Arens simultaneously factorization ¹

María José Aleandro

Abstract

Let X be a Banach bimodule over a Banach algebra $\mathcal{A}$. We shall seek conditions under which (1) X and its dual factorize on the same side or (2) X factors on the left (or right) if and only if its second conjugate space has a left (or right) unit.

2010 Mathematics Subject Classification: Primary 46L06, Secondary 46L07.

Key words and phrases: Banach algebra, Banach bimodule, mixed units, arens product.

1 Introduction

Let $\pi_l : \mathcal{A} \times X \rightarrow X$, $\pi_r : X \times \mathcal{A} \rightarrow X$ be the left and right module actions of a Banach algebra $\mathcal{A}$ into an $\mathcal{A}$ -Banach module X . As usual, $\pi_l(ab, x) = \pi_l(a, \pi_l(b, x))$, $\pi_r(x, ab) = \pi_r(\pi_r(x, a), b)$ and $\pi_l(a, \pi_r(x, b)) = \pi_r(\pi_l(a, x), b)$ for all $a, b \in \mathcal{A}$ and $x \in X$. The $\mathcal{A}$ bimodule X is called *symmetric* if $\pi_r = \pi_l^t$. We say that X factors on the left (or on the right) with respect to $\mathcal{A}$ if π_r (or π_l) is surjective, respectively. Conditions under which factorization properties hold is a matter of interest and an area of intensive research. A landmark in this issue is Cohen's factorization theorem [3]. It states that X factors on the left (or right) if $\mathcal{A}$ has a right (or left) bounded approximate identity that it is a right (or left) approximate identity for X itself, i.e., there exists a bounded net $\{e_i\}_{i \in I}$ within $\mathcal{A}$ so that $\lim_{i \in I} \|xe_i - x\| = 0$ (or $\lim_{i \in I} \|e_ix - x\| = 0$) if $x \in X$. The converse of Cohen's result is false [10, 9]. Our aim in this article is to investigate conditions under which X and its dual space X^* factorize on the same side. This problem was posed in [2], where the reader can consult about Arens adjoints of bounded bilinear forms between Banach spaces, factorization results and the notation used. In particular, given a bounded bilinear

¹Received 14 November, 2018

Accepted for publication (in revised form) 29 April, 2019

map between normed spaces over the same field $\beta : E \times F \rightarrow G$ we recall the basic operations of Arens adjointness $\beta^* : G^* \times E \rightarrow F^*$ and transposition $\beta^t : F \times E \rightarrow G$ that produce bounded bilinear maps so that $\langle f, \beta^*(g^*, e) \rangle = \langle \beta(e, f), g^* \rangle$ and $\beta^t(f, e) = \beta(e, f)$ if $(e, f, g^*) \in E \times F \times G^*$, respectively.

It is worth mentioning that the third Arens duals π_l^{***} and π_r^{***} induce the left and right actions of the Banach algebra $(\mathcal{A}^{**}, \square)$ on X^{**} , where $\square$ denotes the first Arens product. Analogously, π_l^{t***} and π_r^{t***} induce the left and right actions of the Banach algebra $(\mathcal{A}^{**}, \diamond)$ on X^{**} , where $\diamond$ denotes the second Arens product. We shall write

$$Z_{\square}(\mathcal{A}, X) = \{a^{**} \in \mathcal{A}^{**} : \pi_l^{***}(a^{**}, x^{**}) = \pi_r^{***}(x^{**}, a^{**}) \text{ if } x^{**} \in X^{**}\},$$

and

$$Z_{\diamond}(\mathcal{A}, X) = \{a^{**} \in \mathcal{A}^{**} : \pi_l^{t***}(a^{**}, x^{**}) = \pi_r^{t***}(x^{**}, a^{**}) \text{ if } x^{**} \in X^{**}\}.$$

We denote by $U_r(\mathcal{A}^{**}, \square)$ the right units of $\mathcal{A}^{**}$ with respect to the first Arens product and by $U_l(\mathcal{A}^{**}, \diamond)$ the left units of $\mathcal{A}^{**}$ with respect to the second Arens product. We shall also denote $U(\mathcal{A}^{**}, \square)$ and $U(\mathcal{A}^{**}, \diamond)$ to the set of units of $(\mathcal{A}^{**}, \square)$ and of $(\mathcal{A}^{**}, \diamond)$, respectively. Besides, $\mathfrak{M}(\mathcal{A}^{**})$ denotes the class of mixed units of $\mathcal{A}^{**}$.

