FORMALIZED MATHEMATICS
Vol. 26, No. 1, Pages 33–48, 2018
DOI: 10.2478/forma-2018-0004

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Klein-Beltrami Model. Part II

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Summary. Tim Makarios (with Isabelle/HOL[1]) and John Harrison (with HOL-Light[2]) have shown that "the Klein-Beltrami model of the hyperbolic plane satisfy all of Tarski's axioms except his Euclidean axiom" [2, 3, 15, 4].

With the Mizar system [1], [10] we use some ideas are taken from Tim Makarios' MSc thesis [12] for formalized some definitions (like the tangent) and lemmas necessary for the verification of the independence of the parallel postulate. This work can be also treated as a further development of Tarski's geometry in the formal setting [9].

MSC: 51A05 51M10 03B35

Keywords: Tarski's geometry axioms; foundations of geometry; Klein-Beltrami model

MML identifier: BKMODEL2, version: 8.1.07 5.47.1318

1. Beltrami-Cayley-Klein Disk Model

The BK-model yielding a non empty subset of the projective space over $\mathcal{E}^3_\mathrm{T}$ is defined by the term

(Def. 1) the interior of the conic for 1, 1, -1, 0, 0 and 0.

Now we state the propositions:

(1) The BK-model misses the absolute.

(2) Let us consider an element P of the projective space over $\mathcal{E}^3_\mathrm{T}$, and a non zero element u of $\mathcal{E}^3_\mathrm{T}$. Suppose $P =$ the direction of u and $P \in$ the BK-model. Then $u(3) \neq 0$.

[1] https://www.isa-afp.org/entries/Tarskis_Geometry.html
[2] https://github.com/jrh13/hol-light/blob/master/100/independence.ml

ISSN 1426–2630(Print), 1898-9934(Online)

Let P be an element of the BK-model. The functor BK-to-REAL2(P) yielding an element of the inside of circle(0,0,1) is defined by

(Def. 2) there exists a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = P$ and $u(3) = 1$ and $it = [u(1), u(2)]$.

Let Q be an element of the inside of circle(0,0,1). The functor REAL2-to-BK(Q) yielding an element of the BK-model is defined by

(Def. 3) there exists an element P of $\mathcal{E}^2_{\mathrm{T}}$ such that $P = Q$ and $it =$ the direction of $[(P)_{\mathbf{1}}, (P)_{\mathbf{2}}, 1]$.

Now we state the propositions:

(3) Let us consider an element P of the BK-model. Then REAL2-to-BK(BK-to-REAL2(P)) $= P$.
PROOF: Consider u being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = P$ and $u(3) = 1$ and BK-to-REAL2(P) $= [u(1), u(2)]$. Consider Q being an element of $\mathcal{E}^2_{\mathrm{T}}$ such that $Q =$ BK-to-REAL2(P) and REAL2-to-BK(BK-to-REAL2(P)) $=$ the direction of $[(Q)_{\mathbf{1}}, (Q)_{\mathbf{2}}, 1]$. $[(Q)_{\mathbf{1}}, (Q)_{\mathbf{2}}, 1]$ and u are proportional. □

(4) Let us consider elements P, Q of the BK-model. Then $P = Q$ if and only if BK-to-REAL2(P) $=$ BK-to-REAL2(Q).

(5) Let us consider an element Q of the inside of circle(0,0,1). Then BK-to-REAL2(REAL2-to-BK(Q)) $= Q$.

(6) Let us consider elements P, Q of the BK-model, and elements P_1, P_2, P_3 of the absolute. Suppose $P \neq Q$ and $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and P, Q and P_3 are collinear. Then

  (i) $P_3 = P_1$, or

  (ii) $P_3 = P_2$.

PROOF: $P_3 = P_1$ or $P_3 = P_2$. □

(7) Let us consider an element P of the BK-model, an element Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P \neq Q$ and $Q =$ the direction of v and $v(3) = 1$. Then there exists an element P_1 of the absolute such that P, Q and P_1 are collinear.
PROOF: Consider u being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = P$ and $u(3) = 1$ and BK-to-REAL2(P) $= [u(1), u(2)]$. Reconsider $s = [u(1), u(2)]$, $t = [v(1), v(2)]$ as a point of $\mathcal{E}^2_{\mathrm{T}}$. Set $a = 0$. Set $b = 0$. Set $r = 1$. Reconsider $S = s$, $T = t$, $X = [a, b]$ as an element of $\mathcal{R}^2$. Reconsider $w_1 = \frac{-2 \cdot |(t-s, s-[a,b])| + \sqrt{\Delta(\sum({}^2(T-S)), 2 \cdot |(t-s, s-[a,b])|, \sum({}^2(S-X)) - r^{\mathbf{2}})}}{2 \cdot \sum^2 (T-S)}$ as a real number. $s \neq t$. Consider e_1 being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that

$\{e_1\} = \mathrm{HalfLine}(s,t) \cap \mathrm{circle}(a,b,r)$ and $e_1 = (1 - w_1) \cdot s + w_1 \cdot t$. Reconsider $f = [(e_1)_{\mathbf{1}}, (e_1)_{\mathbf{2}}, 1]$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $e_3 = f$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. $1 \cdot e_3 + (-(1 - w_1)) \cdot u + (-w_1) \cdot v = 0_{\mathcal{E}^3_{\mathrm{T}}}$. □

(8) Let us consider an element P of the BK-model, and a line L of Inc-ProjSp (the real projective plane). Then there exists an element Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that

  (i) $P \neq Q$, and

  (ii) $Q \in L$.

