# Implicit Function Theorem. Part I ${ }^{1}$ 

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#### Abstract

Summary. In this article, we formalize in Mizar [1, [3] the existence and uniqueness part of the implicit function theorem. In the first section, some composition properties of Lipschitz continuous linear function are discussed. In the second section, a definition of closed ball and theorems of several properties of open and closed sets in Banach space are described. In the last section, we formalized the existence and uniqueness of continuous implicit function in Banach space using Banach fixed point theorem. We referred to 7 , 8 , and 2 in this formalization.


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## 1. Properties of Lipschitz Continuous Linear Function

From now on $S, T, W, Y$ denote real normed spaces, $f, f_{1}, f_{2}$ denote partial functions from $S$ to $T, Z$ denotes a subset of $S$, and $i$, $n$ denote natural numbers.

Now we state the propositions:
(1) Let us consider real normed spaces $X, Y$, a point $x$ of $X$, a point $y$ of $Y$, and a point $z$ of $X \times Y$. Suppose $z=\langle x, y\rangle$. Then $\|z\|=\sqrt{\|x\|^{2}+\|y\|^{2}}$.

[^0](2) Let us consider real normed spaces $X, Y$, a point $x$ of $X$, and a point $z$ of $X \times Y$. Suppose $z=\left\langle x, 0_{Y}\right\rangle$. Then $\|z\|=\|x\|$. The theorem is a consequence of (1).
(3) Let us consider real normed spaces $X, Y$, a point $y$ of $Y$, and a point $z$ of $X \times Y$. Suppose $z=\left\langle 0_{X}, y\right\rangle$. Then $\|z\|=\|y\|$. The theorem is a consequence of (1).
(4) Let us consider real normed spaces $X, Y, Z, W$, a Lipschitzian linear operator $f$ from $Z$ into $W$, a Lipschitzian linear operator $g$ from $Y$ into $Z$, and a Lipschitzian linear operator $h$ from $X$ into $Y$. Then $f \cdot(g \cdot h)=$ $(f \cdot g) \cdot h$.
(5) Let us consider real normed spaces $X, Y, Z$, a Lipschitzian linear operator $g$ from $X$ into $Y$, a Lipschitzian linear operator $f$ from $Y$ into $Z$, and a Lipschitzian linear operator $h$ from $X$ into $Z$. Then $h=f \cdot g$ if and only if for every vector $x$ of $X, h(x)=f(g(x))$.
(6) Let us consider real normed spaces $X, Y$, and a Lipschitzian linear operator $f$ from $X$ into $Y$. Then
(i) $f \cdot \mathrm{id}_{\alpha}=f$, and
(ii) $\operatorname{id}_{\beta} \cdot f=f$,
where $\alpha$ is the carrier of $X$ and $\beta$ is the carrier of $Y$.
(7) Let us consider real normed spaces $X, Y, Z, W$, an element $f$ of $\operatorname{BdLinOps}(Z, W)$, an element $g$ of $\operatorname{BdLinOps}(Y, Z)$, and an element $h$ of $\operatorname{BdLinOps}(X, Y)$. Then $f \cdot(g \cdot h)=(f \cdot g) \cdot h$.
(8) Let us consider real normed spaces $X, Y$, and an element $f$ of $\operatorname{BdLinOps}(X, Y)$. Then
(i) $f \cdot \operatorname{FuncUnit}(X)=f$, and
(ii) $\operatorname{FuncUnit}(Y) \cdot f=f$.

The theorem is a consequence of (6).
(9) Let us consider real normed spaces $X, Y, Z$, an element $f$ of the real norm space of bounded linear operators from $Y$ into $Z$, and elements $g, h$ of the real norm space of bounded linear operators from $X$ into $Y$. Then $f \cdot(g+h)=f \cdot g+f \cdot h$.
Proof: Set $m_{1}=\operatorname{PartFuncs}(f, Y, Z)$. Set $m_{2}=\operatorname{PartFuncs}(g, X, Y)$. Set $m_{4}=\operatorname{PartFuncs}(h, X, Y)$. Set $m_{3}=\operatorname{PartFuncs}(g+h, X, Y)$. For every vector $x$ of $X,\left(m_{1} \cdot m_{3}\right)(x)=\left(m_{1} \cdot m_{2}\right)(x)+\left(m_{1} \cdot m_{4}\right)(x)$ by [9, (35)], (5).
(10) Let us consider real normed spaces $X, Y, Z$, an element $f$ of the real norm space of bounded linear operators from $X$ into $Y$, and elements $g, h$
of the real norm space of bounded linear operators from $Y$ into $Z$. Then $(g+h) \cdot f=g \cdot f+h \cdot f$.
Proof: Set $m_{1}=\operatorname{PartFuncs}(f, X, Y)$. Set $m_{2}=\operatorname{PartFuncs}(g, Y, Z)$. Set $m_{4}=\operatorname{PartFuncs}(h, Y, Z)$. Set $m_{3}=\operatorname{PartFuncs}(g+h, Y, Z)$. For every vector $x$ of $X,\left(m_{3} \cdot m_{1}\right)(x)=\left(m_{2} \cdot m_{1}\right)(x)+\left(m_{4} \cdot m_{1}\right)(x)$.
(11) Let us consider real normed spaces $X, Y, Z$, an element $f$ of the real norm space of bounded linear operators from $Y$ into $Z$, an element $g$ of the real norm space of bounded linear operators from $X$ into $Y$, and real numbers $a, b$. Then $(a \cdot b) \cdot(f \cdot g)=a \cdot f \cdot(b \cdot g)$.
Proof: Set $m_{1}=\operatorname{PartFuncs}(f, Y, Z)$. Set $m_{2}=\operatorname{PartFuncs}(g, X, Y)$. Set $m_{5}=\operatorname{PartFuncs}(a \cdot f, Y, Z)$. Set $m_{6}=\operatorname{PartFuncs}(b \cdot g, X, Y)$. For every vector $x$ of $X,\left(m_{5} \cdot m_{6}\right)(x)=a \cdot b \cdot\left(m_{1} \cdot m_{2}\right)(x)$.
(12) Let us consider real normed spaces $X, Y, Z$, an element $f$ of the real norm space of bounded linear operators from $Y$ into $Z$, an element $g$ of the real norm space of bounded linear operators from $X$ into $Y$, and a real number $a$. Then $a \cdot(f \cdot g)=(a \cdot f) \cdot g$. The theorem is a consequence of (11).

## 2. Properties of Open and Closed Sets in Banach Space

Let $M$ be a real normed space, $p$ be an element of $M$, and $r$ be a real number. The functor $\overline{\operatorname{Ball}}(p, r)$ yielding a subset of $M$ is defined by the term
(Def. 1) $\{q$, where $q$ is an element of $M:\|p-q\| \leqslant r\}$.
Let us consider an element $p$ of $S$ and a real number $r$. Now we state the propositions:
(13) If $0<r$, then $p \in \operatorname{Ball}(p, r)$ and $p \in \overline{\operatorname{Ball}}(p, r)$.
(14) If $0<r$, then $\operatorname{Ball}(p, r) \neq \emptyset$ and $\overline{\operatorname{Ball}}(p, r) \neq \emptyset$.