In Th. 1 we shall give some simple conditions that allow simultaneous factorizations. We shall consider these conditions in some concrete situations, for instance in Ex. 1 (with $\mathcal{A} = X = c_0(\mathbb{N})$), in Ex. 2 (with $\mathcal{A} = X = K(H)$ and $\mathcal{A} = K(H)$, $X = N(H)$) and on Banach modules over some group algebras in Corollary 1. In Prop. 1 (resp. Prop. 2) we shall seek conditions that allow to deduce factorization of X (resp. of X^*) with respect to $\mathcal{A}$ assuming that X^* (resp. X) factors with respect to $\mathcal{A}$. In Th. 2 we will consider the case in which $\mathcal{A}^{**}$ has lateral units subject to some additional condition so that X factors on one side if and only if X^* factors on the same side. Finally, in Th. 3 we consider the same problem assuming the existence of mixed units.

2 On some simultaneous factorizations

Theorem 1 1. An element $a^{**} \in \mathcal{A}^{**}$ belongs to $Z_{\square}(\mathcal{A}, X)$ if and only if

$$\pi_r^{**}(a^{**}, x^*) = w - \lim_{\alpha \in A} \pi_l^*(x^*, a_{\alpha})$$

in X^* for all $x^* \in X^*$, where $\{a_{\alpha}\}_{\alpha \in A}$ is any bounded net in $\mathcal{A}$ so that $a^{**} = w^* - \lim_{\alpha \in A} a_{\alpha}$ in $\mathcal{A}^{**}$.

2. If $\mathcal{A}$ has a right bounded approximate identity and there is some $a^{**} \in Z_{\square}(\mathcal{A}, X)$ so that $\pi_r^{**}(a^{**}, x^*) = x^*$ for any $x^* \in X^*$ then X and X^* factors on the left with respect to $\mathcal{A}$.

3. An element $a^{**} \in \mathcal{A}^{**}$ belongs to $Z_{\diamond}(\mathcal{A}, X)$ if and only if

$$\pi_l^{t**}(a^{**}, x^*) = w - \lim_{\alpha \in A} \pi_r^{t*}(x^*, a_{\alpha})$$

in X^* for all $x^* \in X^*$, where $\{a_{\alpha}\}_{\alpha \in A}$ is any bounded net in $\mathcal{A}$ so that $a^{**} = w^* - \lim_{\alpha \in A} a_{\alpha}$ in $\mathcal{A}^{**}$.

4. If $\mathcal{A}$ has a left bounded approximate identity and there is an element $a^{**} \in Z_{\diamond}(\mathcal{A}, X)$ so that $\pi_l^{t**}(a^{**}, x^*) = x^*$ for any $x^* \in X^*$ then X and X^* factorize on the right with respect to $\mathcal{A}$.

Proof.

1. Let $a^{**} = w^* - \lim_{\alpha \in A} a_{\alpha}$ be given in $Z_{\square}(\mathcal{A}, X)$. For every $x^* \in X^*$ and $x^{**} \in X^{**}$ we have

$$\begin{aligned} \langle \pi_r^{**}(a^{**}, x^*), x^{**} \rangle &= \langle x^*, \pi_r^{***}(x^{**}, a^{**}) \rangle \\ &= \langle x^*, \pi_l^{***}(a^{**}, x^{**}) \rangle \\ &= \langle \pi_l^{**}(x^{**}, x^*), a^{**} \rangle \\ &= \lim_{\alpha \in A} \langle a_{\alpha}, \pi_l^{**}(x^{**}, x^*) \rangle \\ &= \lim_{\alpha \in A} \langle \pi_l^{*}(x^*, a_{\alpha}), x^{**} \rangle \end{aligned}$$

and the condition is necessary. Now, if the condition holds we see that

$$\begin{aligned} \langle x^*, \pi_r^{***}(x^{**}, a^{**}) \rangle &= \langle \pi_r^{**}(a^{**}, x^*), x^{**} \rangle \\ &= \lim_{\alpha \in A} \langle \pi_l^{*}(x^*, a_{\alpha}), x^{**} \rangle \\ &= \lim_{\alpha \in A} \langle a_{\alpha}, \pi_l^{**}(x^{**}, x^*) \rangle \\ &= \langle \pi_l^{**}(x^{**}, x^*), a^{**} \rangle \\ &= \langle x^*, \pi_l^{***}(a^{**}, x^{**}) \rangle. \end{aligned}$$

2. Let $a^{**} \in Z_{\square}(\mathcal{A}, X)$ so that $\pi_r^{**}(a^{**}, x^*) = x^*$ if $x^* \in X^*$. For every $x \in X$ and $x^* \in X^*$ then

$$\begin{aligned} \langle x, x^* \rangle &= \langle x, \pi_r^{**}(a^{**}, x^*) \rangle \\ &= \langle \pi_r^{*}(x^*, x), a^{**} \rangle \\ &= \lim_{\alpha \in A} \langle a_{\alpha}, \pi_r^{*}(x^*, x) \rangle \\ &= \lim_{\alpha \in A} \langle \pi_r(x, a_{\alpha}), x^* \rangle. \end{aligned}$$

