentiability of the functions and not only, to which the derivatives have been calculated, was also considered in papers like [1], [3], [4], [5], [6], [8].

2 Main results

Theorem 6 *Let* $a, b \in \mathbb{R}$ *with* $a < b$ *and let* f, g *be two four times differentiable functions on* $[a, b]$. *If*

(a) *the function* f *is decreasing on* $[a, b]$,

(b) $f(x) \geq 0$, *for all* $x \in [a, b]$,

(c) f''' *and* g''' *are continuous at* $x = a$,

(d) $f(a)g(a) \neq 0$,

then

1° *The function* $\overline{\theta}$ *is three times differentiable at* $x = a$ *and*

$$\overline{\theta}'''(a) =$$

$$= \frac{1}{(2f(a)g(a))^2}\left[4f(a)f'(a)(g'(a))^2 - 3(f'(a))^2 g(a)g'(a) - \right.$$

$$\left. -f(a)f''(a)g(a)g'(a) - 3f(a)f'(a)g(a)g''(a) + f(a)f'''(a)g^2(a)\right].$$

2° *The function* $\overline{c}$ *is four differentiable at* $x = a$ *and*

$$\overline{c}^{iv}(a) =$$

$$= \frac{1}{(f(a)g(a))^2}\left[4f(a)f'(a)(g'(a))^2 - 3(f'(a))^2 g(a)g'(a) - \right.$$

$$\left. -f(a)f''(a)g(a)g'(a) - 3f(a)f'(a)g(a)g''(a) + f(a)f'''(a)g^2(a)\right].$$

Proof. 1° By Taylor's theorem, for each $x \in (a, b]$, there are ξ_x, $\eta_x \in (a, x)$ such that

$$F(x) = F(a) + \frac{F'(a)}{1!}(x-a) + \frac{F''(a)}{2!}(x-a)^2 + \frac{F'''(a)}{3!}(x-a)^3 + \frac{F^{iv}(\xi_x)}{4!}(x-a)^4$$

$$G(x) = G(a) + \frac{G'(a)}{1!}(x-a) + \frac{G''(a)}{2!}(x-a)^2 + \frac{G'''(a)}{3!}(x-a)^3 + \frac{G^{iv}(\eta_x)}{4!}(x-a)^4$$

We replace these formulas in (5) and divide by $(x-a)$ get

$$\tfrac{F'(a)}{1!}+\tfrac{F''(a)}{2!}(x-a)+\tfrac{F'''(a)}{3!}(x-a)^2+\tfrac{F^{iv}(\xi_x)}{4!}(x-a)^3=$$
$$=f(a)\big[\tfrac{G'(a)}{1!}\theta(x)+\tfrac{G''(a)}{2!}(x-a)\theta^2(x)+\tfrac{G'''(a)}{3!}(x-a)^2\theta^3(x)+\tfrac{G^{iv}(\eta_x)}{4!}(x-a)^3\theta^4(x)\big].$$

For each $x\in[a,b]$, we have

$$F(x)=\int_a^x f(t)g(t)dt, F'(x)=f(x)\,g(x),$$
$$F''(x)=f'(x)\,g(x)+f(x)\,g'(x),$$
$$F'''(x)=f''(x)g(x)+2f'(x)g'(x)+f(x)g''(x),$$
$$F^{iv}(x)=f'''(x)g(x)+3f''(x)g'(x)+3f'(x)g''(x)+f(x)g'''(x),$$
$$G(x)=\int_a^x g(t)dt,\quad G'(x)=g(x),\quad G''(x)=g'(x),$$
$$G'''(x)=g''(x),\quad G^{iv}(x)=g'''(x),$$

so the last equality becomes

(6)
$$f(a)g(a)[\theta(x)-1]=$$
$$=\frac{1}{2!}\big[f'(a)g(a)+f(a)g'(a)-f(a)g'(a)\theta^2(x)\big](x-a)+$$
$$+\frac{1}{3!}\big[f''(a)g(a)+2f'(a)g'(a)+f(a)g''(a)-f(a)g''(a)\theta^3(x)\big](x-a)^2+$$
$$+\frac{1}{4!}\big[f'''(\xi_x)g(\xi_x)+3f''(\xi_x)g'(\xi_x)+3f'(\xi_x)g''(\xi_x)+$$
$$+f(\xi_x)g'''(\xi_x)-f(a)g'''(\eta_x)\theta^4(x)\big](x-a)^3.$$

In relation (6) we let $x\to a$ and obtain

$$\lim_{x\to a}\theta(x)=1.$$

When $x\in I\setminus\{a\}$, we divide relation (6) by $(x-a)$ and get

(7)
$$\frac{\theta(x)-1}{x-a}=$$
$$\frac{1}{2!f(a)g(a)}\big[f'(a)g(a)+f(a)g'(a)-f(a)g'(a)\theta^2(x)\big]+$$

$$+\frac{1}{3!f(a)g(a)}\left[f''(a)g(a)+2f'(a)g'(a)+f(a)g''(a)-f(a)g''(a)\theta^3(x)\right](x-a)+$$

$$+\frac{1}{4!f(a)g(a)}\left[f'''(\xi_x)g(\xi_x)+3f''(\xi_x)g'(\xi_x)+3f'(\xi_x)g''(\xi_x)+\right.$$

$$\left.+f(\xi_x)g'''(\xi_x)-f(a)g'''(\eta_x)\theta^4(x)\right](x-a)^2.$$

In relation (7) we let $x \to a$ and obtain

$$\overline{\theta}'(a)=\lim_{x\to a}\frac{\theta(x)-1}{x-a}=\frac{f'(a)}{2f(a)}.$$