(9) Let us consider real numbers a, b, c, d, e, and elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $u = [a, b, e]$ and $v = [c, d, 0]$ and $w = [a + c, b + d, e]$. Then $\langle|u, v, w|\rangle = 0$.

(10) Let us consider real numbers a, b, and a non zero real number c. Then $[a, b, c]$ is a non zero element of $\mathcal{E}^3_{\mathrm{T}}$.

(11) Let us consider elements u, v of $\mathcal{E}^3_{\mathrm{T}}$, and real numbers a, b, c, d, e. Suppose $u = [a, b, c]$ and $v = [d, e, 0]$ and u and v are proportional. Then $c = 0$.

(12) Let us consider elements P, Q, R of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $(u)_{\mathbf{3}} \neq 0$ and $(v)_{\mathbf{3}} = 0$ and $w = [(u)_{\mathbf{1}} + (v)_{\mathbf{1}}, (u)_{\mathbf{2}} + (v)_{\mathbf{2}}, (u)_{\mathbf{3}}]$. Then

  (i) $R \neq P$, and

  (ii) $R \neq Q$.

(13) Let us consider a line L of Inc-ProjSp(the real projective plane), and elements P, Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. If $P \neq Q$ and P, $Q \in L$, then $L = \mathrm{Line}(P, Q)$.

(14) Let us consider a line L of Inc-ProjSp(the real projective plane), elements P, Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$. Suppose P, $Q \in L$ and $P =$ the direction of u and $Q =$ the direction of v and $(u)_{\mathbf{3}} \neq 0$ and $(v)_{\mathbf{3}} = 0$. Then

  (i) $P \neq Q$, and

  (ii) the direction of $[(u)_{\mathbf{1}} + (v)_{\mathbf{1}}, (u)_{\mathbf{2}} + (v)_{\mathbf{2}}, (u)_{\mathbf{3}}] \in L$.

PROOF: $P \neq Q$. Reconsider $w = [(u)_{\mathbf{1}} + (v)_{\mathbf{1}}, (u)_{\mathbf{2}} + (v)_{\mathbf{2}}, (u)_{\mathbf{3}}]$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. $\langle|u, v, w|\rangle = 0$. □

(15) Let us consider elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $(v)_{\mathbf{3}} = 0$ and $w = [(u)_{\mathbf{1}} + (v)_{\mathbf{1}}, (u)_{\mathbf{2}} + (v)_{\mathbf{2}}, (u)_{\mathbf{3}}]$. Then $\langle|u, v, w|\rangle = 0$.

(16) Let us consider a line L of Inc-ProjSp(the real projective plane), an element P of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$.

Suppose $P =$ the direction of u and $P \in L$ and $u(3) \neq 0$. Then there exists an element Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ and there exists a non zero element v of $\mathcal{E}^3_{\mathrm{T}}$ such that $Q =$ the direction of v and $Q \in L$ and $P \neq Q$ and $v(3) \neq 0$. The theorem is a consequence of (15).

(17) Let us consider an element P of the BK-model, and a line L of Inc-ProjSp (the real projective plane). Suppose $P \in L$. Then there exists an element Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that

- (i) $P \neq Q$, and
- (ii) $Q \in L$, and
- (iii) for every non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that $Q =$ the direction of u holds $u(3) \neq 0$.

The theorem is a consequence of (16).

(18) Let us consider non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$, and a non zero real number k. Suppose $u = k \cdot v$. Then the direction of $u =$ the direction of v.

(19) Let us consider an element P of the BK-model, and an element Q of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P \neq Q$. Then there exists an element P_1 of the absolute such that P, Q and P_1 are collinear.

PROOF: Reconsider $L = \mathrm{Line}(P, Q)$ as a line of Inc-ProjSp(the real projective plane). Consider R being an element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P \neq R$ and $R \in L$ and for every non zero element u of $\mathcal{E}^3_{\mathrm{T}}$ such that $R =$ the direction of u holds $u(3) \neq 0$. Consider u being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = P$ and $u(3) = 1$ and BK-to-REAL2(P) $= [u(1), u(2)]$. Consider v' being an element of $\mathcal{E}^3_{\mathrm{T}}$ such that v' is not zero and the direction of $v' = R$. Reconsider $k = \frac{1}{(v')_{\mathbf{3}}}$ as a non zero real number. $k \cdot v'$ is not zero. Reconsider $v = k \cdot v'$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. the direction of $v = R$ and $v(3) = 1$. Reconsider $s = [u(1), u(2)]$, $t = [v(1), v(2)]$ as a point of $\mathcal{E}^2_{\mathrm{T}}$. Set $a = 0$. Set $b = 0$. Set $r = 1$. Reconsider $S = s$, $T = t$, $X = [a, b]$ as an element of $\mathcal{R}^2$. Reconsider $w_1 = \frac{-2 \cdot |(t-s, s-[a,b])| + \sqrt{\Delta(\sum({}^2(T-S)), 2 \cdot |(t-s, s-[a,b])|, \sum({}^2(S-X)) - r^{\mathbf{2}})}}{2 \cdot \sum({}^2(T-S)}$ as a real number. $s \neq t$. Consider e_1 being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $\{e_1\} = \mathrm{HalfLine}(s, t) \cap \mathrm{circle}(a, b, r)$ and $e_1 = (1 - w_1) \cdot s + w_1 \cdot t$. Reconsider $f = [(e_1)_{\mathbf{1}}, (e_1)_{\mathbf{2}}, 1]$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $e_3 = f$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. $1 \cdot e_3 + (-(1 - w_1)) \cdot u + (-w_1) \cdot v = 0_{\mathcal{E}^3_{\mathrm{T}}}$. □

(20) Let us consider elements P, Q of the BK-model. Suppose $P \neq Q$. Then there exist elements P_1, P_2 of the absolute such that

- (i) $P_1 \neq P_2$, and
- (ii) P, Q and P_1 are collinear, and

(iii) P, Q and P_2 are collinear.