Thus $x = w - \lim_{\alpha \in A} \pi_r(x, a_{\alpha})$ and the net $\{a_{\alpha}\}_{\alpha \in A}$ becomes a weak right bounded approximate identity of X within $\mathcal{A}$. Hence $X\mathcal{A}$ becomes weakly dense in X . Further, the linear span of $X\mathcal{A}$ becomes strongly closed (cf. [6],

V.3.7, p. 420) and by Cohen's factorization theorem X factors on the left with respect to $\mathcal{A}$. Moreover, π_l^* becomes a right action of $\mathcal{A}$ on X^* and as $a^{**} \in Z_\square(\mathcal{A}, X)$ then $X^*\mathcal{A}$ is also weakly dense in X^* . Finally we argue as before to infer that X^* factors on the left with respect to $\mathcal{A}$.

3. It follows as 1.

4. It follows as 2.

Let us consider some concrete applications of Th. 1.

Example 1 Let $\mathcal{A} = c_0(\mathbb{N})$ with pointwise product and $X = \mathcal{A}$. If for $n \in \mathbb{N}$, δ_n is the usual Kronecker delta, i.e., $\delta_n(m) = 1$ if $m = n$ and 0 otherwise, then $\{\delta_1 + \dots + \delta_n\}_{n \in \mathbb{N}}$ becomes a bounded approximate identity for $\mathcal{A}$.

Given $F \in l^\infty(\mathbb{N})$ and a function $\tau : \mathbb{N} \rightarrow \mathbb{N}$ it is a well defined element $F_\tau = \{F(\tau(n))\}_{n \in \mathbb{N}}$ in $l^\infty(\mathbb{N})$. Indeed, if τ is bijective $l^1(\mathbb{N})$ and $\mathcal{A}$ are invariant under the linear bounded map $\Psi_\tau : F \mapsto F_\tau$. Let be given bijective functions $\lambda, \varrho : \mathbb{N} \rightarrow \mathbb{N}$. We write $\pi_l(a, x) = a_\lambda x$ and $\pi_r(x, a) = a_\varrho x$ for $a \in \mathcal{A}$ and $x \in X$. It is readily seen that π_l and π_r are well defined actions of $\mathcal{A}$ on X . As $X^* \approx l^1(\mathbb{N})$ it is straightforward to see that

$$(1) \quad \pi_r^{**}(a^{**}, x^*) = \sum_{n=1}^{\infty} a_{\varrho(n)}^{**} x_n^* e^n$$

in X^* , where $a^{**} \in \mathcal{A}^{**}$, $x^* \in X^*$ and for each $n \in \mathbb{N}$ we write $e^n = \{\delta_n\}_{n \in \mathbb{N}}$ in $l^1(\mathbb{N})$. Besides $X^{**} \approx l^\infty(\mathbb{N})$. So, for every $a^{**} \in \mathcal{A}^{**}$ and $x^{**} \in X^{**}$ we obtain

$$\pi_l^{***}(a^{**}, x^{**}) = \Psi_\lambda^{**}(a^{**}) \square x^{**}$$

and

$$\pi_r^{***}(x^{**}, a^{**}) = x^{**} \square \Psi_\varrho^{**}(a^{**}).$$

But $\mathcal{A}$ is Arens regular and its second dual space fully identifies with $l^\infty(\mathbb{N})$ endowed with pointwise multiplication. So, if $\lambda = \varrho$ (or $\pi_l = \pi_r^t$) then $\mathcal{A}^{**} = Z_\square(\mathcal{A}, X)$. Finally, by (1) we see that, for instance, $\pi_r^{**}(\mathbf{1}, u^*) = u^*$ for any $u \in X^*$, where $\mathbf{1} = w^* - \lim_{n \rightarrow \infty} (\delta_1 + \dots + \delta_n)$.

Example 2 If H is a complex separable Hilbert space let $B(H)$, $K(H)$ and $N(H)$ be the classes of bounded linear, compact and nuclear operators on H respectively. As it is well known, $N(H)^* \approx B(H)$ and $K(H)$ is a two-sided ideal of $B(H)$. So, if we consider $N(H)$ as a $K(H)$ -Banach bimodule its conjugate space do not factors with respect to $K(H)$ unless H be finite dimensional.