Let us consider a real normed space $M$, an element $p$ of $M$, and real numbers $r_{1}, r_{2}$. Now we state the propositions:
(15) Suppose $r_{1} \leqslant r_{2}$. Then
(i) $\overline{\operatorname{Ball}}\left(p, r_{1}\right) \subseteq \overline{\operatorname{Ball}}\left(p, r_{2}\right)$, and
(ii) $\operatorname{Ball}\left(p, r_{1}\right) \subseteq \overline{\operatorname{Ball}}\left(p, r_{2}\right)$, and
(iii) $\operatorname{Ball}\left(p, r_{1}\right) \subseteq \operatorname{Ball}\left(p, r_{2}\right)$.
(16) If $r_{1}<r_{2}$, then $\overline{\operatorname{Ball}}\left(p, r_{1}\right) \subseteq \operatorname{Ball}\left(p, r_{2}\right)$.

Let us consider an element $p$ of $S$ and a real number $r$. Now we state the propositions:
(17) $\operatorname{Ball}(p, r)=\{y$, where $y$ is a point of $S:\|y-p\|<r\}$.

Proof: Define $\mathcal{F}($ object $)=\$_{1}$. Define $\mathcal{P}[$ element of $S] \equiv\left\|p-\$_{1}\right\|<r$. Define $\mathcal{Q}$ element of $S] \equiv\left\|\$_{1}-p\right\|<r .\{\mathcal{F}(y)$, where $y$ is an element of the carrier of $S: \mathcal{P}[y]\}=\{\mathcal{F}(y)$, where $y$ is an element of the carrier of $S: \mathcal{Q}[y]\}$.
(18) $\overline{\operatorname{Ball}}(p, r)=\{y$, where $y$ is a point of $S:\|y-p\| \leqslant r\}$.

Proof: Define $\mathcal{F}$ (object) $=\$_{1}$. Define $\mathcal{P}[$ element of $S] \equiv\left\|p-\$_{1}\right\| \leqslant r$. Define $\mathcal{Q}$ [element of $S] \equiv\left\|\$_{1}-p\right\| \leqslant r$. $\{\mathcal{F}(y)$, where $y$ is an element of the carrier of $S: \mathcal{P}[y]\}=\{\mathcal{F}(y)$, where $y$ is an element of the carrier of $S: \mathcal{Q}[y]\}$.
(19) If $0<r$, then $\operatorname{Ball}(p, r)$ is a neighbourhood of $p$. The theorem is a consequence of (17).
Let $X$ be a real normed space, $x$ be a point of $X$, and $r$ be a real number. One can check that $\operatorname{Ball}(x, r)$ is open and $\overline{\operatorname{Ball}}(x, r)$ is closed.

Now we state the propositions:
(20) Let us consider a real normed space $X$, and a subset $V$ of $X$. Then $V$ is open if and only if for every point $x$ of $X$ such that $x \in V$ there exists a real number $r$ such that $r>0$ and $\operatorname{Ball}(x, r) \subseteq V$.
(21) Let us consider real normed spaces $X, Y$, a point $x$ of $X$, a point $y$ of $Y$, and a point $z$ of $X \times Y$. Suppose $z=\langle x, y\rangle$. Then
(i) $\|x\| \leqslant\|z\|$, and
(ii) $\|y\| \leqslant\|z\|$.

The theorem is a consequence of (1).
(22) Let us consider real normed spaces $X, Y$, a point $x$ of $X$, a point $y$ of $Y$, a point $z$ of $X \times Y$, and a real number $r_{1}$. Suppose $0<r_{1}$ and $z=\langle x, y\rangle$. Then there exists a real number $r_{2}$ such that
(i) $0<r_{2}<r_{1}$, and
(ii) $\operatorname{Ball}\left(x, r_{2}\right) \times \operatorname{Ball}\left(y, r_{2}\right) \subseteq \operatorname{Ball}\left(z, r_{1}\right)$.

Proof: $\operatorname{Ball}\left(x, r_{2}\right) \times \operatorname{Ball}\left(y, r_{2}\right) \subseteq \operatorname{Ball}\left(z, r_{1}\right)$.
(23) Let us consider real normed spaces $X, Y$, a point $x$ of $X$, a point $y$ of $Y$, and a subset $V$ of $X \times Y$. Suppose $V$ is open and $\langle x, y\rangle \in V$. Then there exists a real number $r$ such that
(i) $0<r$, and
(ii) $\operatorname{Ball}(x, r) \times \operatorname{Ball}(y, r) \subseteq V$.

The theorem is a consequence of (20) and (22).
(24) Let us consider real normed spaces $X, Y$, a point $x$ of $X$, a point $y$ of $Y$, a subset $V$ of $X \times Y$, and a real number $r$. Suppose $V=\operatorname{Ball}(x, r) \times$ $\operatorname{Ball}(y, r)$. Then $V$ is open.
Proof: For every point $z$ of $X \times Y$ such that $z \in V$ there exists a real number $s$ such that $s>0$ and $\operatorname{Ball}(z, s) \subseteq V$ by [5, (18)].
(25) Let us consider real normed spaces $E, F$, a linear operator $Q$ from $E$ into $F$, and a point $v$ of $E$. If $Q$ is one-to-one, then $Q(v)=0_{F}$ iff $v=0_{E}$.
Let us consider a real normed space $E$, subsets $X, Y$ of $E$, and a point $v$ of $E$. Now we state the propositions:
(26) If $X$ is open and $Y=\{x+v$, where $x$ is a point of $E: x \in X\}$, then $Y$ is open.
Proof: Define $\mathcal{C}$ (point of $E)=1 \cdot \$_{1}+-v$. Consider $H$ being a function from $E$ into $E$ such that for every point $p$ of $E, H(p)=\mathcal{C}(p)$. For every object $s, s \in H^{-1}(X)$ iff $s \in Y$.
(27) If $X$ is open and $Y=\{x-v$, where $x$ is a point of $E: x \in X\}$, then $Y$ is open.
Proof: Set $w=-v .\{x+w$, where $x$ is a point of $E: x \in X\} \subseteq$ the carrier of $E$. Define $\mathcal{F}$ (point of $E)=\$_{1}+w$. Define $\mathcal{G}$ (point of $\left.E\right)=$ $\$_{1}-v$. Define $\mathcal{P}$ [point of $\left.E\right] \equiv \$_{1} \in X .\left\{\mathcal{F}\left(v_{1}\right)\right.$, where $v_{1}$ is an element of the carrier of $\left.E: \mathcal{P}\left[v_{1}\right]\right\}=\left\{\mathcal{G}\left(v_{2}\right)\right.$, where $v_{2}$ is an element of the carrier of $\left.E: \mathcal{P}\left[v_{2}\right]\right\}$.