PROOF: Consider u being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u = P$ and $u(3) = 1$ and BK-to-REAL2$(P) = [u(1), u(2)]$. Consider v being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $v = Q$ and $v(3) = 1$ and BK-to-REAL2$(Q) = [v(1), v(2)]$. Reconsider $s = [u(1), u(2)]$, $t = [v(1), v(2)]$ as a point of $\mathcal{E}^2_{\mathrm{T}}$. Set $a = 0$. Set $b = 0$. Set $r = 1$. Reconsider $S = s$, $T = t$, $X = [a, b]$ as an element of $\mathcal{R}^2$. Reconsider $w_1 = \frac{-2 \cdot |(t-s, s-[a,b])| + \sqrt{\Delta(\sum({}^2(T-S)), 2 \cdot |(t-s, s-[a,b])|, \sum({}^2(S-X)) - r^{\mathbf{2}})}}{2 \cdot (\sum({}^2(T-S)))}$ as a real number. Consider e_1 being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $\{e_1\} = \mathrm{HalfLine}(s, t) \cap \mathrm{circle}(a, b, r)$ and $e_1 = (1 - w_1) \cdot s + w_1 \cdot t$. Reconsider $w_2 = \frac{-2 \cdot |(s-t, t-[a,b])| + \sqrt{\Delta(\sum({}^2(S-T)), 2 \cdot |(s-t, t-[a,b])|, \sum({}^2(T-X)) - r^{\mathbf{2}})}}{2 \cdot (\sum({}^2(S-T)))}$ as a real number. Consider e_2 being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $\{e_2\} = \mathrm{HalfLine}(t, s) \cap \mathrm{circle}(a, b, r)$ and $e_2 = (1 - w_2) \cdot t + w_2 \cdot s$. Reconsider $f = [(e_1)_{\mathbf{1}}, (e_1)_{\mathbf{2}}, 1]$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $e_3 = f$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $P_1 =$ the direction of e_3 as a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. $1 \cdot e_3 + (-(1 - w_1)) \cdot u + (-w_1) \cdot v = 0_{\mathcal{E}^3_{\mathrm{T}}}$. Reconsider $g = [(e_2)_{\mathbf{1}}, (e_2)_{\mathbf{2}}, 1]$ as an element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $e_4 = g$ as a non zero element of $\mathcal{E}^3_{\mathrm{T}}$. Reconsider $P_2 =$ the direction of e_4 as a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$. $1 \cdot e_4 + (-(1 - w_2)) \cdot v + (-w_2) \cdot u = 0_{\mathcal{E}^3_{\mathrm{T}}}$. $P_1 \neq P_2$. □

(21) Let us consider elements P, Q, R of the real projective plane, non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$, and real numbers a, b, c, d. Suppose $P \in$ the BK-model and $Q \in$ the absolute and $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $u = [a, b, 1]$ and $v = [c, d, 1]$ and $w = [\frac{a+c}{2}, \frac{b+d}{2}, 1]$. Then

(i) $R \in$ the BK-model, and

(ii) $R \neq P$, and

(iii) P, R and Q are collinear.

PROOF: Reconsider $P_6 = P$ as an element of the BK-model. Consider u_2 being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that the direction of $u_2 = P_6$ and $u_2(3) = 1$ and BK-to-REAL2$(P_6) = [u_2(1), u_2(2)]$. Consider p being a point of $\mathcal{E}^2_{\mathrm{T}}$ such that $[v(1), v(2)] = p$ and $|p - [0, 0]| = 1$. Reconsider $R_1 = [w(1), w(2)]$ as an element of $\mathcal{E}^2_{\mathrm{T}}$. $|R_1 - [0, 0]|^{\mathbf{2}} < 1$. Consider P_1 being an element of $\mathcal{E}^2_{\mathrm{T}}$ such that $P_1 = R_1$ and REAL2-to-BK$(R_1) =$ the direction of $[(P_1)_{\mathbf{1}}, (P_1)_{\mathbf{2}}, 1]$. $P \neq R$ by [13, (29)]. □

(22) Let us consider elements P, Q of the real projective plane. Suppose $P \in$ the absolute and $Q \in$ the BK-model. Then there exists an element R of the real projective plane such that

(i) $R \in$ the BK-model, and

(ii) $Q \neq R$, and

(iii) R, Q and P are collinear.

The theorem is a consequence of (21).

(23) Let us consider a line L of Inc-ProjSp(the real projective plane), points p, q of Inc-ProjSp(the real projective plane), and elements P, Q of the real projective plane. Suppose $p = P$ and $q = Q$ and $P \in$ the BK-model and $Q \in$ the absolute and q lies on L and p lies on L. Then there exist points p_1, p_2 of Inc-ProjSp(the real projective plane) and there exist elements P_1, P_2 of the real projective plane such that $p_1 = P_1$ and $p_2 = P_2$ and $P_1 \neq P_2$ and $P_1, P_2 \in$ the absolute and p_1 lies on L and p_2 lies on L. The theorem is a consequence of (1), (22), and (20).