If $\mathcal{A} = K(H)$ let $\{e_n\}_{n \in \mathbb{N}}$ be a fixed orthonormal basis and $E_n = \sum_{j=1}^n e_j \odot e_j$ if $n \in \mathbb{N}$, where for $x, y, z \in H$ we write $(x \odot y)(z) = \langle z, y \rangle x$. It is easily seen that $\{E_n\}_{n \in \mathbb{N}}$ is a left bounded approximate identity for $\mathcal{A}$. The algebra $\mathcal{A}$ is Arens regular [13]. In this case, $Z_\square(K(H), K(H))$ and $Z_\diamond(K(H), K(H))$ coincide with the center

$Z(\mathcal{A}^{**})$ of $\mathcal{A}^{**}$ and $Z(\mathcal{A}^{**}) \approx \mathbb{C}Id_H$ as $\mathcal{A}^{**} \approx B(H)$ [5]. Besides $\mathcal{A}^* \approx N(H)$. As $N(H)$ is a left ideal of $K(H)$ by Cohen's factorization theorem we see that $K(H) = K(H)K(H)$ and $N(H) = K(H)N(H)$. Moreover, let $E = w^* - \lim_{n \rightarrow \infty} E_n$ in $\mathcal{A}^{**}$. Then $E \in Z_{\diamond}(K(H), K(H))$ and $\pi_l^{t**}(E, \eta) = \eta$ if $\eta \in N(H)$. For, let $\eta \in N(H)$ and $k^{**} \in B(H)$, say $k^{**} = w^* - \lim_{s \in S} k_s$ where $\{k_s\}_{s \in S}$ is some bounded net of $K(H)$. Then

$$(2) \quad \langle \eta, E \diamond k^{**} \rangle = \lim_{s \in S} \lim_{n \rightarrow \infty} \langle E_n k_s, \eta \rangle = \lim_{s \in S} \langle k_s, \eta \rangle = \langle \eta, k^{**} \rangle.$$

If $tr[\circ]$ denotes the usual trace functional there is an isometric isomorphism $\bar{\eta} \rightarrow tr[\bar{\eta} \circ]$ from $N(H)$ onto $K(H)^*$ [11]. Thus we also have

$$\begin{aligned} \langle \eta, k^{**} \diamond E \rangle &= \langle \eta, k^{**} \square E \rangle \\ &= \lim_{s \in S} \lim_{n \rightarrow \infty} \langle k_s E_n, \eta \rangle \\ &= \lim_{s \in S} \lim_{n \rightarrow \infty} tr[\eta k_s E_n] \\ &= \lim_{s \in S} \lim_{n \rightarrow \infty} tr[E_n \eta k_s] \\ &= \lim_{s \in S} tr[\eta k_s] \\ &= \lim_{s \in S} \langle k_s, \eta \rangle \\ &= \langle \eta, k^{**} \rangle. \end{aligned} \quad (3)$$

By (2) and (3) is $E \diamond k^{**} = k^{**} \diamond E = k^{**}$ and so $E \in Z_{\diamond}(K(H), K(H))$. Moreover, if $k \in K(H)$ and $\eta \in N(H)$ we see that

$$\begin{aligned} \langle k, \pi_l^{t**}(E, \eta) \rangle &= \langle \pi_l^{t*}(\eta, k), E \rangle \\ &= \lim_{n \rightarrow \infty} \langle E_n, \pi_l^{t*}(\eta, k) \rangle \\ &= \lim_{n \rightarrow \infty} \langle \pi_l^t(k, E_n), \eta \rangle \\ &= \lim_{n \rightarrow \infty} \langle \pi_l(E_n, k), \eta \rangle \\ &= \langle k, \eta \rangle. \end{aligned}$$

By Th. 1 4. we infer that $N(H) = K(H)N(H)$.

Proposition 1 *Let $\mathcal{A}$ be a Banach algebra with a right bounded approximate identity. If X is symmetric, X^* factors on the left with respect to $\mathcal{A}$ and the right action of $\mathcal{A}$ on X^* is faithful then X factors on the left with respect to $\mathcal{A}$.*

Proof. If X do not factors on the left with respect to $\mathcal{A}$ by Cohen's factorization theorem the linear span of $X\mathcal{A}$, that is necessarily closed, must be a proper subspace of X . By the Hahn-Banach theorem there exists $\lambda \in X^* - \{0_{X^*}\}$ with null restriction

on $[X\mathcal{A}]$. Let $x_0^* \in X^*$ and $a_0 \in \mathcal{A}$ so that $\lambda = \pi_l^*(x_0^*, a_0)$. Since X is symmetric, if $a \in \mathcal{A}$ and $x \in X$ we have

$$0 = \langle \pi_l(a, x), \lambda \rangle = \langle \pi_l(a_0, \pi_l(a, x)), x_0^* \rangle = \langle \pi_l(a_0 a, x), x_0^* \rangle = \langle x, \pi_l^*(x_0^*, a_0 a) \rangle,$$

i.e.,

$$\pi_l^*(x_0^*, a_0 a) = \pi_l^*(\lambda, a) = 0_{X^*}$$

which is not possible if $\lambda \neq 0_{X^*}$ and $\mathcal{A}$ acts faithfully on the right on X^* .