## 3. Existence and Uniqueness of Continuous Implicit Function

Now we state the propositions:
(28) Let us consider a real Banach space $X$, a non empty subset $S$ of $X$, and a partial function $f$ from $X$ to $X$. Suppose $S$ is closed and $\operatorname{dom} f=S$ and rng $f \subseteq S$ and there exists a real number $k$ such that $0<k<1$ and for every points $x, y$ of $X$ such that $x, y \in S$ holds $\left\|f_{x}-f_{y}\right\| \leqslant k \cdot\|x-y\|$. Then
(i) there exists a point $x_{0}$ of $X$ such that $x_{0} \in S$ and $f\left(x_{0}\right)=x_{0}$, and
(ii) for every points $x_{0}, y_{0}$ of $X$ such that $x_{0}, y_{0} \in S$ and $f\left(x_{0}\right)=x_{0}$ and $f\left(y_{0}\right)=y_{0}$ holds $x_{0}=y_{0}$.
Proof: Consider $x_{0}$ being an object such that $x_{0} \in S$. Consider $K$ being a real number such that $0<K$ and $K<1$ and for every points $x, y$ of $X$ such that $x, y \in S$ holds $\left\|f_{x}-f_{y}\right\| \leqslant K \cdot\|x-y\|$. Define $\mathcal{G}$ (set, set) $=f\left(\$_{2}\right)$. Consider $g$ being a function such that $\operatorname{dom} g=\mathbb{N}$ and $g(0)=x_{0}$ and for every natural number $n, g(n+1)=\mathcal{G}(n, g(n))$. Define $\mathcal{P}$ [natural number] $\equiv$
$g\left(\$_{1}\right) \in S$ and $g\left(\$_{1}\right)$ is an element of $X$. For every natural number $k$ such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number $n, \mathcal{P}[n]$. For every object $n$ such that $n \in \mathbb{N}$ holds $g(n) \in$ the carrier of $X$. For every natural number $n,\|g(n+1)-g(n)\| \leqslant\|g(1)-g(0)\| \cdot\left(K^{n}\right)$. For every natural numbers $k, n,\|g(n+k)-g(n)\| \leqslant\|g(1)-g(0)\| \cdot\left(\frac{K^{n}-K^{n+k}}{1-K}\right)$. For every natural numbers $k, n,\|g(n+k)-g(n)\| \leqslant\|g(1)-g(0)\| \cdot\left(\frac{K^{n}}{1-K}\right)$. For every real number $e$ such that $e>0$ there exists a natural number $n$ such that for every natural number $m$ such that $n \leqslant m$ holds $\left\|(g \uparrow 1)(m)-f_{\lim g}\right\|<e$. For every points $x_{0}, y_{0}$ of $X$ such that $x_{0}, y_{0} \in S$ and $f\left(x_{0}\right)=x_{0}$ and $f\left(y_{0}\right)=y_{0}$ holds $x_{0}=y_{0}$.
(29) Let us consider a real normed space $E$, a real Banach space $F$, a non empty subset $E_{0}$ of $E$, a non empty subset $F_{0}$ of $F$, and a partial function $F_{1}$ from $E \times F$ to $F$. Suppose $F_{0}$ is closed and $E_{0} \times F_{0} \subseteq \operatorname{dom} F_{1}$ and for every point $x$ of $E$ and for every point $y$ of $F$ such that $x \in E_{0}$ and $y \in F_{0}$ holds $F_{1}(x, y) \in F_{0}$ and for every point $y$ of $F$ such that $y \in F_{0}$ for every point $x_{0}$ of $E$ such that $x_{0} \in E_{0}$ for every real number $e$ such that $0<e$ there exists a real number $d$ such that $0<d$ and for every point $x_{1}$ of $E$ such that $x_{1} \in E_{0}$ and $\left\|x_{1}-x_{0}\right\|<d$ holds $\left\|F_{1}{ }_{\left\langle x_{1}, y\right\rangle}-F_{1}{ }_{\left\langle x_{0}, y\right\rangle}\right\|<e$ and there exists a real number $k$ such that $0<k<1$ and for every point $x$ of $E$ such that $x \in E_{0}$ for every points $y_{1}, y_{2}$ of $F$ such that $y_{1}, y_{2} \in F_{0}$ holds $\left\|F_{1}{ }_{\left\langle x, y_{1}\right\rangle}-F_{1}{ }_{\left\langle x, y_{2}\right\rangle}\right\| \leqslant k \cdot\left\|y_{1}-y_{2}\right\|$. Then
(i) for every point $x$ of $E$ such that $x \in E_{0}$ holds there exists a point $y$ of $F$ such that $y \in F_{0}$ and $F_{1}(x, y)=y$ and for every points $y_{1}, y_{2}$ of $F$ such that $y_{1}, y_{2} \in F_{0}$ and $F_{1}\left(x, y_{1}\right)=y_{1}$ and $F_{1}\left(x, y_{2}\right)=y_{2}$ holds $y_{1}=y_{2}$, and
(ii) for every point $x_{0}$ of $E$ and for every point $y_{0}$ of $F$ such that $x_{0} \in E_{0}$ and $y_{0} \in F_{0}$ and $F_{1}\left(x_{0}, y_{0}\right)=y_{0}$ for every real number $e$ such that $0<e$ there exists a real number $d$ such that $0<d$ and for every point $x_{1}$ of $E$ and for every point $y_{1}$ of $F$ such that $x_{1} \in E_{0}$ and $y_{1} \in F_{0}$ and $F_{1}\left(x_{1}, y_{1}\right)=y_{1}$ and $\left\|x_{1}-x_{0}\right\|<d$ holds $\left\|y_{1}-y_{0}\right\|<e$.
Proof: Consider $k$ being a real number such that $0<k<1$ and for every point $x$ of $E$ such that $x \in E_{0}$ for every points $y_{1}, y_{2}$ of $F$ such that $y_{1}, y_{2} \in F_{0}$ holds $\left\|F_{1}{ }_{\left\langle x, y_{1}\right\rangle}-F_{1}{\left\langle x, y_{2}\right\rangle}\right\| \leqslant k \cdot\left\|y_{1}-y_{2}\right\|$. For every point $x$ of $E$ such that $x \in E_{0}$ holds there exists a point $y$ of $F$ such that $y \in F_{0}$ and $F_{1}(x, y)=y$ and for every points $y_{1}, y_{2}$ of $F$ such that $y_{1}$, $y_{2} \in F_{0}$ and $F_{1}\left(x, y_{1}\right)=y_{1}$ and $F_{1}\left(x, y_{2}\right)=y_{2}$ holds $y_{1}=y_{2}$. For every point $x_{0}$ of $E$ and for every point $y_{0}$ of $F$ such that $x_{0} \in E_{0}$ and $y_{0} \in F_{0}$ and $F_{1}\left(x_{0}, y_{0}\right)=y_{0}$ for every real number $e$ such that $0<e$ there exists a real number $d$ such that $0<d$ and for every point $x_{1}$ of $E$ and for every
point $y_{1}$ of $F$ such that $x_{1} \in E_{0}$ and $y_{1} \in F_{0}$ and $F_{1}\left(x_{1}, y_{1}\right)=y_{1}$ and $\left\|x_{1}-x_{0}\right\|<d$ holds $\left\|y_{1}-y_{0}\right\|<e$.
(30) Let us consider a real normed space $E$, a real Banach space $F$, a non empty subset $A$ of $E$, a non empty subset $B$ of $F$, and a partial function $F_{1}$ from $E \times F$ to $F$. Suppose $B$ is closed and $A \times B \subseteq \operatorname{dom} F_{1}$ and for every point $x$ of $E$ and for every point $y$ of $F$ such that $x \in A$ and $y \in B$ holds $F_{1}(x, y) \in B$ and for every point $y$ of $F$ such that $y \in B$ for every point $x_{0}$ of $E$ such that $x_{0} \in A$ for every real number $e$ such that $0<e$ there exists a real number $d$ such that $0<d$ and for every point $x_{1}$ of $E$ such that $x_{1} \in A$ and $\left\|x_{1}-x_{0}\right\|<d$ holds $\left\|F_{1}{ }_{\left\langle x_{1}, y\right\rangle}-F_{1}{ }_{\left\langle x_{0}, y\right\rangle}\right\|<e$ and there exists a real number $k$ such that $0<k<1$ and for every point $x$ of $E$ such that $x \in A$ for every points $y_{1}, y_{2}$ of $F$ such that $y_{1}, y_{2} \in B$ holds $\left\|F_{1\left\langle x, y_{1}\right\rangle}-F_{1\left\langle x, y_{2}\right\rangle}\right\| \leqslant k \cdot\left\|y_{1}-y_{2}\right\|$. Then
(i) there exists a partial function $g$ from $E$ to $F$ such that $g$ is continuous on $A$ and $\operatorname{dom} g=A$ and $\operatorname{rng} g \subseteq B$ and for every point $x$ of $E$ such that $x \in A$ holds $F_{1}(x, g(x))=g(x)$, and
(ii) for every partial functions $g_{1}, g_{2}$ from $E$ to $F$ such that dom $g_{1}=A$ and $\operatorname{rng} g_{1} \subseteq B$ and $\operatorname{dom} g_{2}=A$ and $\operatorname{rng} g_{2} \subseteq B$ and for every point $x$ of $E$ such that $x \in A$ holds $F_{1}\left(x, g_{1}(x)\right)=g_{1}(x)$ and for every point $x$ of $E$ such that $x \in A$ holds $F_{1}\left(x, g_{2}(x)\right)=g_{2}(x)$ holds $g_{1}=g_{2}$.