(24) Let us consider an element P of the BK-model, and an element Q of the absolute. Then there exists an element R of the absolute such that

(i) $Q \neq R$, and

(ii) Q, P and R are collinear.

The theorem is a consequence of (1) and (23).

(25) Let us consider an element P of the BK-model, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u and $u(3) = 1$. Then $(u(1))^{\mathbf{2}} + (u(2))^{\mathbf{2}} < 1$.

(26) Let us consider elements P_1, P_2 of the absolute, an element Q of the BK-model, and non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose the direction of $u = P_1$ and the direction of $v = P_2$ and the direction of $w = Q$ and $u(3) = 1$ and $v(3) = 1$ and $w(3) = 1$ and $v(1) = -u(1)$ and $v(2) = -u(2)$ and P_1, Q and P_2 are collinear. Then there exists a real number a such that

(i) $-1 < a < 1$, and

(ii) $w(1) = a \cdot u(1)$, and

(iii) $w(2) = a \cdot u(2)$.

The theorem is a consequence of (25).

2. Tangent

Let P be an element of the absolute. The functor PoleInfty(P) yielding an element of the real projective plane is defined by

(Def. 4) there exists a non zero element u of $\mathcal{E}^3_{\rm T}$ such that $P =$ the direction of u and $u(3) = 1$ and $(u(1))^{\mathbf{2}} + (u(2))^{\mathbf{2}} = 1$ and *it* = the direction of $[-u(2), u(1), 0]$.

Now we state the propositions:

(27) Let us consider an element P of the absolute. Then $P \neq$ PoleInfty(P).

(28) Let us consider elements P_1, P_2 of the absolute. Suppose PoleInfty(P_1) = PoleInfty(P_2). Then

(i) $P_1 = P_2$, or

(ii) there exists a non zero element u of $\mathcal{E}^3_{\rm T}$ such that $P_1 =$ the direction of u and $P_2 =$ the direction of $[-(u)_{\mathbf{1}}, -(u)_{\mathbf{2}}, 1]$ and $(u)_{\mathbf{3}} = 1$.

Proof: Consider u_1 being a non zero element of $\mathcal{E}^3_{\rm T}$ such that $P_1 =$ the direction of u_1 and $u_1(3) = 1$ and $u_1(1)^{\mathbf{2}} + u_1(2)^{\mathbf{2}} = 1$ and PoleInfty (P_1) = the direction of $[-u_1(2), u_1(1), 0]$. Consider u_2 being a non zero element of $\mathcal{E}^3_{\rm T}$ such that $P_2 =$ the direction of u_2 and $u_2(3) = 1$ and $(u_2(1))^{\mathbf{2}} + (u_2(2))^{\mathbf{2}} = 1$ and PoleInfty(P_2) = the direction of $[-u_2(2), u_2(1), 0]$. Reconsider $w_1 = [-u_1(2), u_1(1), 0]$ as a non zero element of $\mathcal{E}^3_{\rm T}$. Reconsider $w_2 = [-u_2(2), u_2(1), 0]$ as a non zero element of $\mathcal{E}^3_{\rm T}$. Consider a being a real number such that $a \neq 0$ and $w_1 = a \cdot w_2$. If $a = 1$, then $P_1 = P_2$. If $a = -1$, then there exists a non zero element u of $\mathcal{E}^3_{\rm T}$ such that $P_1 =$ the direction of u and $P_2 =$ the direction of $[-(u)_{\mathbf{1}}, -(u)_{\mathbf{2}}, 1]$ and $(u)_{\mathbf{3}} = 1$. □

Let P be an element of the absolute. The functor tangent(P) yielding a line of the real projective plane is defined by

(Def. 5) there exists an element p of the real projective plane such that $p = P$ and *it* = Line(p, PoleInfty(P)).

Let us consider an element P of the absolute. Now we state the propositions:

(29) $P \in$ tangent(P).

(30) tangent(P) $\cap$ (the absolute) = $\{P\}$.

Proof: $\{P\} \subseteq$ tangent(P) $\cap$ (the absolute). tangent(P) $\cap$ (the absolute) $\subseteq \{P\}$. □

(31) Let us consider elements P_1, P_2 of the absolute. If tangent(P_1) = tangent(P_2), then $P_1 = P_2$. The theorem is a consequence of (30).

(32) Let us consider elements P, Q of the absolute. Then there exists an element R of the real projective plane such that

(i) $R \in \text{tangent}(P)$, and

(ii) $R \in \text{tangent}(Q)$.

(33) Let us consider elements P_1, P_2 of the absolute. Suppose $P_1 \neq P_2$. Then there exists an element P of the real projective plane such that $\text{tangent}(P_1) \cap \text{tangent}(P_2) = \{P\}$. The theorem is a consequence of (31).

(34) Let us consider a square matrix M over $\mathbb{R}$ of dimension 3, an element P of the absolute, an element Q of the real projective plane, non zero elements u, v of $\mathcal{E}^3_{\mathrm{T}}$, and finite sequences f_3, f_7 of elements of $\mathbb{R}$. Suppose $M = \text{symmetric3}(1, 1, -1, 0, 0, 0)$ and $P =$ the direction of u and $Q =$ the direction of v and $u = f_3$ and $v = f_7$ and $Q \in \text{tangent}(P)$. Then $\text{SumAll QuadraticForm}(f_7, M, f_3) = 0$.