Proposition 2 *Let $\mathcal{A}$ be a Banach algebra endowed with a bounded approximate identity.*

1. *Suppose that X factors on the left with respect to $\mathcal{A}$. If there is a bounded weak right approximate identity $\{e_\alpha\}_{\alpha \in A}$ for X in $\mathcal{A}$ that define an element $e^{**} = w^* - \lim_{\alpha \in A} e_\alpha$ in $Z_\square(\mathcal{A}, X)$ then X^* also factors on the left with respect to $\mathcal{A}$.*
2. *Suppose that X factors on the right with respect to $\mathcal{A}$. If there is a bounded weak left approximate identity $\{f_\beta\}_{\beta \in B}$ for X in $\mathcal{A}$ that define an element $f^{**} = w^* - \lim_{\beta \in B} f_\beta$ in $Z_\diamond(\mathcal{A}, X)$ then X^* also factors on the right with respect to $\mathcal{A}$.*

Proof.

1. Since $\mathcal{A}$ has a bounded approximate identity and X factors on the left (resp. right) with respect to $\mathcal{A}$ there is a weak right (resp. left) approximate identity for X in $\mathcal{A}$ (cf. [2], Th. 3.3 (i) and (ii)). Now, in the present conditions let us assume that X^* do not factors on the left with respect to $\mathcal{A}$. By Cohen's theorem the linear span of $X^*\mathcal{A}$ is closed. By the Hahn-Banach theorem let $x_0^{**} \in X^{**} - \{0_{X^{**}}\}$ that annihilates on $X^*\mathcal{A}$. For every $x \in X$ and $x^* \in X^*$ we have

$$\begin{aligned} \langle x, \pi_r^{**}(e^{**}, x^*) \rangle &= \langle \pi_r^*(x^*, x), e^{**} \rangle \\ &= \lim_{\alpha \in A} \langle e_\alpha, \pi_r^*(x^*, x) \rangle \\ &= \lim_{\alpha \in A} \langle \pi_r(x, e_\alpha), x^* \rangle \\ &= \langle x, x^* \rangle, \end{aligned}$$

i.e., $\pi_r^{**}(e^{**}, x^*) = x^*$. Moreover,

$$\langle x^*, \pi_l^{***}(e^{**}, x_0^{**}) \rangle = \langle \pi_l^{**}(x_0^{**}, x^*), e^{**} \rangle = \lim_{\alpha \in A} \langle \pi_l^*(x^*, e_\alpha), x_0^{**} \rangle = 0$$

i.e., $\pi_l^{***}(e^{**}, x_0^{**}) = 0_{X^{**}}$. So, $\pi_r^{***}(x_0^{**}, e^{**}) = 0_{X^{**}}$ as $e^{**} \in Z_\square$. Then

$$\langle x^*, x_0^{**} \rangle = \langle \pi_r^{**}(e^{**}, x^*), x_0^{**} \rangle = \langle x^*, \pi_r^{***}(x_0^{**}, e^{**}) \rangle = 0$$

for every $x^* \in X^*$, which is not possible because $x_0^{**} \neq 0_{X^{**}}$.

2. Analogously, if X^* do not factors on the right with respect to $\mathcal{A}$ we choose $x_1^{**} \in X^{**} - \{0_{X^{**}}\}$ that annihilates on $\mathcal{A}X^*$. If $x^* \in X^*$ it is seen that $\pi_l^{t**}(f^{**}, x^*) = x^*$ and $\pi_r^{t***}(f^{**}, x_1^{**}) = 0_{X^{**}}$. Hence

$$0_{X^{**}} = \pi_r^{t***}(x_1^{**}, f^{**}) = \pi_l^{t***}(f^{**}, x_1^{**}).$$

Therefore

$$\langle x^*, x_1^{**} \rangle = \langle \pi_l^{t**}(f^{**}, x^*), x_1^{**} \rangle = \langle x^*, \pi_l^{t***}(f^{**}, x_1^{**}) \rangle = 0$$

for every $x^* \in X^*$, which is not possible.

Remark 1 If e^{**} is a right unit in $(\mathcal{A}^{**}, \square)$ it is known that X factors on the left with respect to $\mathcal{A}$ if and only if e^{**} is a right unit for X^{**} (cf. [2], Th. 3.4). Analogously, if $e^{**} \in U_l(\mathcal{A}^{**}, \diamond)$ then X factors on the right with respect to $\mathcal{A}$ if and only if e^{**} is a left unit for X^{**} .

Theorem 2 Let X be a Banach bimodule over a Banach algebra $\mathcal{A}$.

1. If $U_r(\mathcal{A}^{**}, \square) \cap Z_{\square}(\mathcal{A}, X) \neq \emptyset$ the following assertions are equivalent:
 - (a) X factors on the left with respect to $\mathcal{A}$.
 - (b) X^* factors on the left with respect to $\mathcal{A}$.
 - (c) X^{**} has a left unit.
2. If $U_l(\mathcal{A}^{**}, \diamond) \cap Z_{\diamond}(\mathcal{A}, X) \neq \emptyset$ the following assertions are equivalent:
 - (a) X factors on the right with respect to $\mathcal{A}$.
 - (b) X^* factors on the right with respect to $\mathcal{A}$.
 - (c) X^{**} has a right unit.