Proof: There exists a partial function $g$ from $E$ to $F$ such that $g$ is continuous on $A$ and $\operatorname{dom} g=A$ and $\operatorname{rng} g \subseteq B$ and for every point $x$ of $E$ such that $x \in A$ holds $F_{1}(x, g(x))=g(x)$ by (29), [4, (19)]. For every object $x$ such that $x \in \operatorname{dom} g_{1}$ holds $g_{1}(x)=g_{2}(x)$.
Let us consider real normed spaces $E, F$ and points $s_{1}, s_{2}$ of $E \times F$. Now we state the propositions:
(31) If $\left(s_{1}\right)_{\mathbf{2}}=\left(s_{2}\right)_{\mathbf{2}}$, then reproj1 $\left(s_{1}\right)=\operatorname{reproj} 1\left(s_{2}\right)$.
(32) If $\left(s_{1}\right)_{\mathbf{1}}=\left(s_{2}\right)_{\mathbf{1}}$, then reproj2 $\left(s_{1}\right)=\operatorname{reproj} 2\left(s_{2}\right)$.
(33) Let us consider a real normed space $E$, a real number $r$, and points $z$, $y_{1}, y_{2}$ of $E$. Suppose $y_{1}, y_{2} \in \overline{\operatorname{Ball}}(z, r)$. Then $\left[y_{1}, y_{2}\right] \subseteq \overline{\operatorname{Ball}}(z, r)$.
(34) Let us consider a real normed space $E$, points $x, b$ of $E$, and a neighbourhood $N$ of $x$. Then $\{z-b$, where $z$ is a point of $E: z \in N\}$ is neighbourhood of $x-b$ and neighbourhood of $x+b$.
Proof: Consider $g$ being a real number such that $0<g$ and $\{y$, where $y$ is a point of $E:\|y-x\|<g\} \subseteq N .\{z-b$, where $z$ is a point of $E: z \in N\} \subseteq$ the carrier of $E .\{z+b$, where $z$ is a point of $E: z \in$ $N\} \subseteq$ the carrier of $E .\{y$, where $y$ is a point of $E:\|y-(x-b)\|<$ $g\} \subseteq\{z-b$, where $z$ is a point of $E: z \in N\} .\{y$, where $y$ is a point of
$E:\|y-(x+b)\|<g\} \subseteq\{z+b$, where $z$ is a point of $E: z \in N\}$.
Let us consider real normed spaces $E, G$, a real Banach space $F$, a subset $Z$ of $E \times F$, a partial function $f$ from $E \times F$ to $G$, a point $a$ of $E$, a point $b$ of $F$, a point $c$ of $G$, and a point $z$ of $E \times F$. Now we state the propositions:
(35) Suppose $Z$ is open and $\operatorname{dom} f=Z$ and $f$ is continuous on $Z$ and $f$ is partially differentiable on $Z$ w.r.t. 2 and $f \vdash^{2} Z$ is continuous on $Z$ and $z=\langle a, b\rangle$ and $z \in Z$ and $f(a, b)=c$ and $\operatorname{partdiff}(f, z)$ w.r.t. 2 is one-toone and (partdiff $(f, z)$ w.r.t. 2) $)^{-1}$ is a Lipschitzian linear operator from $G$ into $F$. Then there exist real numbers $r_{1}, r_{2}$ such that
(i) $0<r_{1}$, and
(ii) $0<r_{2}$, and
(iii) $\operatorname{Ball}\left(a, r_{1}\right) \times \overline{\operatorname{Ball}}\left(b, r_{2}\right) \subseteq Z$, and
(iv) for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ there exists a point $y$ of $F$ such that $y \in \overline{\operatorname{Ball}}\left(b, r_{2}\right)$ and $f(x, y)=c$, and
(v) for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ for every points $y_{1}, y_{2}$ of $F$ such that $y_{1}, y_{2} \in \overline{\operatorname{Ball}}\left(b, r_{2}\right)$ and $f\left(x, y_{1}\right)=c$ and $f\left(x, y_{2}\right)=c$ holds $y_{1}=y_{2}$, and
(vi) there exists a partial function $g$ from $E$ to $F$ such that $g$ is continuous on $\operatorname{Ball}\left(a, r_{1}\right)$ and $\operatorname{dom} g=\operatorname{Ball}\left(a, r_{1}\right)$ and $\operatorname{rng} g \subseteq \overline{\operatorname{Ball}}\left(b, r_{2}\right)$ and $g(a)=b$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ holds $f(x, g(x))=c$, and
(vii) for every partial functions $g_{1}, g_{2}$ from $E$ to $F$ such that dom $g_{1}=$ $\operatorname{Ball}\left(a, r_{1}\right)$ and $\operatorname{rng} g_{1} \subseteq \overline{\operatorname{Ball}}\left(b, r_{2}\right)$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ holds $f\left(x, g_{1}(x)\right)=c$ and $\operatorname{dom} g_{2}=\operatorname{Ball}\left(a, r_{1}\right)$ and $\operatorname{rng} g_{2} \subseteq \overline{\operatorname{Ball}}\left(b, r_{2}\right)$ and for every point $x$ of $E$ such that $x \in$ $\operatorname{Ball}\left(a, r_{1}\right)$ holds $f\left(x, g_{2}(x)\right)=c$ holds $g_{1}=g_{2}$.
Proof: Consider $Q_{1}$ being a Lipschitzian linear operator from $G$ into $F$ such that $Q_{1}=(\operatorname{partdiff}(f, z) \text { w.r.t. } 2)^{-1}$. Reconsider $Q=Q_{1}$ as a point of the real norm space of bounded linear operators from $G$ into $F$. Reconsider $z_{1}=\left\langle a, 0_{F}\right\rangle$ as a point of $E \times F$. Reconsider $e_{0}=\left\langle 0_{E}, b\right\rangle$ as a point of $E \times$ $F$. Define $\mathcal{C}$ (point of $E \times F)=1 \cdot \$_{1}+-e_{0}$. Consider $H$ being a function from the carrier of $E \times F$ into the carrier of $E \times F$ such that for every point $p$ of $E \times F, H(p)=\mathcal{C}(p)$. For every point $x$ of $E$ and for every point $y$ of $F, H(x, y)=\langle x, y-b\rangle$. Define $\mathcal{D}$ (point of $E \times F)=1 \cdot \$_{1}+e_{0}$. Consider $K$ being a function from the carrier of $E \times F$ into the carrier of $E \times F$ such that for every point $p$ of $E \times F, K(p)=\mathcal{D}(p)$. For every point $p$ of $E \times F, K \cdot H(p)=p$. For every point $p$ of $E \times F, H \cdot K(p)=p$. Reconsider $Z_{1}=H^{\circ} Z$ as a subset of $E \times F$. For every point $x$ of $E$ and for every
point $y$ of $F,\langle x, y+b\rangle \in Z$ iff $\langle x, y\rangle \in Z_{1}$. Reconsider $e_{0}=\left\langle 0_{E}, b\right\rangle$ as a point of $E \times F$. For every object $p, p \in Z_{1}$ iff $p \in\left\{y-e_{0}\right.$, where $y$ is a point of $E \times F: y \in Z\} . Z_{1}$ is open. Define $\mathcal{J}[$ object, object $] \equiv$ there exists a point $x$ of $E$ and there exists a point $y$ of $F$ such that $\$_{1}=\langle x$, $y\rangle$ and $\$_{2}=f_{\langle x, y+b\rangle}-c$. For every object $p$ such that $p \in Z_{1}$ there exists an object $w$ such that $w \in$ the carrier of $G$ and $\mathcal{J}[p, w]$. Consider $f_{1}$ being a function from $Z_{1}$ into $G$ such that for every object $p$ such that $p \in Z_{1}$ holds $\mathcal{J}\left[p, f_{1}(p)\right]$. For every point $x$ of $E$ and for every point $y$ of $F$ such that $\langle x, y\rangle \in Z_{1}$ holds $f_{1}(x, y)=f_{\langle x, y+b\rangle}-c$. Define $\mathcal{O}$ [object, object $] \equiv$ there exists a point $x$ of $E$ and there exists a point $y$ of $F$ such that $\$_{1}=\langle x$, $y\rangle$ and $\$_{2}=Q\left(f_{1}(x, y)\right)$. For every object $p$ such that $p \in Z_{1}$ there exists an object $w$ such that $w \in$ the carrier of $F$ and $\mathcal{O}[p, w]$. Consider $f_{2}$ being a function from $Z_{1}$ into $F$ such that for every object $p$ such that $p \in Z_{1}$ holds $\mathcal{O}\left[p, f_{2}(p)\right]$. For every point $x$ of $E$ and for every point $y$ of $F$ such that $\langle x, y\rangle \in Z_{1}$ holds $f_{2}(x, y)=Q\left(f_{1}(x, y)\right)$. Define $\mathcal{U}[$ object, object $] \equiv$ there exists a point $x$ of $E$ and there exists a point $y$ of $F$ such that $\$_{1}=\langle x, y\rangle$ and $\$_{2}=y-f_{2}\langle x, y\rangle$. For every object $p$ such that $p \in Z_{1}$ there exists an object $w$ such that $w \in$ the carrier of $F$ and $\mathcal{U}[p, w]$.