PROOF: Consider p being an element of the real projective plane such that $p = P$ and $\text{tangent}(P) = \text{Line}(p, \text{PoleInfty}(P))$. Consider w being a non zero element of $\mathcal{E}^3_{\mathrm{T}}$ such that $P =$ the direction of w and $w(3) = 1$ and $(w(1))^{\mathbf{2}} + (w(2))^{\mathbf{2}} = 1$ and $\text{PoleInfty}(P) =$ the direction of $[-w(2), w(1), 0]$. Consider a_1 being a real number such that $a_1 \neq 0$ and $w = a_1 \cdot u$. $w(1) = a_1 \cdot ((u)_{\mathbf{1}})$ and $w(2) = a_1 \cdot ((u)_{\mathbf{2}})$ and $w(3) = a_1 \cdot ((u)_{\mathbf{3}})$. $\text{len}\, f_3 = \text{width}\, M$ and $\text{len}\, f_7 = \text{len}\, M$ and $\text{len}\, f_3 = \text{len}\, M$ and $\text{len}\, f_7 = \text{width}\, M$ and $\text{len}\, f_3 > 0$ and $\text{len}\, f_7 > 0$. $\square$

(35) Let us consider elements P, Q, R of the absolute, and points P_1, P_2, P_3, P_4 of the real projective plane. Suppose P, Q, R are mutually different and $P_1 = P$ and $P_2 = Q$ and $P_3 = R$ and $P_4 \in \text{tangent}(P)$ and $P_4 \in \text{tangent}(Q)$. Then

(i) P_1, P_2 and P_3 are not collinear, and

(ii) P_1, P_2 and P_4 are not collinear, and

(iii) P_1, P_3 and P_4 are not collinear, and

(iv) P_2, P_3 and P_4 are not collinear.

PROOF: $P_4 \notin$ the absolute. Consider p being an element of the real projective plane such that $p = P$ and $\text{tangent}(P) = \text{Line}(p, \text{PoleInfty}(P))$. Consider q being an element of the real projective plane such that $q = Q$ and $\text{tangent}(Q) = \text{Line}(q, \text{PoleInfty}(Q))$. P_1, P_2 and P_4 are not collinear. P_1, P_3 and P_4 are not collinear. P_2, P_3 and P_4 are not collinear. $\square$

(36) Let us consider elements P, Q of the absolute, an element R of the real projective plane, and non zero elements u, v, w of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $P =$ the direction of u and $Q =$ the direction of v and $R =$ the direction of w and $R \in \text{tangent}(P)$ and $R \in \text{tangent}(Q)$ and $u(3) = 1$ and $v(3) = 1$ and $w(3) = 0$. Then

(i) $P = Q$, or

(ii) $u(1) = -v(1)$ and $u(2) = -v(2)$.

The theorem is a consequence of (34).

(37) Let us consider an element P of the absolute, an element R of the real projective plane, and a non zero element u of $\mathcal{E}^3_{\mathrm{T}}$. Suppose $R \in \mathrm{tangent}(P)$ and $R =$ the direction of u and $u(3) = 0$. Then $R = \mathrm{PoleInfty}(P)$. The theorem is a consequence of (34).

(38) Let us consider a non zero real number a, and an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $N = \mathrm{symmetric3}(a, a, -a, 0, 0, 0)$. Then (the homography of N)$^\circ$(the absolute) = the absolute.
Proof: (The homography of N)$^\circ$(the absolute) $\subseteq$ the absolute by [8, (8)]. The absolute $\subseteq$ (the homography of N)$^\circ$(the absolute) by [11, (4), (3)], [7, (89)]. □

(39) Let us consider a non zero element r_1 of $\mathbb{R}_{\mathrm{F}}$, and invertible square matrices M, O over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $O = \mathrm{symmetric3}(1, 1, -1, 0, 0, 0)$ and $M = r_1 \cdot O$. Then (the homography of M)$^\circ$(the absolute) = the absolute.
Proof: $r_1 \neq 0$ by [14, (34)]. □

(40) Let us consider an element P of the absolute. Then $\mathrm{tangent}(P)$ misses the BK-model. The theorem is a consequence of (29), (23), and (30).

(41) Let us consider elements P, P_3, P_4 of the real projective plane, elements P_1, P_2 of the absolute, and an element Q of the real projective plane. Suppose $P_1 \neq P_2$ and $P_3 = P_1$ and $P_4 = P_2$ and $P \in$ the BK-model and P, P_3 and P_4 are collinear and $Q \in \mathrm{tangent}(P_1)$ and $Q \in \mathrm{tangent}(P_2)$. Then there exists an element R of the real projective plane such that

(i) $R \in$ the absolute, and

(ii) P, Q and R are collinear.

The theorem is a consequence of (40), (7), (37), (28), and (26).

(42) Let us consider elements P, R, S of the real projective plane, and an element Q of the absolute. Suppose $P \in$ the BK-model and $R \in \mathrm{tangent}(Q)$ and P, S and R are collinear and $R \neq S$. Then $Q \neq S$. The theorem is a consequence of (29), (23), and (30).

3. Subgroup of K-Isometry

Let h be an element of EnsHomography3. We say that h is K-isometry if and only if

(Def. 6) there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that $h =$ the homography of N and (the homography of N)$^\circ$(the absolute) = the absolute.