Proof.

1. (a) $\Rightarrow$ (b) As $(\mathcal{A}^{**}, \square)$ has a right unit e^{**} there is a right bounded approximate identity in $\mathcal{A}$ (cf. [4], Prop. 2.9.16(ii)). Hence, if X^* do not factors on the left with respect to $\mathcal{A}$ by Cohen's theorem the linear span of $X^*\mathcal{A}$ is a proper closed subspace of X^* . As in Prop. 2 1. we choose $x_0^{**} \in X^{**} - \{0_{X^{**}}\}$ that annihilates on $X^*\mathcal{A}$. If besides $e^{**} \in Z_{\square}(\mathcal{A}, X)$ then $x_0^{**} = \pi_l^{***}(e^{**}, x_0^{**})$. Further, for every $a \in \mathcal{A}$ and $x^* \in X^*$ we see that

$$\langle \pi_l^*(x^*, a), x_0^{**} \rangle = \langle a, \pi_l^{**}(x_0^{**}, x^*) \rangle = 0$$

i.e., $\pi_l^{**}(x_0^{**}, x^*) = 0_{\mathcal{A}^*}$. Hence

$$0 = \langle \pi_l^{**}(x_0^{**}, x^*), e^{**} \rangle = \langle x^*, \pi_l^{***}(e^{**}, x_0^{**}) \rangle = \langle x^*, x_0^{**} \rangle,$$

which is impossible.

(b) $\Rightarrow$ (a) Assume that X^* factors on the left with respect to $\mathcal{A}$ and let $e^{**} \in U_r(\mathcal{A}^{**}, \square) \cap Z_\square(\mathcal{A}, X)$. Now, if $x^{**} \in X^{**}$ and $x^{***} \in X^{***}$ then $\pi_l^{***}(x^{***}, e^{**}) = x^{***}$ and so

$$\langle x^{**}, x^{***} \rangle = \langle \pi_l^{***}(e^{**}, x^{**}), x^{***} \rangle = \langle \pi_r^{***}(x^{**}, e^{**}), x^{***} \rangle,$$

i.e., $\pi_r^{***}(x^{**}, e^{**}) = x^{**}$ and we can conclude that X factors on the left with respect to $\mathcal{A}$.

(a) $\Rightarrow$ (c) It suffices to observe that by Remark 1 any element of

$$U_r(\mathcal{A}^{**}, \square) \cap Z_\square(\mathcal{A}, X) \neq \emptyset$$

is a left unit for X^{**} .

(c) $\Rightarrow$ (a) If X^{**} has a left unit then it has a weak right approximate identity and the linear span of $X^*\mathcal{A}$ becomes weakly dense within X^* . Since $\mathcal{A}$ has a right bounded approximate identity the conclusion follows by ([6], V.3.7) and Cohen's factorization theorem.

2. (a) $\Leftrightarrow$ (b) Let $e^{**} \in U_l(\mathcal{A}^{**}, \diamond) \cap Z_\diamond(\mathcal{A}, X) \neq \emptyset$ and suppose that X factors on the right with respect to $\mathcal{A}$. In order to prove that X^* also factors on the right by Remark 1 it will suffice to see that e^{**} is a left unit for X^{***} . Thus, as the left action of $\mathcal{A}$ in X^* is provided by the bilinear functional π_r^{t*} it will suffice to see that $(\pi_r^{t*})^{t***}(e^{**}, x^{***}) = x^{***}$ for any $x^{***} \in X^{***}$. For, as $e^{**} \in Z_\diamond(\mathcal{A}, X)$ it is a left unit for X^{**} for every $x^{**} \in X^{**}$, $x^{***} \in X^{***}$ we obtain

$$\begin{aligned} \langle x^{**}, x^{***} \rangle &= \langle \pi_r^{t***}(x^{**}, e^{**}), x^{***} \rangle \\ &= \langle \pi_r^{t***}(e^{**}, x^{**}), x^{***} \rangle \\ &= \langle x^{**}, \pi_r^{t***t}(e^{**}, x^{***}) \rangle \end{aligned}$$

and the claim holds. The converse assertion follows analogously.

(a) $\Leftrightarrow$ (c) It follows as 1.

Remark 2 Following [2] if $e^{**} \in \mathcal{A}^{**}$ let $Z_{e^{**}}(\pi_r^t)$ be the set of elements $x^{**} \in X^{**}$ so that $\pi_r^{***}(x^{**}, e^{**}) = \pi_r^{t***}(x^{**}, e^{**})$. If e^{**} is a mixed unit X^* factors on the left with respect to $\mathcal{A}$ if and only if e^{**} is a left unit for X^{**} . Further, if $Z_{e^{**}}(\pi_r^t) = X^{**}$, X^* factors on the right with respect to $\mathcal{A}$ if and only if e^{**} is a right unit for X^{**} (cf. [2], Th. 3.6).