Consider $F_{1}$ being a function from $Z_{1}$ into $F$ such that for every object $p$ such that $p \in Z_{1}$ holds $\mathcal{U}\left[p, F_{1}(p)\right]$. For every point $x$ of $E$ and for every point $y$ of $F$ such that $\langle x, y\rangle \in Z_{1}$ holds $F_{1}(x, y)=y-f_{2}\langle x, y\rangle$. For every point $z_{0}$ of $E \times F$ and for every real number $r$ such that $z_{0} \in Z_{1}$ and $0<r$ there exists a real number $s$ such that $0<s$ and for every point $z_{1}$ of $E \times F$ such that $z_{1} \in Z_{1}$ and $\left\|z_{1}-z_{0}\right\|<s$ holds $\left\|F_{1 z_{1}}-F_{1 z_{0}}\right\|<r$. For every point $w_{0}$ of $E \times F$ such that $w_{0} \in Z$ holds $f \cdot\left(\operatorname{reproj} 2\left(w_{0}\right)\right)$ is differentiable in $\left(w_{0}\right)_{\mathbf{2}}$. For every point $w_{0}$ of $E \times F$ such that $w_{0} \in Z$ there exists a neighbourhood $N$ of $\left(w_{0}\right)_{2}$ such that $N \subseteq \operatorname{dom} f \cdot\left(\operatorname{reproj} 2\left(w_{0}\right)\right)$ and there exists a rest $R$ of $F, G$ such that for every point $w_{1}$ of $F$ such that $w_{1} \in N$ holds $f \cdot\left(\operatorname{reproj} 2\left(w_{0}\right)\right)_{w_{1}}-f \cdot\left(\operatorname{reproj} 2\left(w_{0}\right)\right)_{\left(w_{0}\right)_{2}}=$ $f \cdot\left(\operatorname{reproj} 2\left(w_{0}\right)\right)^{\prime}\left(\left(w_{0}\right)_{\mathbf{2}}\right)\left(w_{1}-\left(w_{0}\right)_{\mathbf{2}}\right)+R_{w_{1}-\left(w_{0}\right)_{2}}$. For every point $z_{0}$ of $E \times F$ such that $z_{0} \in Z_{1}$ holds $F_{1} \cdot\left(\operatorname{reproj} 2\left(z_{0}\right)\right)$ is differentiable in $\left(z_{0}\right)_{2}$ and there exist points $L_{0}, I$ of the real norm space of bounded linear operators from $F$ into $F$ such that $L_{0}=Q \cdot\left(\left(f \upharpoonright^{2} Z\right)_{z_{0}+e_{0}}\right)$ and $I=\mathrm{id}_{\alpha}$ and $F_{1} \cdot\left(\operatorname{reproj} 2\left(z_{0}\right)\right)^{\prime}\left(\left(z_{0}\right)_{2}\right)=I-L_{0}$, where $\alpha$ is the carrier of $F$. $\operatorname{dom}\left(F_{1} \upharpoonright^{2}\right.$ $\left.Z_{1}\right)=Z_{1}$ and for every point $z$ of $E \times F$ such that $z \in Z_{1}$ there exist points $L, I$ of the real norm space of bounded linear operators from $F$ into $F$ such that $L=Q \cdot\left(\left(f \upharpoonright^{2} Z\right)_{z+e_{0}}\right)$ and $I=\mathrm{id}_{\alpha}$ and $\left(F_{1} \upharpoonright^{2} Z_{1}\right)_{z}=I-L$, where $\alpha$ is the carrier of $F$. Set $F_{2}=F_{1} \upharpoonright^{2} Z_{1}$. For every point $z_{0}$ of $E \times$ $F$ and for every real number $r$ such that $z_{0} \in Z_{1}$ and $0<r$ there exists
a real number $s$ such that $0<s$ and for every point $z_{1}$ of $E \times F$ such that $z_{1} \in Z_{1}$ and $\left\|z_{1}-z_{0}\right\|<s$ holds $\left\|F_{2 z_{1}}-F_{2 z_{0}}\right\|<r . F_{1}\left(a, 0_{F}\right)=0_{F}$ by [6, (3)]. Reconsider $a_{0}=\left\langle a, 0_{F}\right\rangle$ as a point of $E \times F$. Consider $r_{4}$ being a real number such that $0<r_{4}$ and for every point $s$ of $E \times F$ such that $s \in Z_{1}$ and $\left\|s-a_{0}\right\|<r_{4}$ holds $\left\|\left(F_{1} \upharpoonright^{2} Z_{1}\right)_{s}-\left(F_{1} \upharpoonright^{2} Z_{1}\right)_{a_{0}}\right\|<\frac{1}{4}$. Consider $r_{5}$ being a real number such that $0<r_{5}$ and $\operatorname{Ball}\left(a_{0}, r_{5}\right) \subseteq Z_{1}$. Reconsider $r_{6}=\min \left(r_{4}, r_{5}\right)$ as a real number. $\operatorname{Ball}\left(a_{0}, r_{6}\right) \subseteq \operatorname{Ball}\left(a_{0}, r_{5}\right)$.