Now we state the proposition:

(43) Let us consider an element h of EnsHomography3.
Suppose $h =$ the homography of $I^{3\times 3}_{\mathbb{R}_{\mathrm{F}}}$. Then h is K-isometry.

The set of K-isometries yielding a non empty subset of EnsHomography3 is defined by the term

(Def. 7) $\{h$, where h is an element of EnsHomography3 : h is K-isometry$\}$.

The subgroup of K-isometries yielding a strict subgroup of GroupHomography3 is defined by

(Def. 8) the carrier of *it* = the set of K-isometries.

Now we state the propositions:

(44) Let us consider an element h of the set of K-isometries, and an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $h =$ the homography of N. Then (the homography of N)$^{\circ}$(the absolute) = the absolute.

(45) (i) the homography of $I^{3\times 3}_{\mathbb{R}_{\mathrm{F}}} = \mathbf{1}_{\mathrm{GroupHomography3}}$, and

(ii) the homography of $I^{3\times 3}_{\mathbb{R}_{\mathrm{F}}} = \mathbf{1}_{\alpha}$,

where α is the subgroup of K-isometries.

(46) Let us consider invertible square matrices N_1, N_2 over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, and elements h_1, h_2 of the subgroup of K-isometries. Suppose $h_1 =$ the homography of N_1 and $h_2 =$ the homography of N_2. Then

(i) $h_1 \cdot h_2$ is an element of the subgroup of K-isometries, and

(ii) $h_1 \cdot h_2 =$ the homography of $N_1 \cdot N_2$.

(47) Let us consider an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3, and an element h of the subgroup of K-isometries.
Suppose $h =$ the homography of N. Then

(i) $h^{-1} =$ the homography of $N^{\smile}$, and

(ii) the homography of $N^{\smile}$ is an element of the subgroup of K-isometries.

The theorem is a consequence of (45).

(48) Let us consider an element s of the projective space over $\mathcal{E}^3_{\mathrm{T}}$, and elements p, q, r of the absolute. Suppose p, q, r are mutually different and $s \in \mathrm{tangent}(p) \cap \mathrm{tangent}(q)$. Then there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that

(i) (the homography of N)$^{\circ}$(the absolute) = the absolute, and

(ii) (the homography of N)(Dir101) $= p$, and

(iii) (the homography of N)(Dirm101) $= q$, and

(iv) (the homography of N)(Dir011) $= r$, and

(v) (the homography of N)(Dir010) $= s$.

PROOF: Reconsider $P_1 = p$, $P_2 = q$, $P_3 = r$, $P_4 = s$ as a point of the real projective plane. P_1, P_2 and P_3 are not collinear and P_1, P_2 and P_4 are not collinear and P_1, P_3 and P_4 are not collinear and P_2, P_3 and P_4 are not collinear. Consider N being an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that (the homography of N)(Dir101) $= P_1$ and (the homography of N)(Dirm101) $= P_2$ and (the homography of N)(Dir011) $= P_3$ and (the homography of N)(Dir010) $= P_4$. Consider n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8, n_9 being elements of $\mathbb{R}_{\mathrm{F}}$ such that $N = \langle\langle n_1, n_2, n_3\rangle, \langle n_4, n_5, n_6\rangle, \langle n_7, n_8, n_9\rangle\rangle$. Reconsider $b = -1$ as an element of $\mathbb{R}_{\mathrm{F}}$. Reconsider $a = 1$ as an element of $\mathbb{R}_{\mathrm{F}}$. Reconsider $a = 1$, $b = -1$ as a non zero element of $\mathbb{R}_{\mathrm{F}}$. Reconsider $N_1 = \langle\langle a, 0, 0\rangle, \langle 0, a, 0\rangle, \langle 0, 0, b\rangle\rangle$ as an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Reconsider $M = N^{\mathrm{T}} \cdot N_1 \cdot N$ as an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Consider v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9 being elements of $\mathbb{R}_{\mathrm{F}}$ such that $M = \langle\langle v_1, v_2, v_3\rangle, \langle v_4, v_5, v_6\rangle, \langle v_7, v_8, v_9\rangle\rangle$. Reconsider $r_1 = v_1$, $r_2 = v_2$, $r_3 = v_3$, $r_4 = v_5$, $r_5 = v_6$, $r_6 = v_9$ as a real number. Consider Q being a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that Dir101 $= Q$ and for every element u of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and $Q =$ the direction of u holds qfconic$(r_1, r_4, r_6, 2 \cdot r_2, 2 \cdot r_3, 2 \cdot r_5, u) = 0$. Consider Q being a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that Dirm101 $= Q$ and for every element u of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and $Q =$ the direction of u holds qfconic$(r_1, r_4, r_6, 2 \cdot r_2, 2 \cdot r_3, 2 \cdot r_5, u) = 0$. Consider Q being a point of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that Dir011 $= Q$ and for every element u of $\mathcal{E}^3_{\mathrm{T}}$ such that u is not zero and $Q =$ the direction of u holds qfconic$(r_1, r_4, r_6, 2 \cdot r_2, 2 \cdot r_3, 2 \cdot r_5, u) = 0$. $r_3 = 0$ and $r_1 = -r_6$ and $r_2 = 0$ and $r_5 = 0$ and $r_1 = r_4$. $r_1 \neq 0$. (The homography of M)$^\circ$(the absolute) = the absolute. □