Theorem 3 Let X be a Banach bimodule over a Banach algebra $\mathcal{A}$.

1. If the class $\mathfrak{M}(\mathcal{A}^{**})$ of mixed units of $\mathcal{A}^{**}$ intersects $Z_\square(\mathcal{A}, X)$ in a non empty set, X factors on the left with respect to $\mathcal{A}$ if and only if X^* factors to the left with respect to $\mathcal{A}$.

2. If X is symmetric and $Z_\diamond(\mathcal{A}, X)$ contains some mixed unit X factors on the right with respect to $\mathcal{A}$ if and only if X^* factors on the right with respect to $\mathcal{A}$.

Proof.

1. Let $e^{**} \in \mathfrak{M}(\mathcal{A}^{**}) \cap Z_\square(\mathcal{A}, X)$. If X factors on the left with respect to $\mathcal{A}$ we know that e^{**} becomes a right unit for X^{**} . Since $e^{**} \in Z_\square(\mathcal{A}, X)$ then e^{**} becomes a left unit for X^{**} . Thus X^* factors on the left with respect to $\mathcal{A}$. Conversely, if X^* factors on the left with respect to $\mathcal{A}$ then e^{**} is a left unit for X^{**} . As $e^{**} \in Z_\square(\mathcal{A}, X)$ it is a right unit for X^{**} and the claim follows.
2. Assume that X is symmetric and let $e^{**} \in \mathfrak{M}(\mathcal{A}^{**}) \cap Z_\diamond(\mathcal{A}, X)$. So $\pi_r = \pi_l^t$ and for every $x^{**} \in X^{**}$ we have

$$\pi_r^{t***}(x^{**}, e^{**}) = \pi_l^{t***}(e^{**}, x^{**}) = \pi_r^{***}(x^{**}, e^{**})$$

and $x^{**} \in Z_{e^{**}}(\pi_r^t)$, i.e., $Z_{e^{**}}(\pi_r^t) = X^{**}$. Hence, if X factors on the right with respect to $\mathcal{A}$ by Remark 1 we see that

$$\pi_r^{***}(x^{**}, e^{**}) = \pi_l^{t***}(e^{**}, x^{**}) = x^{**}$$

and X^* factors on the right by Remark 2. The converse follows similarly.

Corollary 1 *Let G be a non discrete locally compact group with left invariant Haar measure λ .*

1. $\mathfrak{M}(L^1(G)^{**}) \cap [Z_\square(L^1(G), L^1(G)) \cup Z_\diamond(L^1(G), L^1(G))] = \emptyset$. In particular, $L^\infty(G)$ do not factors to any side respect to $L^1(G)$.
2. $\mathfrak{M}(L^1(G)^{**}) \cap Z_\square(L^1(G), C_0(G)) = \emptyset$.
3. If G is abelian, $\mathfrak{M}(L^1(G)^{**}) \cap Z_\diamond(L^1(G), C_0(G)) = \emptyset$.

Proof.

1. The Banach uniform-norm algebra $U(G)$ of uniformly continuous functions on G becomes an $M(G)$ -Banach bimodule if, for $\mu \in M(G)$, $x \in U(G)$ and $a \in G$ we set $\Pi_l(\mu, x)(a) = \langle l_a x, \mu \rangle$ and $\Pi_r(x, \mu)(a) = \langle r_a x, \mu \rangle$, where $l_a x(b) = x(ab)$ and $r_a x(b) = x(ba)$ if $b \in G$. Consequently, $C_0(G)$ inherits a Banach bimodule structure on $L^1(G)$. Further, $L^1(G)$ has a bounded approximate identity $\{x_i\}_{i \in I}$ of $\|\cdot\|_1$ -unitary elements of $C_{00}(G)^+$ that is an approximate identity for $C_0(G)$ (cf. [4], Th. 3.3.23).

Now, with respect to the usual duality for every $f \in L^1(G)$ and $F \in L^\infty(G)$ we see that $Ff \in L^1(G)^*$ and for almost every $a \in G$ we have

$$(Ff)(a) = \Delta_G(a^{-1}) \int_G f(ba^{-1})F(b)d\lambda(b).$$

By the continuity of the modular function Δ_G of G , the continuity of the inversion of G and the continuity of the map $c \rightarrow r_cf$ of G into $L^1(G)$ it is easy to see that $Ff \in C_b(G)$ (see [7], Th. 15.11 and Th. 20.4). Therefore $L^\infty(G)$ do not factors on the left (and analogously to the right) with respect to $L^1(G)$. Since by Cohen's theorem $L^1(G)L^1(G) = L^1(G)$, 1. follows by Th. 3.