Next, we rewrite relation (7) as,

$$\frac{\theta(x)-1}{x-a}-\overline{\theta}'(a)=$$

$$=-\frac{f'(a)}{2f(a)}+\frac{1}{2!f(a)g(a)}\left[f'(a)g(a)+f(a)g'(a)-f(a)g'(a)\theta^2(x)\right]+$$

$$+\frac{1}{3!f(a)g(a)}\left[f''(a)g(a)+2f'(a)g'(a)+f(a)g''(a)-f(a)g''(a)\theta^3(x)\right](x-a)+$$

$$+\frac{1}{4!f(a)g(a)}\left[f'''(\xi_x)g(\xi_x)+3f''(\xi_x)g'(\xi_x)+3f'(\xi_x)g''(\xi_x)+\right.$$

$$\left.+f(\xi_x)g'''(\xi_x)-f(a)g'''(\eta_x)\theta^4(x)\right](x-a)^2.$$

or, equivalently,

(8)
$$\frac{\frac{\theta(x)-1}{x-a}-\overline{\theta}'(a)}{x-a}=$$

$$=\frac{1}{2!f(a)g(a)}\frac{f(a)g'(a)-f(a)g'(a)\theta^2(x)}{x-a}+$$

$$+\frac{1}{3!f(a)g(a)}\left[f''(a)g(a)+2f'(a)g'(a)+f(a)g''(a)-f(a)g''(a)\theta^3(x)\right]+$$

$$+\frac{1}{4!f(a)g(a)}\left[f'''(\xi_x)g(\xi_x)+3f''(\xi_x)g'(\xi_x)+3f'(\xi_x)g''(\xi_x)+\right.$$

$$\left.+f(\xi_x)g'''(\xi_x)-f(a)g'''(\eta_x)\theta^4(x)\right](x-a)$$

From here result that

$$\overline{\theta}''(a)=\frac{f''(a)\,g(a)-f'(a)\,g'(a)}{3f(a)\,g(a)}.$$

From relation (8) we obtain

$$\frac{\frac{\theta(x)-1}{x-a}-\overline{\theta}'(a)}{x-a}-\frac{\overline{\theta}''(a)}{2!}=$$

$$=-\frac{f(a)g'(a)}{2!f(a)g(a)}\frac{\theta(x)-1}{x-a}\big[\theta(x)+1\big]-\frac{f''(a)g(a)-f'(a)g'(a)}{3!f(a)g(a)}+$$

$$+\frac{1}{3!f(a)g(a)}\big[f''(a)g(a)+2f'(a)g'(a)+f(a)g''(a)-f(a)g''(a)\theta^3(x)\big]+$$

$$+\frac{1}{4!f(a)g(a)}\big[f'''(\xi_x)g(\xi_x)+3f''(\xi_x)g'(\xi_x)+3f'(\xi_x)g''(\xi_x)+$$

$$+f(\xi_x)g'''(\xi_x)-f(a)g'''(\eta_x)\theta^4(x)\big](x-a).$$

Here we divide by $x-a$, where $x\neq a$, and we have

$$\frac{\frac{\frac{\theta(x)-1}{x-a}-\overline{\theta}'(a)}{x-a}-\frac{\overline{\theta}''(a)}{2!}}{x-a}=$$

$$=\frac{1}{3!f(a)g(a)}\frac{3f'(a)g'(a)-f(a)g''(a)\left[\theta^3(x)-1\right]-3f(a)g'(a)\frac{\theta^2(x)-1}{x-a}}{x-a}+$$

$$+\frac{1}{4!f(a)g(a)}\big[f'''(\xi_x)g(\xi_x)+3f''(\xi_x)g'(\xi_x)+3f'(\xi_x)g''(\xi_x)+$$

$$+f(\xi_x)g'''(\xi_x)-f(a)g'''(\eta_x)\theta^4(x)\big].$$

Now, we let $x\to a$, and get

$$\overline{\theta}'''(a)=$$

$$=\frac{1}{(2f(a)\,g(a))^2}\big[4f(a)\,f'(a)\,(g'(a))^2-3\,(f'(a))^2\,g(a)\,g'(a)-$$

$$-f(a)\,f''(a)\,g(a)\,g'(a)-3f(a)\,f'(a)\,g(a)\,g''(a)+f(a)\,f'''(a)\,g^2(a)\big].$$

2° Follows immediately from 1°.

3 Conclusions and further challenges

In this paper we introduced conditions for the functions f and g such that the intermediate point of functions $\overline{c}$ and $\overline{\theta}$ to be derivable in the point a and we provided the derivative $\overline{\theta}'''(a)$ and $\overline{c}^{iv}(a)$.

In future we want to see in which conditions the functions $\overline{c}$ and $\overline{\theta}$ are derivable of order n in the point a and to calculate the corresponding derivative.

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