Consider $r_{1}$ being a real number such that $0<r_{1}<r_{6}$ and $\operatorname{Ball}\left(a, r_{1}\right) \times \operatorname{Ball}\left(0_{F}, r_{1}\right) \subseteq \operatorname{Ball}\left(a_{0}, r_{6}\right)$. For every point $x$ of $E \times F$ such that $x \in Z_{1}$ holds $\left(F_{1} \upharpoonright^{2} Z_{1}\right)_{x}=F_{1} \cdot(\operatorname{reproj} 2(x))^{\prime}\left((x)_{2}\right) . a \in \operatorname{Ball}\left(a, r_{1}\right)$. $0_{F} \in \operatorname{Ball}\left(0_{F}, r_{1}\right)$. Reconsider $a_{0}=\left\langle a, 0_{F}\right\rangle$ as a point of $E \times F$. Consider $L_{1}, I_{1}$ being points of the real norm space of bounded linear operators from $F$ into $F$ such that $L_{1}=Q \cdot\left(\left(f \upharpoonright^{2} Z\right)_{a_{0}+e_{0}}\right)$ and $I_{1}=\mathrm{id}_{\alpha}$ and $\left(F_{1} \upharpoonright^{2} Z_{1}\right)_{a_{0}}=I_{1}-L_{1}$, where $\alpha$ is the carrier of $F$. For every point $x$ of $E$ and for every point $y$ of $F$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ and $y \in$ $\operatorname{Ball}\left(0_{F}, r_{1}\right)$ holds $\left\|\left(F_{1} \upharpoonright^{2} Z_{1}\right)_{\langle x, y\rangle}\right\|<\frac{1}{4}$. Reconsider $r_{2}=\frac{r_{1}}{2}$ as a real number. Consider $a_{2}$ being a real number such that $0<a_{2}$ and for every point $s$ of $E \times F$ such that $s \in Z_{1}$ and $\left\|s-a_{0}\right\|<a_{2}$ holds $\| F_{1 s}-$ $F_{1 a_{0}} \|<\left(\frac{1}{2}\right) \cdot r_{2}$. Consider $a_{4}$ being a real number such that $0<a_{4}<a_{2}$ and $\operatorname{Ball}\left(a, a_{4}\right) \times \operatorname{Ball}\left(0_{F}, a_{4}\right) \subseteq \operatorname{Ball}\left(a_{0}, a_{2}\right)$. Reconsider $a_{3}=\min \left(a_{2}, a_{4}\right)$ as a real number. $\operatorname{Ball}\left(a, a_{3}\right) \subseteq \operatorname{Ball}\left(a, a_{4}\right)$. Reconsider $a_{1}=\min \left(a_{3}, r_{1}\right)$ as a real number. $\operatorname{Ball}\left(a, a_{1}\right) \subseteq \operatorname{Ball}\left(a, r_{1}\right) . \operatorname{Ball}\left(a, a_{1}\right) \subseteq \operatorname{Ball}\left(a, a_{3}\right)$. For every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, a_{1}\right)$ holds $\| F_{1}\left\langle_{\left.x, 0_{F}\right\rangle} \| \leqslant\left(\frac{1}{2}\right) \cdot r_{2}\right.$. Reconsider $r_{0}=\min \left(\frac{r_{1}}{2}, a_{1}\right)$ as a real number. $\operatorname{Ball}\left(a, r_{0}\right) \subseteq \operatorname{Ball}\left(a, r_{1}\right)$. For every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{0}\right)$ holds $\left\|F_{1\left\langle x, 0_{F}\right\rangle}\right\| \leqslant$ $\left(\frac{1}{2}\right) \cdot r_{2} \cdot \overline{\operatorname{Ball}}\left(0_{F}, r_{2}\right) \subseteq \operatorname{Ball}\left(0_{F}, r_{1}\right)$. For every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{0}\right)$ for every points $y_{1}, y_{2}$ of $F$ such that $y_{1}, y_{2} \in \operatorname{Ball}\left(0_{F}, r_{2}\right)$ holds $\left\|F_{1}{ }_{\left\langle x, y_{1}\right\rangle}-F_{1}{ }_{\left\langle x, y_{2}\right\rangle}\right\| \leqslant\left(\frac{1}{2}\right) \cdot\left\|y_{1}-y_{2}\right\|$. For every point $x$ of $E$ and for every point $y$ of $F$ such that $x \in \operatorname{Ball}\left(a, r_{0}\right)$ and $y \in \overline{\operatorname{Ball}}\left(0_{F}, r_{2}\right)$ holds $F_{1}(x, y) \in \overline{\operatorname{Ball}}\left(0_{F}, r_{2}\right) . \operatorname{Ball}\left(a, r_{0}\right) \neq \emptyset \cdot \overline{\operatorname{Ball}}\left(0_{F}, r_{2}\right) \neq \emptyset$. For every point $y$ of $F$ such that $y \in \overline{\operatorname{Ball}}\left(0_{F}, r_{2}\right)$ for every point $x_{0}$ of $E$ such that $x_{0} \in \operatorname{Ball}\left(a, r_{0}\right)$ for every real number $e$ such that $0<e$ there exists a real number $d$ such that $0<d$ and for every point $x_{1}$ of $E$ such that $x_{1} \in \operatorname{Ball}\left(a, r_{0}\right)$ and $\left\|x_{1}-x_{0}\right\|<d$ holds $\left\|F_{1}{ }_{\left\langle x_{1}, y\right\rangle}-F_{1}{ }_{{ }_{\left.x_{0}, y\right\rangle}}\right\|<e$.