(49) Let us consider elements p_1, q_1, r_1, p_2, q_2, r_2 of the absolute, and elements s_1, s_2 of the real projective plane. Suppose p_1, q_1, r_1 are mutually different and p_2, q_2, r_2 are mutually different and $s_1 \in \mathrm{tangent}(p_1) \cap \mathrm{tangent}(q_1)$ and $s_2 \in \mathrm{tangent}(p_2) \cap \mathrm{tangent}(q_2)$. Then there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that

(i) (the homography of N)$^\circ$(the absolute) = the absolute, and

(ii) (the homography of N)$(p_1) = p_2$, and

(iii) (the homography of N)$(q_1) = q_2$, and

(iv) (the homography of N)$(r_1) = r_2$, and

(v) (the homography of N)$(s_1) = s_2$.

The theorem is a consequence of (48) and (47).

(50) Let us consider elements p_1, q_1, r_1, p_2, q_2, r_2 of the absolute. Suppose p_1, q_1, r_1 are mutually different and p_2, q_2, r_2 are mutually different. Then there exists an invertible square matrix N over $\mathbb{R}_\mathrm{F}$ of dimension 3 such that

(i) (the homography of N)$^\circ$(the absolute) = the absolute, and

(ii) (the homography of N)$(p_1) = p_2$, and

(iii) (the homography of N)$(q_1) = q_2$, and

(iv) (the homography of N)$(r_1) = r_2$.

The theorem is a consequence of (33), (48), and (47).

(51) Let us consider a collinearity space C, and elements p, q, r, s of C. If $\mathrm{Line}(p, q) = \mathrm{Line}(r, s)$, then r, s and p are collinear.

(52) Let us consider a collinearity space C, and elements p, q of C. Then $\mathrm{Line}(p, q) = \mathrm{Line}(q, p)$.
PROOF: $\mathrm{Line}(p, q) \subseteq \mathrm{Line}(q, p)$. $\mathrm{Line}(q, p) \subseteq \mathrm{Line}(p, q)$. □

(53) Let us consider an invertible square matrix N over $\mathbb{R}_\mathrm{F}$ of dimension 3, and elements p, q, r, s of the projective space over $\mathcal{E}^3_\mathrm{T}$.
Suppose Line((the homography of N)(p), (the homography of N)(q)) = Line((the homography of N)(r), (the homography of N)(s)). Then

(i) p, q and r are collinear, and

(ii) p, q and s are collinear, and

(iii) r, s and p are collinear, and

(iv) r, s and q are collinear.

The theorem is a consequence of (51) and (52).

Let us consider an invertible square matrix N over $\mathbb{R}_\mathrm{F}$ of dimension 3 and elements p, q, r, s, t, u, n_1, n_2, n_3, n_4 of the real projective plane. Now we state the propositions:

(54) Suppose $p \neq q$ and $r \neq s$ and $n_1 \neq n_2$ and $n_3 \neq n_4$ and p, q and t are collinear and r, s and t are collinear and $n_1 =$ (the homography of N)(p) and $n_2 =$ (the homography of N)(q) and $n_3 =$ (the homography of N)(r) and $n_4 =$ (the homography of N)(s) and n_1, n_2 and u are collinear and n_3, n_4 and u are collinear. Then

(i) $u =$ (the homography of N)(t), or

(ii) $\mathrm{Line}(n_1, n_2) = \mathrm{Line}(n_3, n_4)$.

(55) Suppose $p \neq q$ and $r \neq s$ and $n_1 \neq n_2$ and $n_3 \neq n_4$ and p, q and t are collinear and r, s and t are collinear and $n_1 =$ (the homography of N)(p) and $n_2 =$ (the homography of N)(q) and $n_3 =$ (the homography of N)(r) and $n_4 =$ (the homography of N)(s) and n_1, n_2 and u are collinear and n_3, n_4 and u are collinear and p, q and r are not collinear. Then $u =$ (the homography of N)(t). The theorem is a consequence of (54) and (53).

(56) Let us consider elements p, q of the absolute, and elements a, b of the BK-model. Then there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that

(i) (the homography of N)$^\circ$(the absolute) = the absolute, and

(ii) (the homography of N)(a) $= b$, and

(iii) (the homography of N)(p) $= q$.

Proof: Consider p' being an element of the absolute such that $p \neq p'$ and p, a and p' are collinear. Consider q' being an element of the absolute such that $q \neq q'$ and q, b and q' are collinear. Consider t being an element of the real projective plane such that tangent(p)$\cap$tangent(p') $= \{t\}$. Consider u being an element of the real projective plane such that tangent(q) $\cap$ tangent(q') $= \{u\}$. Reconsider $a' = a$ as an element of the real projective plane. Consider R_1 being an element of the real projective plane such that $R_1 \in$ the absolute and a', t and R_1 are collinear. Reconsider $b' = b$ as an element of the real projective plane. Consider R_2 being an element of the real projective plane such that $R_2 \in$ the absolute and b', u and R_2 are collinear. p, p', R_1 are mutually different. Consider N being an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that (the homography of N)$^\circ$(the absolute) = the absolute and (the homography of N)(p) $= q$ and (the homography of N)(p') $= q'$ and (the homography of N)(R_1) $= R_2$ and (the homography of N)(t) $= u$. Reconsider $p_5 = p$, $p_6 = p'$, $p_7 = R_1$, $p_8 = t$, $p_9 = a$, $n_1 = q$, $n_2 = q'$, $n_3 = R_2$, $n_4 = u$, $n_5 = b$ as an element of the real projective plane. $n_5 =$ (the homography of N)(p_9). □

(57) Let us consider elements p, q, r, s of the absolute. Suppose p, q, r are mutually different and q, p, s are mutually different. Then there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that

(i) (the homography of N)$^\circ$(the absolute) = the absolute, and

(ii) (the homography of N)(p) $= q$, and

(iii) (the homography of N)(q) $= p$, and

(iv) (the homography of N)(r) $= s$, and

(v) for every element t of the real projective plane such that $t \in \text{tangent}(p) \cap \text{tangent}(q)$ holds (the homography of N)$(t) = t$.