2. Likewise, let $\pi_l : L^1(G) \times C_0(G) \rightarrow C_0(G)$ be the left action of $L^1(G)$ on $C_0(G)$ induced by Π_l . Then $\pi_l^* : M(G) \times L^1(G) \rightarrow M(G)$ and for every $x \in C_0(G)$ and $f \in L^1(G)$ for almost every $a \in G$ we have

$$\pi_l(f, x)(a) = \Pi_l(f d\lambda, x)(a) = \int_G x(ab)f(b)d\lambda(b) = \int_G x(c)f(a^{-1}c)d\lambda(c).$$

Hence if $\mu \in M(G)$ we see that

$$\begin{aligned} \langle x, \pi_l^*(\mu, f) \rangle &= \langle \pi_l(f, x), \mu \rangle \\ &= \int_G \int_G x(c)f(a^{-1}c)d\lambda(c)d\mu(a) \\ &= \int_G \int_G f(a^{-1}c)d\mu(a)x(c)d\lambda(c). \end{aligned}$$

Thus $\pi_l^*(\mu, f)(c) = (\mu * f)(c)d\lambda(c)$ for almost every $c \in G$. Then $M(G)$ do not factors on the left with respect to $L^1(G)$ as the closed ideal of absolutely continuous measures of G is proper in $M(G)$. Since $C_0(G)$ factors with respect to $L^1(G)$, 2. follows by Th. 3.

3. If G is abelian the action of $L^1(G)$ on $C_0(G)$ becomes symmetric and the conclusion follows analogously by Th. 3.

Remark 3 For an alternative proof of Corollary 1 1. we can argue as follows: let $e^{**} \in \mathfrak{M}(L^1(G)^{**})$. If besides $e^{**} \in Z_\square(L^1(G), L^1(G))$ it is seen that $e^{**} \in U(L^1(G)^{**}, \square)$. As $e^{**} \in U_l(L^1(G)^{**}, \diamond)$ then e^{**} belongs to the left topological center $Z_l(L^1(G)^{**})$ of $L^1(G)^{**}$. But $Z_l(L^1(G)^{**}) = L^1(G)$ (cf. [8], Corollary 5.5). Hence $e^{**} \in U(L^1(G))$ while $L^1(G)$ has no units (cf. [12], Th. 1.10). Thus $\mathfrak{M}(L^1(G)^{**}) \cap Z_\square(L^1(G), L^1(G)) = \emptyset$. Similarly, $\mathfrak{M}(L^1(G)^{**}) \cap Z_\diamond(L^1(G), L^1(G)) = \emptyset$.

References

- [1] S. Ishikawa, *Fixed point and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc., vol. 73, no. 2, 1976, 65-71.
- [2] K. Haghnejad Azar, *Arens regularity of bilinear forms and unital Banach module spaces*, Bull. Iranian Math. Soc., vol. 40, no. 2, 2014, 505-520.
- [3] P. Cohen, *Factorization in group algebras*, Duke Math. J., vol. 26, no. 2, 1959, 199-205.

- [4] H. Dales, *Banach algebras and automatic continuity*, Oxford University Press, 2000.
- [5] J. Dixmier, *Les fonctionnelles linéaires sur l'ensemble des opérateurs bornées d'un espace de Hilbert*, Ann. of Math., vol. 51, no. 2, 1950, 387-408.
- [6] N. Dunford, J. Schwartz, *Linear operators I. General Theory*, Interscience Publishers, 1958.
- [7] E. Hewitt, K. Ross, *Abstract harmonic analysis*, Die Grundlehren der mathematischen Wissenschaften, 1963.
- [8] A. Lau, A. Ülger, *Topological centers of certain Banach algebras*, Trans. Amer. Math. Soc., vol. 348, no. 3, 1996, 1191-1212.
- [9] M. Leinert, *A commutative Banach algebra which factorizes but has no approximate units*, Proc. Amer. Math. Soc., vol. 55, no. 2, 1976, 245-246.
- [10] W. Paschke, *A factorable Banach algebra without bounded approximate unit*, Pacific J. Math., vol. 46, no. 1, 1973, 249-251.
- [11] R. Schatten, *A theory of cross-spaces*, Princeton University Press, 1950.
- [12] I. Segal, *The group algebra of a locally compact group*, Trans. Amer. Math. Soc., vol. 61, no. 1, 1947, 69-105.
- [13] N. Young, *Periodicity of functional and representations of normed algebras on reflexive spaces*, Proc. Edinburgh Math. Soc., vol. 20, no. 2, 1976, 99-120.

María José Aleandro

Universidad Nacional del Centro de la Provincia de Buenos Aires

Facultad de Ciencias Exactas

Departamento de Matemática

Campus Universitario, Argentina

e-mail: majo.aleandro@gmail.com, aleandro@exa.unicen.edu.ar