Consider $\Psi$ being a partial function from $E$ to $F$ such that $\Psi$ is continuous on $\operatorname{Ball}\left(a, r_{0}\right)$ and dom $\Psi=\operatorname{Ball}\left(a, r_{0}\right)$ and $\operatorname{rng} \Psi \subseteq \overline{\operatorname{Ball}}\left(0_{F}, r_{2}\right)$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{0}\right)$ holds $F_{1}(x, \Psi(x))=$ $\Psi(x)$. For every object $z, z \in \overline{\operatorname{Ball}}\left(b, r_{2}\right)$ iff $z \in\{y+b$, where $y$ is a point of $\left.F: y \in \overline{\operatorname{Ball}}\left(0_{F}, r_{2}\right)\right\}$. For every object $y, y \in \operatorname{Ball}\left(a, r_{0}\right) \times \overline{\operatorname{Ball}}\left(b, r_{2}\right)$
iff there exists an object $x$ such that $x \in \operatorname{dom} K$ and $x \in \operatorname{Ball}\left(a, r_{0}\right) \times$ $\overline{\operatorname{Ball}}\left(0_{F}, r_{2}\right)$ and $y=K(x)$. Define $\mathcal{W}($ object $)=\Psi_{\$_{1}}+b$. For every object $y$ such that $y \in \operatorname{Ball}\left(a, r_{0}\right)$ holds $\mathcal{W}(y) \in \overline{\operatorname{Ball}}\left(b, r_{2}\right)$. Consider $E_{3}$ being a function from $\operatorname{Ball}\left(a, r_{0}\right)$ into $\overline{\operatorname{Ball}}\left(b, r_{2}\right)$ such that for every object $y$ such that $y \in \operatorname{Ball}\left(a, r_{0}\right)$ holds $E_{3}(y)=\mathcal{W}(y) . \overline{\operatorname{Ball}}\left(b, r_{2}\right) \neq \emptyset$. For every point $x_{0}$ of $E$ and for every real number $r$ such that $x_{0} \in \operatorname{Ball}\left(a, r_{0}\right)$ and $0<r$ there exists a real number $s$ such that $0<s$ and for every point $x_{1}$ of $E$ such that $x_{1} \in \operatorname{Ball}\left(a, r_{0}\right)$ and $\left\|x_{1}-x_{0}\right\|<s$ holds $\left\|E_{3 x_{1}}-E_{3 x_{0}}\right\|<r$. For every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{0}\right)$ holds $f\left(x, E_{3}(x)\right)=c$. For every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{0}\right)$ there exists a point $y$ of $F$ such that $y \in \overline{\operatorname{Ball}}\left(b, r_{2}\right)$ and $f(x, y)=c$. For every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{0}\right)$ for every points $y_{1}, y_{2}$ of $F$ such that $y_{1}, y_{2} \in \overline{\operatorname{Ball}}\left(b, r_{2}\right)$ and $f\left(x, y_{1}\right)=c$ and $f\left(x, y_{2}\right)=c$ holds $y_{1}=y_{2} . a \in \operatorname{Ball}\left(a, r_{0}\right)$ and $b \in$ $\overline{\operatorname{Ball}}\left(b, r_{2}\right) . E_{3}(a) \in \operatorname{rng} E_{3} . f\left(a, E_{3}(a)\right)=c$. For every partial functions $E_{1}$, $E_{2}$ from $E$ to $F$ such that dom $E_{1}=\operatorname{Ball}\left(a, r_{0}\right)$ and $\operatorname{rng} E_{1} \subseteq \overline{\operatorname{Ball}}\left(b, r_{2}\right)$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{0}\right)$ holds $f\left(x, E_{1}(x)\right)=c$ and $\operatorname{dom} E_{2}=\operatorname{Ball}\left(a, r_{0}\right)$ and $\operatorname{rng} E_{2} \subseteq \overline{\operatorname{Ball}}\left(b, r_{2}\right)$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{0}\right)$ holds $f\left(x, E_{2}(x)\right)=c$ holds $E_{1}=E_{2}$.
(36) Suppose $Z$ is open and $\operatorname{dom} f=Z$ and $f$ is continuous on $Z$ and $f$ is partially differentiable on $Z$ w.r.t. 2 and $f \vdash^{2} Z$ is continuous on $Z$ and $z=\langle a, b\rangle$ and $z \in Z$ and $f(a, b)=c$ and $\operatorname{partdiff}(f, z)$ w.r.t. 2 is one-toone and (partdiff $(f, z)$ w.r.t. 2) ${ }^{-1}$ is a Lipschitzian linear operator from $G$ into $F$. Then there exist real numbers $r_{1}, r_{2}$ such that
(i) $0<r_{1}$, and
(ii) $0<r_{2}$, and
(iii) $\operatorname{Ball}\left(a, r_{1}\right) \times \overline{\operatorname{Ball}}\left(b, r_{2}\right) \subseteq Z$, and
(iv) for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ there exists a point $y$ of $F$ such that $y \in \operatorname{Ball}\left(b, r_{2}\right)$ and $f(x, y)=c$, and
(v) for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ for every points $y_{1}, y_{2}$ of $F$ such that $y_{1}, y_{2} \in \operatorname{Ball}\left(b, r_{2}\right)$ and $f\left(x, y_{1}\right)=c$ and $f\left(x, y_{2}\right)=c$ holds $y_{1}=y_{2}$, and
(vi) there exists a partial function $g$ from $E$ to $F$ such that $g$ is continuous on $\operatorname{Ball}\left(a, r_{1}\right)$ and $\operatorname{dom} g=\operatorname{Ball}\left(a, r_{1}\right)$ and $r n g g \subseteq \operatorname{Ball}\left(b, r_{2}\right)$ and $g(a)=b$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ holds $f(x, g(x))=c$, and
(vii) for every partial functions $g_{1}, g_{2}$ from $E$ to $F$ such that dom $g_{1}=$ $\operatorname{Ball}\left(a, r_{1}\right)$ and $\operatorname{rng} g_{1} \subseteq \operatorname{Ball}\left(b, r_{2}\right)$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ holds $f\left(x, g_{1}(x)\right)=c$ and $\operatorname{dom} g_{2}=\operatorname{Ball}\left(a, r_{1}\right)$
and $\operatorname{rng} g_{2} \subseteq \operatorname{Ball}\left(b, r_{2}\right)$ and for every point $x$ of $E$ such that $x \in$ $\operatorname{Ball}\left(a, r_{1}\right)$ holds $f\left(x, g_{2}(x)\right)=c$ holds $g_{1}=g_{2}$.