The theorem is a consequence of (33), (48), and (47).

Let us consider elements P, Q of the BK-model. Now we state the propositions:

(58) Suppose $P \neq Q$. Then there exist elements P_1, P_2, P_3, P_4 of the absolute and there exists an element P_5 of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and P, P_5 and P_3 are collinear and Q, P_5 and P_4 are collinear and P_1, P_2, P_3 are mutually different and P_1, P_2, P_4 are mutually different and $P_5 \in \text{tangent}(P_1) \cap \text{tangent}(P_2)$. The theorem is a consequence of (20), (32), (41), (30), (42), (29), (40), and (7).

(59) Suppose $P \neq Q$. Then there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that

(i) (the homography of N)$^\circ$(the absolute) = the absolute, and

(ii) (the homography of N)$(P) = Q$, and

(iii) (the homography of N)$(Q) = P$, and

(iv) there exist elements P_1, P_2 of the absolute such that $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and (the homography of N)$(P_1) = P_2$ and (the homography of N)$(P_2) = P_1$.

Proof: Consider P_1, P_2, P_3, P_4 being elements of the absolute, P_5 being an element of the projective space over $\mathcal{E}^3_{\mathrm{T}}$ such that $P_1 \neq P_2$ and P, Q and P_1 are collinear and P, Q and P_2 are collinear and P, P_5 and P_3 are collinear and Q, P_5 and P_4 are collinear and P_1, P_2, P_3 are mutually different and P_1, P_2, P_4 are mutually different and $P_5 \in \text{tangent}(P_1) \cap \text{tangent}(P_2)$. Consider N_1 being an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that (the homography of N_1)$^\circ$(the absolute) = the absolute and (the homography of N_1)(Dir101) $= P_1$ and (the homography of N_1)(Dirm 101) $= P_2$ and (the homography of N_1)(Dir011) $= P_3$ and (the homography of N_1)(Dir010) $= P_5$. Consider N_2 being an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that (the homography of N_2)$^\circ$(the absolute) = the absolute and (the homography of N_2)(Dir101) $= P_2$ and (the homography of N_2)(Dirm101) $= P_1$ and (the homography of N_2)(Dir011) $= P_4$ and (the homography of N_2)(Dir010) $= P_5$. Reconsider $N = N_2 \cdot (N_1{}^{\smile})$ as an invertible square matrix over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Reconsider $h_1 =$ the homography of N_1 as an element of EnsHomography3. Reconsider $h_5 = h_1$ as an element of the subgroup of K-isometries. Reconsider $h_2 =$ the homography of N_2 as an element of EnsHomography3. Reconsider

$h_6 = h_2$ as an element of the subgroup of K-isometries. Reconsider $h_3 =$ the homography of $N_1{}^\smile$ as an element of EnsHomography3. ${h_5}^{-1} = h_3$. Reconsider $h_7 = h_3$ as an element of the subgroup of K-isometries. Reconsider $h_4 = h_6 \cdot h_7$ as an element of the subgroup of K-isometries. Consider h being an element of EnsHomography3 such that $h_4 = h$ and h is K-isometry. (the homography of N)(P) $= Q$ and (the homography of N)(Q) $= P$ by [5, (102), (57)], [6, (15)]. □

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Now we state the propositions:

(60) Let us consider elements P, Q of the BK-model. Then there exists an element h of the subgroup of K-isometries and there exists an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that $h =$ the homography of N and (the homography of N)(P) $= Q$ and (the homography of N)(Q) $= P$. The theorem is a consequence of (43) and (59).

(61) Let us consider elements P, Q, R, S, T, U of the BK-model. Suppose there exist elements h_1, h_2 of the subgroup of K-isometries and there exist invertible square matrices N_1, N_2 over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that $h_1 =$ the homography of N_1 and $h_2 =$ the homography of N_2 and (the homography of N_1)(P) $= R$ and (the homography of N_1)(Q) $= S$ and (the homography of N_2)(R) $= T$ and (the homography of N_2)(S) $= U$. Then there exists an element h_3 of the subgroup of K-isometries and there exists an invertible square matrix N_3 over $\mathbb{R}_{\mathrm{F}}$ of dimension 3 such that $h_3 =$ the homography of N_3 and (the homography of N_3)(P) $= T$ and (the homography of N_3)(Q) $= U$. The theorem is a consequence of (46).

(62) Let us consider elements P, Q, R of the BK-model, an element h of the subgroup of K-isometries, and an invertible square matrix N over $\mathbb{R}_{\mathrm{F}}$ of dimension 3. Suppose $h =$ the homography of N and (the homography of N)(P) $= R$ and (the homography of N)(Q) $= R$. Then $P = Q$.

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Received March 27, 2018