Proof: Consider $r_{1}, r_{2}$ being real numbers such that $0<r_{1}$ and $0<r_{2}$ and $\operatorname{Ball}\left(a, r_{1}\right) \times \overline{\operatorname{Ball}}\left(b, r_{2}\right) \subseteq Z$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ there exists a point $y$ of $F$ such that $y \in \overline{\operatorname{Ball}\left(b, r_{2}\right) \text { and }}$ $f(x, y)=c$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ for every points $y_{1}, y_{2}$ of $F$ such that $y_{1}, y_{2} \in \overline{\operatorname{Ball}\left(b, r_{2}\right) \text { and } f\left(x, y_{1}\right)=c \text { and }}$ $f\left(x, y_{2}\right)=c$ holds $y_{1}=y_{2}$ and there exists a partial function $g$ from $E$ to $F$ such that $g$ is continuous on $\operatorname{Ball}\left(a, r_{1}\right)$ and $\operatorname{dom} g=\operatorname{Ball}\left(a, r_{1}\right)$ and $\operatorname{rng} g \subseteq \overline{\operatorname{Ball}}\left(b, r_{2}\right)$ and $g(a)=b$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ holds $f(x, g(x))=c$ and for every partial functions $g_{1}, g_{2}$ from $E$ to $F$ such that $\operatorname{dom} g_{1}=\operatorname{Ball}\left(a, r_{1}\right)$ and $\operatorname{rng} g_{1} \subseteq \overline{\operatorname{Ball}\left(b, r_{2}\right) \text { and }}$ for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ holds $f\left(x, g_{1}(x)\right)=c$ and $\operatorname{dom} g_{2}=\operatorname{Ball}\left(a, r_{1}\right)$ and $\operatorname{rng} g_{2} \subseteq \overline{\operatorname{Ball}}\left(b, r_{2}\right)$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ holds $f\left(x, g_{2}(x)\right)=c$ holds $g_{1}=g_{2}$.

Consider $g$ being a partial function from $E$ to $F$ such that $g$ is continuous on $\operatorname{Ball}\left(a, r_{1}\right)$ and $\operatorname{dom} g=\operatorname{Ball}\left(a, r_{1}\right)$ and $\operatorname{rng} g \subseteq \overline{\operatorname{Ball}}\left(b, r_{2}\right)$ and $g(a)=b$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ holds $f(x, g(x))=c$ and for every partial functions $g_{1}, g_{2}$ from $E$ to $F$ such that $\operatorname{dom} g_{1}=\operatorname{Ball}\left(a, r_{1}\right)$ and $\operatorname{rng} g_{1} \subseteq \overline{\operatorname{Ball}}\left(b, r_{2}\right)$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ holds $f\left(x, g_{1}(x)\right)=c$ and $\operatorname{dom} g_{2}=\operatorname{Ball}\left(a, r_{1}\right)$ and $\operatorname{rng} g_{2} \subseteq \overline{\operatorname{Ball}}\left(b, r_{2}\right)$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{1}\right)$ holds $f\left(x, g_{2}(x)\right)=c$ holds $g_{1}=g_{2} . a \in \operatorname{Ball}\left(a, r_{1}\right)$. Consider $r_{3}$ being a real number such that $0<r_{3}$ and for every point $x_{1}$ of $E$ such that $x_{1} \in$ $\operatorname{dom} g$ and $\left\|x_{1}-a\right\|<r_{3}$ holds $\left\|g_{x_{1}}-g_{a}\right\|<r_{2}$. Reconsider $r_{0}=\min \left(r_{1}, r_{3}\right)$ as a real number. $\operatorname{Ball}\left(a, r_{0}\right) \subseteq \operatorname{Ball}\left(a, r_{1}\right)$ and $\operatorname{Ball}\left(a, r_{0}\right) \subseteq \operatorname{Ball}\left(a, r_{3}\right)$. For every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{0}\right)$ there exists a point $y$ of $F$ such that $y \in \operatorname{Ball}\left(b, r_{2}\right)$ and $f(x, y)=c$.

For every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{0}\right)$ for every points $y_{1}, y_{2}$ of $F$ such that $y_{1}, y_{2} \in \operatorname{Ball}\left(b, r_{2}\right)$ and $f\left(x, y_{1}\right)=c$ and $f\left(x, y_{2}\right)=c$ holds $y_{1}=y_{2}$. Reconsider $g_{1}=g \upharpoonright \operatorname{Ball}\left(a, r_{0}\right)$ as a partial function from $E$ to $F$. $\operatorname{dom} g_{1}=\operatorname{Ball}\left(a, r_{0}\right)$. For every object $y$ such that $y \in \operatorname{rng} g_{1}$ holds $y \in \operatorname{Ball}\left(b, r_{2}\right) . g_{1}(a)=b$. For every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{0}\right)$ holds $f\left(x, g_{1}(x)\right)=c$. For every partial functions $g_{1}, g_{2}$ from $E$ to $F$ such that $\operatorname{dom} g_{1}=\operatorname{Ball}\left(a, r_{0}\right)$ and $\operatorname{rng} g_{1} \subseteq \operatorname{Ball}\left(b, r_{2}\right)$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{0}\right)$ holds $f\left(x, g_{1}(x)\right)=c$ and $\operatorname{dom} g_{2}=\operatorname{Ball}\left(a, r_{0}\right)$ and $\operatorname{rng} g_{2} \subseteq \operatorname{Ball}\left(b, r_{2}\right)$ and for every point $x$ of $E$ such that $x \in \operatorname{Ball}\left(a, r_{0}\right)$ holds $f\left(x, g_{2}(x)\right)=c$ holds $g_{1}=g_{2}$.

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